Acta Math. Hungar. 93 (4), (2001), 281-325

Minimizers of energy functionals

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Abstract. We consider a general class of problems of minimization of convex integral functionals (maximization of entropy) subject to linear constraints. Under general assumptions, the minimizing solutions are characterized. Our results improve previous literature on the subject in the following directions:

- a necessary and sufficient condition for the shape of the minimizing density is proved
- without constraint qualification
- under infinitely many linear constraints subject to natural integrability conditions (no topological restrictions).

As an illustration, we give the general shape of the minimizing density for the marginal problem on a product space. Finally, a counterexample of I. Csiszár is clarified. Our proofs mainly rely on convex duality.

1. Introduction

We consider the energy functionals defined on the space: $M(\Omega)$, of the signed measures on the measure space $(\Omega, \mathcal{A})$ which are of the following form

$$I(Q) = \int_\Omega \gamma^* \left(\frac{dQ}{dR} \right) dR \in [0, +\infty], \quad Q \in M(\Omega)$$

if Q is absolutely continuous with respect to the given nonnegative reference measure R, and $I(Q) = +\infty$ otherwise. The function $\gamma^* : \mathbb{R} \to [0, \infty]$ is the convex conjugate of a function γ, hence it is convex and lower semicontinuous. The energy functional I is sometimes called γ^*-divergence or γ^*-entropy. With the special choice: $\gamma(x) = e^x - x - 1$, we get $\gamma^*(x) = (x+1)\log(x+1) - x$ and, P, R being probability measures, $I(P - R)$ is the relative entropy of P with respect to R which is also called Kullback information or Boltzmann-Shannon entropy of P with respect to R.

Let $A : M(\Omega) \to \mathcal{X}$ be a linear operator on $M(\Omega)$ with its values in a vector space $\mathcal{X}$; $A(Q) = x$ is the expression of a linear constraint. We are concerned with the minimum energy problem

$$\inf \{ I(Q) \,;\, Q \in M(\Omega), A(Q) = x_o \} . \tag{1.1}$$

Notice that a solution to (1.1) is absolutely continuous with respect to R. Such a minimization problem is sometimes called a maximum entropy problem: $-I$ may be seen as an entropy.

Keywords: maximum entropy method, convex integral functionals, convex conjugacy, Orlicz spaces, marginal problems.

Mathematics Subject Classification: 49K22, 52A41, 46B10.

The classical moment constraint: $\int_\Omega f_k\,dQ = x_k,\ f_k : \Omega \to \mathrm{I\!R},\ 1 \le k \le n$ corresponds to $\mathcal{X} = \mathrm{I\!R}^n$ and $A(Q) = \left(\int_\Omega f_k\,dQ\right)_{1\le k\le n}$. It naturally extends to infinitely many moments with

$$A(Q) = \left(\int_\Omega f_k\,dQ\right)_{k\ge 1} \in \mathcal{X} = \mathrm{I\!R}^{\{1,2,\dots\}}.$$

Therefore, the problem (1.1) leads to solutions of some moment problems under the additional constraint: $I(Q) < \infty$, which implies in particular that Q is absolutely continuous with respect to R. It is extensively studied in the literature, among others, let us refer to: [BL1-5], [Cs1-3], [DCG], [GG], [LV], [TV] and more recently [CGG].

Some marginal problems may also be of type (1.1). Consider a product measure space $\Omega = \Omega_1 \times \Omega_2$ and find the measures Q on Ω whose marginals Q_1 on Ω_1 and Q_2 on Ω_2 are given. The constraint is $Q_1 = \nu_1 \in M(\Omega_1)$, $Q_2 = \nu_2 \in M(\Omega_2)$ which corresponds to $A(Q) = (Q_1, Q_2) \in \mathcal{X} = M(\Omega_1) \times M(\Omega_2)$. It can also be regarded as a moment constraint: $A(Q) = \left(\int_\Omega f_\theta\,dQ\right)_{\theta\in\Theta} \in \mathrm{I\!R}^\Theta$ where Θ is not countable in the general case. Marginal problems of type (1.1) are considered in [BLN], [CG] and [RüT]. See also [Beu] for a close related problem.

This problem can be extended with an infinite number of marginals. Let us consider the space $\Omega = E^{[0,1]}$ of the E-valued paths $\omega = (\omega_t)_{0\le t\le 1}$. If Q is a probability measure on Ω, let $Q_t \in M(E)$ stand for its marginal at time t : the law of ω_t under Q. The constraint $Q_t = \nu_t, \forall t \in [0,1]$, where $(\nu_t)_{0\le t\le 1}$ is a given flow of probability measures on E, corresponds to $A(Q) = (Q_t)_{0\le t\le 1} \in \mathcal{X} = M(E)^{[0,1]}$. Such problems of reconstruction of laws with given marginal flows appear naturally in the statistical mechanics of large dynamical particle systems (see [DaG], [Föl], [CaL]). The intermediate problem where the only initial and final marginal laws are constrained: $Q_0 = \nu_0$, $Q_1 = \nu_1$ is related to the construction of Schrödinger bridges (see [Zam], [Föl], [DP], [FöG], [Lé2]) and for a recent account on this subject see [CWZ] and the references therein. In the above reconstruction problems motivated by physical questions, the relevant energy functional is the relative entropy.

The formal dual problem associated with (1.1) is

$$\sup\left\{\langle x_o, y\rangle - \int_\Omega \gamma(A^*y)\,dR\,;\ y \in \mathcal{Y}\right\} \tag{1.2}$$

where $\mathcal{Y}$ is a vector space in duality with $\mathcal{X}$, A^* is the formal adjoint of A and A^*y is a measurable function for any $y \in \mathcal{Y}$. In [BL1], J. M. Borwein and A. S. Lewis have proved that if γ is everywhere finite and differentiable, if $\mathcal{X}$ is finite dimensional and if the following constraint qualification:

$$\text{there is a } \hat{Q} \in M(\Omega) \text{ such that } A(\hat{Q}) = x_o \text{ and } \inf \mathit{dom}\,\gamma^* < \inf \frac{d\hat{Q}}{dR} \le \sup \frac{d\hat{Q}}{dR} < \sup \mathit{dom}\,\gamma^*$$

holds, then the value of the primal problem (1.1) equals the value of the dual problem (1.2), there is attainment in (1.2) and the unique optimal solution of (1.1) is given by

$$Q_* = \gamma'(A^*y_*)\cdot R \tag{1.3}$$

where y_* is any solution of (1.2). The representation of the solutions to (1.1) is related to the dual attainment.

If one is only interested in the dual equality: $\inf(1.1) = \sup(1.2)$, and drops the problem of the dual attainment (together with the question of the representation of the primal solutions), no constraint qualification is required. In [Bor], it is proved that the dual equality holds when A is a continuous operator from $L^1(\Omega, R)$ to a normed space $\mathcal{X}$. This equality had already been obtained by R. T. Rockafellar ([Roc], Theorem 23) in the case where $f \mapsto \int_\Omega \gamma(f)\, dR$ is everywhere finite and continuous on some reflexive Orlicz space built on $(\Omega, \mathcal{A}, R)$. Using standard Fenchel duality techniques developped in [Roc], a related dual equality is obtained in ([Lé2], Theorems 3.3 and 3.4) assuming that $\mathcal{X}$ is in duality with some space $\mathcal{Y}$ such that

$$\forall y \in \mathcal{Y}, \exists \lambda > 0 \text{ such that } \int_\Omega \gamma(\lambda A^* y)\, dR + \int_\Omega \gamma(-\lambda A^* y)\, dR < \infty. \tag{1.4}$$

By means of relaxation and penalizing methods, under quite general linear constraints, J. M. Borwein ([Bor]) approximates the solution of (1.1) by sequences whose terms can be represented by a formula of the type (1.3).

The aim of this article is to give an exact (without approximating sequences) representation of the solution of (1.1), not assuming any constraint qualification. Our main result is Theorem 4.5: a necessary and sufficient condition (of the type (1.3)) is given for a measure to be the solution of the minimization problem (1.1). More precisely, one can formally extend the dual problem (1.2) by:

$$\sup_z \left\{ z(x_o) - \int_\Omega \gamma\Big(\overline{A^*} z(\omega)\Big)\, R(d\omega) \right\} \tag{1.5}$$

where $z(x_o)$ may achieve the values $\pm\infty$, $x \mapsto z(x)$ acts linearly on $\{x\,;\, -\infty < z(x) < +\infty\}$, $\overline{A^*}$ extends A^*, $\overline{A^*}z(\omega)$ may achieve the values $\pm\infty$ and $z \mapsto \overline{A^*}z(\omega)$ acts linearly on $\{z\,;\, -\infty < \overline{A^*}z(\omega) < +\infty\}$. If this dual value is finite, then it (formally) exists a z_{x_o} such that

$$\langle x_o, y\rangle = \int_\Omega A^* y(\omega) \gamma'\Big(\overline{A^*} z_{x_o}(\omega)\Big)\, R(d\omega), \ \forall y \in \mathcal{Y} \tag{1.6}$$

(with $\gamma'(\pm\infty) = \lim_{x\to\pm\infty} \gamma'(x)$) which is an extension of (1.3). We characterize those z_{x_o} such that, x_o being defined by (1.6), the value of (1.5) is finite. They will be called admissible force fields.

One obtains this result strengthening (1.4) into:

$$\forall y \in \mathcal{Y}, \int_\Omega \gamma(A^* y)\, dR < \infty. \tag{1.7}$$

In particular, this implies that γ is everywhere finite and excludes the interesting phenomenon of singular solutions (see [BL4], [GG],[Lé2]).

Comparison with already existing results. A characterization of the minimizer Q_* in terms of the cancelling of a gradient is given in [Rüs] and extended in ([LV], Theorem 8.10) and ([TV], Theorem 2). It doesn't lead to the exact shape of the density of the minimizer: $\frac{dQ_*}{dR}$.

In the special case of the relative entropy, ([Cs1], Theorem 3.1) states a necessary condition and a sufficient condition as well, for a density to be $\frac{dQ_*}{dR}$. Except for a finite number of moment constraints, it remains a gap between these conditions to be both necessary and sufficient. Similar conditions in more general situations are obtained in ([Cs2], Lemma 3.4) and ([LV], Theorem 8.20).

For a finite number of qualified constraints, the characterization of $\frac{dQ_*}{dR}$ is given in [BL4] and extended in [CGG]. Let us mention that a qualified constraint is "interior" and it follows from our results that the force field associated with Q_* doesn't take any infinite values.

Our main results: Theorem 4.4, 4.5 and 4.6, close the problem of the characterization of $\frac{dQ_*}{dR}$ under the integrability condition (1.7), without any topological restrictions, for a large class of energy functionals including the relative entropy. They improve the already published related results.

As an application of these abstract results, we shall consider in Section 5, energy minimization problems related to the marginal problem. In this setting, a necessary condition for a density to be $\frac{dQ_*}{dR}$ is stated in ([CG], Theorem 5.4). Our Theorem 5.1 gives a necessary and sufficient condition for a density to be $\frac{dQ_*}{dR}$ under general conditions. In particular, no topological restrictions on the underlying measure space are assumed. This new result is the complete solution to the problem of characterizing the energy minimizer in the marginal problem.

For the general shape of the minimizing density in the important special case of the relative entropy, see (5.9).

Outline of the paper. Our approach mainly relies on classical results of convex conjugacy. We consider algebraic duality rather than any arbitrary topological duality: linear forms are not subject to any a priori regularity restrictions. Of course, this creates difficulties. They have been partly solved in [Lé1], taking advantage of the convex integral functional form of the objective function I. The results of [Lé1] which will be used later are recalled in Section 2.

In Section 3, our representation problem is solved for the "subgradient constraints" and a rough representation of the solutions is proved for the "boundary constraints".

In Section 4, a detailed representation of the solution of (1.1) is derived and a necessary and sufficient condition is obtained in Theorems 4.4, 4.5 and 4.6.

In Section 5, we give some illustrations of our general results in the special case of the marginal problem. Finally, an astonishing counterexample of I. Csiszár is clarified: a natural candidate to be a minimizer is shown not be the correct answer for its related force field isn't admissible.

2. Preliminaries

In the first subsection, some usual results of convex analysis are recalled. In the second subsection, a minimization problem for a convex conjugate (see (2.2)) is introduced in a general setting; then, our approach for solving this minimization problem is described at Proposition 2.3. In the last subsection, the statements of some results of [Lé1] are recalled in Proposition 2.4 and Theorem 2.5; our approach deeply relies on them.

Basic convex analysis. Let X and Y be two vector spaces in separating duality for the bracket: $(x,y)\in X\times Y\mapsto \langle x,y\rangle\in \mathbb{R}$. We consider a function

$$f:x\in X\mapsto f(x)\in]-\infty,+\infty].$$

Its conjugate f^* is defined by

$$f^*:y\in Y\mapsto \sup_{x\in X}\{\langle x,y\rangle-f(x)\}\in]-\infty,+\infty],$$

where we assume that there exists $x_o\in X$ such that $f(x_o)<+\infty$, so that $f^*(y)>-\infty,\forall y\in Y$. In particular, if $f(0)=0$, then $f^*(Y)\subset[0,+\infty]$.

The topology $\sigma(X,Y)$ is the weakest topology on X such that every linear form $\langle\cdot,y\rangle$, $y\in Y$ is continuous and $\sigma(Y,X)$ is the weakest topology on Y such that every linear form $\langle x,\cdot\rangle$, $x\in X$ is continuous.

As a supremum of affine continuous functions, f^* is a convex $\sigma(Y,X)$-lower semicontinuous function.

In this paper, it is understood that all the convex functions are proper convex, that is: convex and $]-\infty,+\infty]$-valued with at least one finite value.

The geometric interior (core) of a subset A of X is the set of those $a\in A$ such that for any $x\in X$, there exists $\lambda>0$ satisfying $[a,\lambda x[\subset A$.

The affine hull of A : *aff* A, is the smallest affine space containing A. The relative interior of A : *ri* A, is the geometric interior of A considered as a subset of its affine hull: *ri* $A=\{a\in A\,;\forall x\in$ *aff* $A,\exists\lambda>0,[a,\lambda x[\subset A\}$.

The relative boundary of A : *rb* A, is defined by: *rb* $A=A\setminus$ *ri* A. Notice that *rb* A is included in A rather than in its closure.

The effective domain of f is *dom* $f:=\{x\in X\,;\,f(x)<+\infty\}$.

The relative interior of *dom* f is denoted by *ridom* f and the relative boundary of *dom* f is denoted by *rbdom* f

If x belongs to *dom* f, the subdifferential of f at x is

$$\partial_Y f(x)=\{y\in Y\,;\,f(x)+\langle h,y\rangle\le f(x+h),\forall h\in X\}.$$

The following basic result will be used later several times.

Proposition 2.1.

(a) *for any* $x\in$ *dom* f *such that* $\partial_Y f(x)\neq\emptyset$ *and any* $y\in Y$,*we have:*
$y\in\partial_Y f(x)\Longleftrightarrow f(x)+f^*(y)=\langle x,y\rangle$

(b) $y\in\partial_Y f(x)\Longrightarrow x\in\partial_X f^*(y)$

Proof. See for instance [EkT], Ch. 1, Proposition 5.1 and Corollary 5.2. ■

Minimization of a convex conjugate under linear constraints. Let U be a vector space and V a vector subspace of U. The algebraic dual and bidual spaces of U and V are denoted by: U^*,U^{**},V^* and V^{**}. We consider a nonnegative convex function

$$\Phi:u\in U\mapsto\Phi(u)\in[0,\infty]$$

such that $\Phi(0) = 0$. Its conjugates are

$$\begin{array}{lclcll} \Phi^* & : & \ell \in U^* & \mapsto & \sup_{u \in U} \{\langle \ell, u\rangle - \Phi(u)\} & \in [0,\infty], \\ \overline{\Phi} & : & \xi \in U^{**} & \mapsto & \sup_{\ell \in U^*} \{\langle \xi, \ell\rangle - \Phi^*(\ell)\} & \in [0,\infty]. \end{array}$$

Let us consider the relations between the vector spaces. We define the equivalence relation on U^* : $\ell \sim \ell'$ for any $\ell, \ell' \in U^*$ if and only if $\ell(u) = \ell'(u), \forall u \in V$. In other words: $\ell \sim \ell' \iff \ell_{|V} = \ell'_{|V}$. We identify V^* with the factor space:

$$V^* = U^* / \sim \tag{2.1}$$

and $\dot{\ell} \in V^*$ stands for the equivalence class of $\ell \in U^*$. Therefore, one can identify V^{**} with a vector subspace of $U^{**} : V^{**} \subset U^{**}$ as follows. For any $\xi \in U^{**}$,

$$\xi \in V^{**} \iff \Big(\forall \ell, \ell' \in U^*, \ell \sim \ell' \implies \langle \xi, \ell - \ell'\rangle = 0\Big).$$

In this article, we are going to solve a minimization problem of the type

$$\inf\{\Phi^*(\ell)\,;\,\ell \in \dot{\ell}_o\} \tag{2.2}$$

with $\dot{\ell}_o \in V^*$. Since $\dot{\ell}_o$ is an affine subspace of U^*, $\ell \in \dot{\ell}_o$ can be interpreted as a linear constraint. In order to solve (2.2), let us introduce the restriction Ψ of Φ to $V \subset U$:

$$\Psi : u \in V \mapsto \Phi(u) \in [0,\infty].$$

Its conjugates are

$$\begin{array}{lclcll} \Psi^* & : & v^* \in V^* & \mapsto & \sup_{u \in V} \{\langle v^*, u\rangle - \Phi(u)\} & \in [0,\infty], \\ \overline{\Psi} & : & \xi \in V^{**} & \mapsto & \sup_{v^* \in V^*} \{\langle \xi, v^*\rangle - \Psi^*(v^*)\} & \in [0,\infty]. \end{array}$$

Proposition 2.2. *For any $\dot{\ell}_o$ in V^*, we have*

$$\Psi^*(\dot{\ell}_o) \leq \inf_{\ell \in \dot{\ell}_o} \Phi^*(\ell).$$

Proof. For any $\dot{\ell}_o \in V^*$ and $\ell \in \dot{\ell}_o$, we have: $\Psi^*(\dot{\ell}_o) = \sup_{u \in V}\{\langle \dot{\ell}_o, u\rangle - \Psi(u)\} = \sup_{u \in V}\{\langle \ell, u\rangle - \Phi(u)\} \leq \sup_{u \in U}\{\langle \ell, u\rangle - \Phi(u)\} = \Phi^*(\ell)$, from which the result follows. ■

For any $\dot{\ell} \in V^*$, $\partial\Psi^*(\dot{\ell})$ stands for the algebraic subdifferential of Ψ^* at $\dot{\ell}$: $\partial\Psi^*(\dot{\ell}) = \partial_{V^{**}}\Psi^*(\dot{\ell}) \subset V^{**}$ and for any $\xi \in V^{**}$, $\partial\overline{\Psi}(\xi)$ stands for the algebraic subdifferential of $\overline{\Psi}$ at ξ : $\partial\overline{\Psi}(\xi) = \partial_{V^*}\overline{\Psi}(\xi) \subset V^*$.

If $\overline{\Psi}$ is Gâteaux differentiable at $\xi \in V^{**}$, $\overline{\Psi}'(\xi)$ stands for the Gâteaux derivative at ξ. Since $\overline{\Psi}$ is convex, $\overline{\Psi}'(\xi)$ belongs to V^*.

As an application of Proposition 2.1, one obtains the following proposition.

Proposition 2.3.

(a) *If $\dot{\ell}_o$ belongs to ridom Ψ^*, then $\partial\Psi^*(\dot{\ell}_o)$ is nonempty.*

(b) *Let $\dot{\ell}_o$ be such that $\partial\Psi^*(\dot{\ell}_o) \neq \emptyset$, for any $\xi_o \in \partial\Psi^*(\dot{\ell}_o)$, we have:*
$\dot{\ell}_o \in \partial\overline{\Psi}(\xi_o)$ and $\Psi^*(\dot{\ell}_o) = \langle \xi_o, \dot{\ell}_o \rangle - \overline{\Psi}(\xi_o)$.
If $\overline{\Psi}$ is Gâteaux differentiable at ξ_o, then:
$\dot{\ell}_o = \overline{\Psi}'(\xi_o)$ and $\Psi^*(\dot{\ell}_o) = \langle \xi_o, \overline{\Psi}'(\xi_o) \rangle - \overline{\Psi}(\xi_o)$.

Proof. The statement (a) is a direct consequence of the geometric version of Hahn-Banach theorem.

Since $\overline{\Psi}$ is the convex conjugate of Ψ^* for the duality (V^*, V^{**}), the first part of (b) follows from Proposition 2.1. The second part follows from the identity $\partial\overline{\Psi}(\xi_o) = \{\overline{\Psi}'(\xi_o)\}$ which holds when $\overline{\Psi}$ is Gâteaux differentiable. ■

The equality $\dot{\ell}_o = \overline{\Psi}'(\xi_o)$ leads us to a belonging relation of the type $\ell_{\xi_o} \in \dot{\ell}_o$, so that if one can rewrite $\Psi^*(\dot{\ell}_o) = \langle \xi_o, \ell_{\xi_o} \rangle - \overline{\Psi}(\xi_o)$ as $\Psi^*(\dot{\ell}_o) = \Phi^*(\ell_{\xi_o})$, with Proposition 2.2, one obtains that ℓ_{ξ_o} is a solution to the minimization problem (2.2). In the case where Φ^* is strictly convex (this property is close to the differentiability of Φ and $\overline{\Psi}$), ℓ_{ξ_o} is the unique solution to (2.2).

The biconjugate of a convex integral functional. In this subsection, we express Φ^* and $\overline{\Psi}$ when Φ is an integral functional.

Let Ω be an arbitrary set, $\mathcal{A}$ a σ-field of subsets of Ω and R a nonnegative measure on $\mathcal{A}$. Let γ be a nonnegative convex function on $\mathbb{R}$; γ^* stands for its convex conjugate.
Our assumptions are

$$(2.3) \qquad \left| \begin{array}{l} \mathcal{A} \text{ is } R\text{-complete,} \\ R \text{ is } \sigma\text{-finite,} \\ \gamma : \mathbb{R} \to [0, +\infty[\text{ is a nonnegative convex function such that } \gamma(0) = 0 \text{ and} \\ dom\ \gamma = \mathbb{R}. \end{array} \right.$$

We consider the integral functional

$$(2.4) \qquad \Phi : u \in U \mapsto \int_\Omega \gamma(u)\, dR \in [0, \infty[$$

defined on

$$(2.5) \qquad U := \left\{ u : \Omega \to \mathbb{R}, \text{measurable such that } \int_\Omega \gamma(\lambda u)\, dR < \infty, \forall \lambda \in \mathbb{R} \right\}$$

where R-almost equal functions are identified.

Proposition 2.4. *Under the assumptions (2.3), Φ and U being defined by (2.4) and (2.5), for any $\ell \in U^*$, we have*

$$\Phi^*(\ell) = \begin{cases} \int_\Omega \gamma^*\left(\frac{d\ell}{dR}\right) dR & \text{if } \ell \ll R \\ +\infty & \text{otherwise,} \end{cases}$$

where $\ell \ll R$ means that there exists a measurable function $\frac{d\ell}{dR} : \Omega \to \mathbb{R}$ such that $\ell = \frac{d\ell}{dR} \cdot R$.

Proof. See for instance ([Lé1], Proposition 6.2). ■

Let θ be a Young function and θ^* be its convex conjugate. The Orlicz space associated with θ is $L_\theta = \{f : \Omega \to \mathbb{R} \ ; \ f$ is $\mathcal{A}$-measurable and $\|f\|_\theta < +\infty\}$ with the Luxemburg norm $\|f\|_\theta = \inf\left\{\beta > 0 \ ; \ \int_\Omega \theta\left(\frac{|f(\omega)|}{\beta}\right) R(d\omega) \le 1\right\}$. The space $(L_\theta, \|\cdot\|_\theta)$ is a Banach space.

Any continuous linear form on $L_\theta : \xi \in L'_\theta$, is uniquely decomposed into the sum $\xi = \xi^a + \xi^s$ where ξ^a belongs to the Orlicz space L_{θ^*} and ξ^s is singular with respect to R, that is

$$L'_\theta = L_{\theta^*} \oplus L^s_\theta.$$

where L^s_θ is the space of all the singular forms with respect to R. The forms ξ^a and ξ^s are called respectively the absolutely continuous and singular parts of ξ with respect to R. For more details, see ([Lé1], Section 5).

If γ satisfies

$$\lim_{t\to+\infty} \gamma(t) = \lim_{t\to-\infty} \gamma(t) = \infty, \tag{2.6}$$

the functions

$$\gamma_+(t) := \gamma(|t|) \quad \text{and} \quad \gamma_-(t) := \gamma(-|t|), \qquad t \in \mathbb{R}$$

are Young functions. The corresponding Luxemburg norms $\|\cdot\|_{\gamma_+}$ and $\|\cdot\|_{\gamma_-}$ are defined respectively on the Orlicz spaces L_{γ_+} and L_{γ_-}. As usual, γ_+^* and γ_-^* are their convex conjugates.

We define the cone W_Φ which consists of all the elements of the form $\xi = \xi_1 - \xi_2 + \xi_3 - \xi_4$, $\xi_1, \xi_2, \xi_3, \xi_4 \ge 0$ with $\xi_1 \in L_{\gamma_+}$, $\xi_2 \in L_{\gamma_-}$, $\xi_3 \in L^s_{\gamma_+^*}$, $\xi_4 \in L^s_{\gamma_-^*}$, where the above spaces appear in the decompositions $L'_{\gamma_+^*} = L_{\gamma_+} \oplus L^s_{\gamma_+^*}$ and $L'_{\gamma_-^*} = L_{\gamma_-} \oplus L^s_{\gamma_-^*}$. Any element of W_Φ can uniquely be written in the form

$$\xi = \xi^a_+ - \xi^a_- + \xi^s_+ - \xi^s_- = \xi^a + \xi^s \tag{2.7}$$

(putting $\xi^a = \xi^a_+ - \xi^a_-$ and $\xi^s = \xi^s_+ - \xi^s_-$), with $\xi^a_+, \xi^a_-, \xi^s_+, \xi^s_- \ge 0$, $\xi^a_+ \in L_{\gamma_+}, \xi^a_- \in L_{\gamma_-}$, $\xi^s_+ \in L^s_{\gamma_+^*}, \xi^s_- \in L^s_{\gamma_-^*}$ and $\xi^a \wedge \xi^s = \xi^a_+ \wedge \xi^a_- = \xi^s_+ \wedge \xi^s_- = 0$.

We define

$$W = \sigma(W_\Phi, dom\ \Phi^*)\text{-closure of } V \text{ in } W_\Phi.$$

Any element of W can uniquely be written in the form (2.7).

Let us denote respectively W^a and W^s the cones of the absolutely continuous and singular forms in $W : W = W^a \oplus W^s$.

Let us define the functions $\overline{\Psi}^a$ and $\overline{\Psi}^s$ on V^{**}, for any $\xi \in V^{**}$, by:

$$\overline{\Psi}^a(\xi) = \begin{cases} \int_\Omega \gamma(\xi)\ dR & \text{if } \xi \in W^a \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\overline{\Psi}^s(\xi) = \begin{cases} \sup\left\{\langle \xi^s_+, v\rangle \,;\, v \geq 0, \int_\Omega \gamma^*(v)\, dR < \infty\right\} & \\ + \sup\left\{\langle \xi^s_-, |v|\rangle \,;\, v \leq 0, \int_\Omega \gamma^*(v)\, dR < \infty\right\} & \text{if } \xi \in W^s \\ & \\ +\infty & \text{otherwise} \end{cases}$$

We are ready to give the expression of $\overline{\Psi}$.

Theorem 2.5. *Under the assumptions (2.3) and (2.6), Φ and U being defined by (2.4) and (2.5), for any $\xi \in V^{**}$, taking the decomposition (2.7) into account, we have*

$$\overline{\Psi}(\xi) = \overline{\Psi}^a(\xi^a) + \overline{\Psi}^s(\xi^s).$$

Proof. See ([Lé1], Theorem 6.4). ■

3. The minimization

The main result of this section is Theorem 3.6 which states the solution to a minimization problem (2.2) where Φ^* is given by Proposition 2.4 and the linear constraints are described below. In the rest of the paper, Φ and U are given by (2.4) and (2.5).

Description of the constraints. We consider a vector space $\mathcal{Y}$ and its algebraic dual and bidual spaces $\mathcal{Y}^* = \mathcal{X}$ and $\mathcal{X}^*$. The space $\mathcal{X}$ is endowed with the Borel σ-field of the weak topology $\sigma(\mathcal{X}, \mathcal{Y})$. Let $\varphi : \Omega \to \mathcal{X}$ be a measurable function which satisfies

$$\text{(3.1)} \qquad \int_\Omega \gamma(\langle y, \varphi(\omega)\rangle)\, R(d\omega) < \infty, \quad \forall y \in \mathcal{Y}.$$

We define the constraints ($\ell \in \dot{\ell}_o$) by means of (2.1) with

$$V = \left\{(\omega \in \Omega \mapsto \langle y, \varphi(\omega)\rangle \in \mathbb{R})\,;\, y \in \mathcal{Y}\right\}.$$

Because of (3.1), V is a vector subspace of U (see (2.5)).

We define the "projection" $\Pi : U^* \to \mathcal{X}$, for all $\ell \in U^*$, by

$$\text{(3.2.a)} \qquad \forall y \in \mathcal{Y}, \left\langle \ell, \langle y, \varphi(\cdot)\rangle \right\rangle_{U^*,U} = \left\langle \Pi(\ell), y\right\rangle_{\mathcal{X},\mathcal{Y}}$$

so that, for any $x_o \in \mathcal{X}$,

$$\Pi(\ell) = x_o$$

is the expression of a linear constraint on $\ell \in U^*$. Similarly, let $\dot{\Pi} : V^* \to \mathcal{X}$ be defined for all $\dot{\ell}_o \in V^*$ by:

$$\text{(3.2.b)} \qquad \dot{\Pi}(\dot{\ell}_o) = \Pi(\ell), \ell \in \dot{\ell}_o :$$

the common value of $\Pi(\ell)$ when ℓ describes $\dot{\ell}_o$.

With the notations of the Section 1, this corresponds to $\langle y, \varphi(\omega)\rangle = A^* y(\omega)$ (see (1.2)). In [Lé2], it is shown that with this description of the linear constraints, one almost doesn't lose any generality (see [Lé2], (3.10)) and that the assumption (3.1) (see also (1.7)) is a continuity hypothesis for the constraint operator A.

The assumptions. The assumptions of the main result of this section: Theorem 3.6, are:

(3.3) $\left|\begin{array}{l}\text{The assumptions (2.3) and (3.1) hold and } \gamma \text{ is differentiable,}\\ \text{if } R(\Omega)=\infty, \text{ the assumption (3.3.a) or (3.3.b) below also holds.}\end{array}\right.$

If $R(\Omega) = +\infty$, we shall need the following additional assumptions.
A Young function θ satisfies the Δ_2-condition, if there exists $K > 0$ and $t_o \geq 0$ such that $\theta(2t) \leq K\theta(t), \quad \forall t \geq t_o$. The Δ_2-condition is said to be global when $t_o = 0$.

When $R(\Omega) = +\infty$, the additional assumptions are

(3.3.a) If $\lim_{t\to+\infty} \gamma(t) = \lim_{t\to-\infty} \gamma(t) = +\infty$, then:
there exists a convex function $\rho : [0, 2] \to [0, \infty[$ such that $\rho(t) \leq \min(\gamma(t), \gamma(-t)), \forall 0 \leq t \leq 2$ which satisfies (3.4).

(3.3.b) If $\lim_{t\to+\infty} \gamma(t) = +\infty$ and $\gamma_- \equiv 0$, then: γ_+ satisfies (3.4).
If $\lim_{t\to-\infty} \gamma(t) = +\infty$ and $\gamma_+ \equiv 0$, then: γ_- satisfies (3.4).

We say that ρ satisfies the Δ_2-condition around zero if

$$(\rho(t) = 0 \iff t = 0) \quad \text{and} \quad \exists K > 0,\ \rho(2t) \leq K\rho(t),\ \forall t \in [0, 1]. \tag{3.4}$$

Note that, if γ is equivalent to $C|t|^p$ around $t = 0$, with $C > 0$ and $1 \leq p < \infty$, there exists a function ρ as in (3.3.a). Similarly, if $\gamma_\pm$ is equivalent to $C|t|^p$ around $t = 0$, with $C > 0$ and $1 \leq p < \infty$, it satisfies (3.4) as in (3.3.b).

Let us give some more comments about the assumption (3.3.a). Let θ be a Young function, we denote $M_\theta = \left\{f \in L_\theta\, ; \forall\alpha > 0, \int_\Omega \theta(\alpha f)\, dR < \infty\right\}$. If $\theta(\mathbb{R}) = [0, \infty[$ and if

$$R(\Omega) = \infty \Longrightarrow (\theta(t) = 0 \iff t = 0), \tag{3.5}$$

the strong dual space of M_θ can be identified, by means of the dual bracket $\langle f, g\rangle = \int_\Omega fg\, dR, \quad \forall f \in L_\theta$, with $L_{\theta^*} : M'_\theta \simeq L_{\theta^*}$, which means that for any $\chi \in M'_\theta$, there exists a unique $g_\chi \in L_{\theta^*}$ such that $\chi(f) = \int_\Omega fg_\chi\, dR$, for all $f \in M_\theta$ (see [RaR], Theorem 4.1.7).
The assumption [(2.6) & (3.3.a)] implies that $\gamma_\pm$ satisfy (3.5). Therefore, under [(2.6) & (3.3.a)], we have

$$M'_{\gamma_\pm} \simeq L_{\gamma^*_\pm}. \tag{3.6}$$

If R is bounded, in order that $M_\theta = L_\theta$, it is enough that θ satisfies the Δ_2-condition, while if R is unbounded (but σ-finite), in order that $M_\theta = L_\theta$, it is enough that θ satisfies the Δ_2-condition globally (see [RaR], Corollary 3.4.5).

Assuming (2.3) and [(2.6) & (3.3.a)], there exists a Young function σ which matches with ρ on $[0, 1]$ and satisfies $\lim_{t\to\infty} \sigma(t)/t < \infty$. Since, it satisfies the Δ_2-condition (globally if $R(\Omega) = \infty$), we obtain:

$$\text{the dual space of } L_\sigma \text{ is } L_{\sigma^*}. \tag{3.7}$$

The Orlicz space L_σ is close to L_1. Indeed, if R is a bounded measure, $L_\sigma = L_1$; while if R is unbounded, L_σ takes into consideration the behavior of its elements around zero.
Similarly, the space L_{σ^*} is close to L_∞. In particular, if R is a bounded measure, then $L_{\sigma^*} = L_\infty$. We clearly have:

$$L_{\gamma_\pm} \subset L_\sigma \quad \text{and} \quad L_{\sigma^*} \subset L_{\gamma^*_\pm}. \tag{3.8}$$

Subgradient constraints. A constraint $\dot{\ell}$ is said to be a subgradient constraint if $\partial\Psi^*(\dot{\ell})$ is nonempty. Any element of the relative interior of the effective domain of Ψ^* : *ridom* Ψ^*, is a subgradient constraint. In Proposition 3.3 below, we give a description of those constraints. In Lemma 3.1 and Lemma 3.2, preliminaries for its proof are established.

Lemma 3.1. *Let us assume (2.3), (2.6) and if $R(\Omega) = \infty$: (3.3.a). Then, the domain of $\overline{\Psi}^a$ consists of the measurable functions $\langle z, \varphi(\cdot)\rangle$, $z \in \mathcal{X}^*$, which satisty $\int_\Omega \gamma(\langle z, \varphi(\cdot)\rangle)\, dR < \infty$ and*

$$\forall \varepsilon > 0, \forall K \geq 1, 1 \leq k \leq K, f_k \in L_{\gamma^*_+}, g_k \in L_{\gamma^*_-}, \exists y \in \mathcal{Y} \text{ such that}$$
$$\sum_{k=1}^{K} \left| \int_\Omega \langle z - y, \varphi(\cdot)\rangle (\mathbb{1}_{\{\langle z,\varphi(\cdot)\rangle \geq 0\}} f_k + \mathbb{1}_{\{\langle z,\varphi(\cdot)\rangle \leq 0\}} g_k)\, dR \right| \leq \varepsilon.$$

Moreover, for any $\langle z, \varphi(\cdot)\rangle \in$ dom $\overline{\Psi}^a$, there exists a sequence $(y_n)_{n\geq 1}$ in $\mathcal{Y}$ such that

$$\lim_{n\to\infty} \|\langle z, \varphi(\cdot)\rangle - \langle y_n, \varphi(\cdot)\rangle\|_\sigma = 0 \quad \text{and}$$
$$\lim_{n\to\infty} \langle y_n, \varphi(\omega)\rangle = \langle z, \varphi(\omega)\rangle, \text{ for } R\text{-almost every } \omega \in \Omega.$$

Similarly, let ρ be a Young function satisfying the Δ_2-condition globally and $\rho(t) \leq \min(\gamma(t), \gamma(-t))$, for all $t \geq 0$. There exists a sequence $(y_n)_{n\geq 1}$ in $\mathcal{Y}$ such that $\lim_{n\to\infty} \|\langle z, \varphi(\cdot)\rangle - \langle y_n, \varphi(\cdot)\rangle\|_\rho = 0$.

In particular, if R is bounded and γ grows faster than any polynomial at infinity, by means of a diagonal sequence argument, it follows that $\langle y_n, \varphi(\cdot)\rangle$ tends to $\langle z, \varphi(\cdot)\rangle$ in all the L_p's.

Proof. We only have to show that an element ξ of *dom* $\overline{\Psi}^a$ satisfies $\lim_{n\to\infty} \|\xi - \langle y_n, \varphi(\cdot)\rangle\|_\sigma = 0$ for some sequence $(y_n)_{n\geq 1}$ in $\mathcal{Y}$. In fact, the first statement follows easily from this together with the inclusion *dom* $\overline{\Psi}^a \subset W^a$.
Thanks to (3.8), V is $\sigma(L_\sigma, L_{\sigma^*})$-dense in *dom* $\overline{\Psi}^a \subset L_\sigma$. Therefore, for any $\xi \in$ *dom* $\overline{\Psi}^a$ and any neighbourhood $\mathcal{T}$ of ξ for the topology $\sigma(L_\sigma, L_{\sigma^*})$, $V \cap \mathcal{T}$ is nonempty: let us pick $\eta_\mathcal{T}$ in $V \cap \mathcal{T}$.
Denote C the convex hull of $\{\eta_\mathcal{T}\, ; \mathcal{T} : \sigma(L_\sigma, L_{\sigma^*})\text{-neighbourhood of } \xi\}$, $\bar{C}^w$ its $\sigma(L_\sigma, L_{\sigma^*})$-closure and $\bar{C}$ its $\|\cdot\|_\sigma$-closure. Because of the assumption [(2.6) & (3.3.a)], we have (3.7) and we obtain that: $\bar{C}^w = \bar{C}$ (see [Rud], Theorem 3.12). But, we have just seen that $\xi \in \bar{C}^w$, so that $\xi \in \bar{C}$. It follows that for any $n \geq 1$, there exists $\eta_n \in C$ such that $\|\eta_n - \xi\|_\sigma \leq \frac{1}{n}$. From $(\eta_n)_{n\geq 1}$, one can extract a subsequence which converges R-a.e. to ξ : there exists a sequence $(y_n)_{n\geq 1}$ in $\mathcal{Y}$ such that

$$\lim_{n\to\infty} \langle y_n, \varphi(\omega)\rangle = \xi(\omega)$$

for R-almost every $\omega \in \Omega$. Hence, there exist a vector subspace: $\mathcal{X}_\xi$, of $\mathcal{X}$ such that $R\circ\varphi^{-1}(\mathcal{X}\setminus\mathcal{X}_\xi) = 0$ and a numerical function $z(\cdot)$ on $\mathcal{X}_\xi$ (taking $\xi = z\circ\varphi$) such that

$$\lim_{n\to\infty} \langle y_n, x\rangle = z(x), \quad \forall x \in \mathcal{X}_\xi.$$

Clearly z is measurable and linear on $\mathcal{X}_\xi$. By Hahn-Banach theorem, it admits a linear extension to $\mathcal{X}$ denoted $\langle z, \cdot\rangle \in \mathcal{X}^*$.

The proof of the statement with ρ is similar since under our assumptions, $L_\gamma \subset L_\rho$, $L_{\rho^*} \in L_{\gamma^*}$ and ${L_\gamma}' \simeq L_{\gamma^*}$. This completes the proof of the lemma. ■

Let γ_o be the Young function defined by $\gamma_o(x) = \max(\gamma(x), \gamma(-x))$. The associated Orlicz space is $L_{\gamma_o} = \{v\,;\, v : \Omega \to \mathbb{R}$ measurable such that $\exists\lambda > 0, \int_\Omega [\gamma(\lambda v) + \gamma(-\lambda v)]\, dR < \infty\}$.

Lemma 3.2. (Characterization of the subgradient constraints). *Let $\xi \in V^{**}$ be such that $\overline{\Psi}(\xi) < \infty$.*

If $\partial_{V^*}\overline{\Psi}(\xi) \neq \emptyset$, then the absolutely continuous part ξ^a of ξ satisfies

$$0 \le \int_\Omega \xi^a \gamma'(\xi^a)\, dR < \infty \quad \textit{and} \quad 0 \le \int_\Omega \gamma^* \circ \gamma'(\xi^a)\, dR < \infty.$$

In addition, for any v in L_{γ_o} : $\int_\Omega |v\gamma'(\xi^a)|\, dR < \infty$.

Conversely, if $\int_\Omega \xi^a \gamma'(\xi^a)\, dR = \infty$, then $\partial_{V^*}\overline{\Psi}(\xi) = \emptyset$.

Proof. The function $g(t) = \int_\Omega \gamma(t\xi^a)\, dR, 0 \le t \le 1$ is convex nondecreasing and $0 = g(0) \le g(t) \le g(1) = \int_\Omega \gamma(\xi^a)\, dR = \overline{\Psi}^a(\xi^a) \le \overline{\Psi}(\xi) < \infty$ (see Theorem 2.5). Since for any $x \in \mathbb{R}, t \in [0,1] \mapsto \gamma(tx)$ is convex, $\frac{\gamma((1-h)x)-\gamma(x)}{-h}$ increases towards $x\gamma'(x) \ge 0$ as h decreases to zero. It follows from the monotone convergence theorem that

$$g'(1^-) = \int_\Omega \xi^a\gamma'(\xi^a)\, dR \in [0, \infty]. \tag{3.9}$$

On the other hand, as $\overline{\Psi}(\xi) < \infty$ and $\partial_{V^*}\overline{\Psi}(\xi) \neq \emptyset$, for any $\dot\ell \in \partial_{V^*}\overline{\Psi}(\xi)$ and any $0 \le h \le 1$, we have $g(1-h) - g(1) = \overline{\Psi}^a((1-h)\xi^a) - \overline{\Psi}^a(\xi^a) = \overline{\Psi}((1-h)\xi^a) - \overline{\Psi}(\xi^a) \ge \langle\dot\ell, -h\xi^a\rangle$. Consequently, $\frac{g(1-h)-g(1)}{-h} \le \langle \dot\ell, \xi^a\rangle < \infty, \forall 0 < h \le 1$. Hence, $g'(1^-) < \infty$. Together with (3.9), this leads us to: $0 \le \int_\Omega \xi^a\gamma'(\xi^a)\, dR < \infty$ and with the identity

$$t\gamma'(t) - \gamma(t) = \gamma^* \circ \gamma'(t), \forall t \in \mathbb{R} \tag{3.10}$$

we get: $\int_\Omega \gamma^* \circ \gamma'(\xi^a)\, dR < \infty$.

For any $v : \Omega \to \mathbb{R}$, $\gamma'(\xi^a)v \le \gamma^*\circ\gamma'(\xi^a) + \gamma(v)$ and $-\gamma'(\xi^a)v \le \gamma^*\circ\gamma'(\xi^a) + \gamma(-v)$. Therefore, for any $\lambda > 0$, $|\gamma'(\xi^a)v| \le \frac{1}{\lambda}[\gamma^*\circ\gamma'(\xi^a) + \gamma(\lambda v) + \gamma(-\lambda v)]$. Choosing $\lambda > 0$ small enough, one deduces that $\int_\Omega |\gamma'(\xi^a)v|\, dR < \infty$ provided that $v \in L_{\gamma_o}$.

The converse part is clear from the above proof. ■

Proposition 3.3. *We assume (3.3) and (2.6). Let $\dot\ell \in$ dom Ψ^* be such that $\partial\Psi^*(\dot\ell) \neq \emptyset$. One can associate a measurable $z_{\dot\ell} \in \mathcal{X}^*$ such that* *(a)* $\gamma'(\langle z_{\dot\ell},\varphi\rangle)\cdot R\in\dot\ell,$

(b) $\Psi^*(\dot\ell)=\Phi^*\Big(\gamma'(\langle z_{\dot\ell},\varphi\rangle)\cdot R\Big)=\int_\Omega\gamma^*\big[\gamma'\left(\langle z_{\dot\ell},\varphi\rangle\right)\big]\,dR<\infty$ *and*

(c) $\langle z_{\dot\ell},\varphi(\cdot)\rangle\in dom\,\overline{\Psi}^a.$

More precisely, any $z_{\dot\ell}$ such that $\langle z_{\dot\ell},\varphi\rangle=\xi^a$ with $\xi\in\partial\Psi^*(\dot\ell)$ is convenient.

Remark. By the geometric version of Hahn-Banach theorem, any $\dot\ell$ in *ridom* Ψ^* satisfies the assumption: $\partial\Psi^*(\dot\ell)\neq\emptyset$.

Proof. Let $\xi\in\partial\Psi^*(\dot\ell)$. By Proposition 2.1: $\xi\in dom\,\overline{\Psi}$ and $\dot\ell\in\partial_{V^*}\overline{\Psi}(\xi)$. Let $\xi=\xi^a+\xi^s$ be the decomposition (2.7). By Theorem 2.5, for any $v\in W^a$,

$$\begin{aligned}\overline{\Psi}(\xi+v)-\overline{\Psi}(\xi)&=[\overline{\Psi}^a(\xi^a+v)-\overline{\Psi}^a(\xi^a)]+[\overline{\Psi}^s(\xi^s+0)-\overline{\Psi}^s(\xi^s)]\\&=\overline{\Psi}^a(\xi^a+v)-\overline{\Psi}^a(\xi^a).\end{aligned}$$

Therefore, $\partial_{V^*}\overline{\Psi}(\xi)=\partial_{V^*}\overline{\Psi}^a(\xi^a)$. But, as Ψ^* is strictly convex (since γ is differentiable), we have $\partial_{V^*}\overline{\Psi}(\xi)=\{\dot\ell\}$. This gives

$$\partial_{V^*}\overline{\Psi}(\xi^a)=\partial_{V^*}\overline{\Psi}^a(\xi^a)=\{\dot\ell\}\tag{3.11}$$

As $\gamma(a+b)\geq\gamma(a)+\gamma'(a)b,\forall a,b\in\mathbb{R}$, for any $v\in L_{\gamma_o}$, $\int\gamma(\xi^a+v)\,dR-\int\gamma(\xi^a)\,dR\geq\int\gamma'(\xi^a)v\,dR$ where the last integral is finite by Lemma 3.2. It follows that $\gamma'(\xi^a)\cdot R\in\partial_{V^*}\overline{\Psi}^a(\xi^a)$. Together with (3.11), this leads us to

$$\gamma'(\xi^a)\cdot R\in\dot\ell.\tag{3.12}$$

As Ψ^* and $\overline{\Psi}$ are conjugate to each other for the duality (V^*,V^{**}), thanks to Proposition 2.1, (3.11) and (3.12), we obtain

$$\begin{aligned}\Psi^*(\dot\ell)&=\langle\xi^a,\dot\ell\rangle-\overline{\Psi}(\xi^a)\\&=\int_\Omega\left[\xi^a\gamma'(\xi^a)-\gamma(\xi^a)\right]dR\\&=\int_\Omega\gamma^*\circ\gamma'(\xi^a)\,dR\\&=\Phi^*\left(\gamma'(\xi^a)\cdot R\right),\end{aligned}\tag{3.13}$$

where we have used (3.10) and Proposition 2.4.

Thanks to Lemma 3.1, there exists $z_{\dot\ell}\in\mathcal{X}^*$ such that $\xi^a=\langle z_{\dot\ell},\varphi\rangle$ belongs to *dom* $\overline{\Psi}^a$. Finally, (3.12) is (a) and (3.13) is (b). ■

Boundary constraints. Now, we focus our attention on those $\dot\ell$ which stand on the relative boundary of *dom* Ψ^* : $\dot\ell\in$ *rbdom* Ψ^*.

Let us sketch an analogy with mechanics. If the elements of $\mathcal{X}$ are the possible states of a system, the elements of $\mathcal{X}^*$ may be interpreted as forces. More precisely, the vector $z_{\dot\ell}$ appearing in Proposition 3.3 is the "force" one has to apply in order that the system stands in the state $\dot\Pi(\dot\ell)$. The equilibrium state of the system is the state $\dot\Pi(\dot\ell)$ such that $\Psi^*(\dot\ell)$ is minimal, that is: $\dot\Pi(\dot\ell) = 0$. The corresponding force is $z_0 = 0$. We are going to establish that, in some circumstances, the force corresponding to boundary states may assume infinite ($\pm\infty$) values. Let us fix some conventions and notations:

$$\begin{aligned}
\lim_{t\to+\infty} \gamma'(t) &= \gamma'(+\infty) \overset{\text{def}}{=} \kappa_+ \in [0,+\infty], \\
\lim_{t\to-\infty} \gamma'(t) &= \gamma'(-\infty) \overset{\text{def}}{=} \kappa_- \in [-\infty,0], \\
\lim_{t\to+\infty} \gamma^*(t) &= \gamma^*(+\infty) = +\infty \quad \text{and} \\
\lim_{t\to-\infty} \gamma^*(t) &= \gamma^*(-\infty) = +\infty,
\end{aligned}$$

so that for any $z(\cdot) : \mathcal{X} \to [-\infty,+\infty]$, $\gamma' \circ z(x) \in [\kappa_-, \kappa_+]$ and $\gamma^* \circ \gamma' \circ z(x) \in [0,+\infty]$ are well defined.

Proposition 3.4. *We assume (3.3) and (2.6). To any $\dot\ell \in rbdom\ \Psi^*$, one can associate a measurable function $z_{\dot\ell} : \mathcal{X} \to [-\infty,+\infty]$, which is the pointwise limit of a sequence in $\mathcal{Y} \subset \mathcal{X}^*$, such that*

(a) $\quad \gamma'(z_{\dot\ell} \circ \varphi) \cdot R \in \dot\ell$ *and*

(b) $\quad \Psi^*(\dot\ell) = \Phi^*\Big(\gamma'(z_{\dot\ell} \circ \varphi) \cdot R\Big) = \int_\Omega \gamma^*\big[\gamma'(z_{\dot\ell} \circ \varphi)\big]\, dR < \infty.$

Remark. Suppose that $z_{\dot\ell}\circ\varphi$ assumes infinite values. As $\Psi^*(\dot\ell)$ is finite, this implies that $\gamma^*(\kappa_+) < \infty$ or $\gamma^*(\kappa_-) < \infty$. It is necessary that $0 \le \kappa_+ < \infty$ or $-\infty < \kappa_- \le 0$ which means that γ_+ or γ_- admits an asymptot with slope κ_+ or κ_-.

Proof. Let $\dot\ell$ be on the boundary $rbdom\ \Psi^*$. Let us define $\dot\ell_n = (1-\frac{1}{n})\dot\ell$. We have: $\lim_{n\to\infty} \dot\ell_n = \dot\ell$ for the topology $\sigma(V^*,V)$ on the segment $[0,\dot\ell]$. Since Ψ^* is convex and $\sigma(V^*,V)$-lower semicontinuous, it comes out that

$$\lim_{n\to\infty} \Psi^*(\dot\ell_n) = \Psi^*(\dot\ell). \tag{3.14}$$

As the $\dot\ell_n$'s are in *ridom* Ψ^* (by (2.6), 0 is interior), by Proposition 3.3, for any $n \ge 1$, there exists $z_n = z_{\dot\ell_n} \in \mathcal{X}^*$ such that

$$\left|\begin{array}{l}
\gamma'(\langle z_n,\varphi\rangle) \cdot R \in \dot\ell_n \\
\langle z_n,\varphi\rangle_+ \in L_{\gamma_+} \text{ and } \langle z_n,\varphi\rangle_- \in L_{\gamma_-} \\
\Psi^*(\dot\ell_n) = \int_\Omega \gamma^* \circ \gamma'(\langle z_n,\varphi\rangle)\, dR < \infty
\end{array}\right. \tag{3.15}$$

Let us consider the function θ defined on $[-\infty,+\infty]$ by $\theta(t) = \begin{cases} \frac{t}{1+|t|} & \text{if } t \in \mathbb{R} \\ +1 & \text{if } t = +\infty \\ -1 & \text{if } t = -\infty \end{cases}$. It is continuous, increasing and bounded. By Tychonov's theorem, one can extract from $(\theta\circ\gamma'(\langle z_n,\cdot\rangle))_{n\ge1}$ a subsequence pointwise converging to a $[-1,+1]$-valued limit $g : \lim_{n\to\infty} \theta\circ\gamma'(\langle z_n,x\rangle) = g(x)$, $\forall x \in \mathcal{X}$. Let us suppose for a while that γ' is increasing, then the reciprocal function $(\theta\circ\gamma')^{-1}$ of $\theta\circ\gamma'$ is continuous. Putting $z_{\dot\ell} = (\theta\circ\gamma')^{-1} \circ g$, for any $x \in \mathcal{X}$, we obtain

$$\begin{cases}
\lim_{n\to\infty} \langle z_n,x\rangle & = & z_{\dot\ell}(x) & \in & [-\infty,+\infty], \\
\lim_{n\to\infty} \gamma'(\langle z_n,x\rangle) & = & \gamma' \circ z_{\dot\ell}(x) & \in & [\kappa_-,\kappa_+].
\end{cases} \tag{3.16}$$

In the general case where γ' is nondecreasing, it may assume constant values c_k on some intervals $I_k = \gamma'^{-1}(c_k)$ with a nonempty interior. There exist at most countably many such I_k's. They are bounded, except possibly for two intervals of the form $[a_+, +\infty[$ and $]-\infty, a_-]$ where γ' may assume the values κ_+ and κ_-. As θ^{-1} is continuous, the limit $\lim_{n\to\infty} \gamma'(\langle z_n, \cdot\rangle) = \theta^{-1} \circ g$ exists. In addition, $\lim_{n\to\infty}\langle z_n, x\rangle \overset{\text{def}}{=} z_{\dot\ell}(x)$ exists for all the x's such that $\theta^{-1} \circ g(x) \notin \{c_k\,;\, k \geq 1\}$. But for any bounded I_k, Tychonov's theorem allows us to extract a pointwise converging subsequence: $\lim_{n\to\infty}\langle z_n, x\rangle \overset{\text{def}}{=} z_{\dot\ell}(x) \in I_k$, for all x such that $\theta^{-1} \circ g(x) = c_k$. For the intervals of the form $[a_+, +\infty[$ and $]-\infty, a_-]$, put $z_{\dot\ell}(x) = \pm\infty$ for all $x \in \mathcal{X}$ such that $\theta^{-1} \circ g(x) = \kappa_\pm$. Finally, (3.16) is obtained by means of a diagonal subsequence procedure.

Now, let us prove the statement (a) of the proposition. By (3.14) and (3.15), possibly dropping the first terms of the sequence, we have

$$\sup_{n\geq 1}\Big(\int_\Omega \gamma_+^* \circ \gamma_+'(\langle z_n, \varphi\rangle_+)\, dR + \int_\Omega \gamma_-^* \circ \gamma_-'(\langle z_n, \varphi\rangle_-)\, dR\Big) \leq \Psi^*(\dot\ell) + 1 < +\infty \tag{3.17}$$

so that, together with (3.6), this allows us to extract from $(\gamma'(\langle z_n, \varphi\rangle))_{n\geq 1}$ a subsequence such that the following limits with respect to the $*$-weak topologies $\sigma(L_{\gamma_\pm^*}, M_{\gamma_\pm})$:
$\lim_{n\to\infty} \gamma_\pm'(\langle z_n, \varphi\rangle_\pm) = h_\pm \in L_{\gamma_\pm^*}$, exist. This means

$$\forall v \in M_{\gamma_\pm}, \lim_{n\to\infty} \int_\Omega v\gamma_\pm'(\langle z_n, \varphi\rangle_\pm)\, dR = \int_\Omega v h_\pm \, dR \tag{3.18}$$

where $M_{\gamma_\pm} = \{f \in L_{\gamma_\pm}\,;\, \forall \alpha > 0, \int_\Omega \gamma_\pm(\alpha f)\, dR < \infty\}$.

In particular, choosing $\tilde v_+ = \mathbb{1}_{\{z_{\dot\ell}\circ\varphi = +\infty\}}$ (which is in $M\gamma_+$ thanks to (3.17)), (3.16) and (3.18) provide us with

$$\begin{aligned}
\kappa_+ R(z_{\dot\ell} \circ \varphi = +\infty) &= \int_\Omega \tilde v_+ \lim_{n\to\infty} \gamma_+'(\langle z_n, \varphi\rangle_+)\, dR \\
&\leq \liminf_{n\to\infty} \int_\Omega \tilde v_+ \gamma_+'(\langle z_n, \varphi\rangle_+)\, dR \qquad \text{(Fatou)} \\
&= \int_\Omega \tilde v_+ h_+ \, dR < +\infty
\end{aligned}$$

from which it comes out (with a similar argument for the nonpositive parts) that

$$[\kappa_\pm > 0 \Rightarrow R(z_{\dot\ell} \circ \varphi = \pm\infty) < \infty] \text{ and } [\kappa_\pm = \pm\infty \Rightarrow R(z_{\dot\ell} \circ \varphi = \pm\infty) = 0] \tag{3.19}$$

Since *dom* $\gamma = \mathbb{R}$, we have: $\lim_{t\to\pm\infty} \gamma^*(t)/|t| = +\infty$. Together with (3.17) and de la Vallée-Poussin's theorem, this leads us to the following statement: for any bounded measurable function v, the families $\{v\gamma_\pm'(\langle z_n, \varphi\rangle_\pm)\,;\, n \geq 1\}$ are uniformly integrable. From (3.16), (3.18) and (3.19), one deduces that for any bounded measurable function v : $\int_\Omega v\gamma_\pm'(\langle z_{\dot\ell}, \varphi\rangle_\pm)\, dR = \int_\Omega v h_\pm \, dR$. It follows that: $L_{\gamma_\pm^*} \ni h_\pm = \gamma_\pm'(\langle z_{\dot\ell}, \varphi\rangle_\pm)$. But, taking the assumption (3.1) into account, for any $y \in \mathcal{Y}$, $\langle y, \varphi(\cdot)\rangle$ belongs to $M_{\gamma_\pm}$ and (3.18) yields

$$\forall y \in \mathcal{Y}, \lim_{n\to\infty} \int_\Omega \langle y, \varphi\rangle \gamma'(\langle z_n, \varphi\rangle)\, dR = \int_\Omega \langle y, \varphi\rangle \gamma'(\langle z_{\dot\ell}, \varphi\rangle)\, dR.$$

On the other hand, $\lim_{n\to\infty} \dot\ell_n = \dot\ell$ and $\gamma'(\langle z_n, \varphi\rangle)\cdot R \in \dot\ell_n$ (see (3.15)) mean that

$$\forall y \in \mathcal{Y}, \lim_{n\to\infty} \int_\Omega \langle y,\varphi\rangle \gamma'(\langle z_n,\varphi\rangle)\, dR = \langle y, \dot\Pi(\dot\ell)\rangle.$$

Putting together the last two limits, we obtain the statement (a) of the proposition.

Now, let us prove the statement (b). By (3.14) and (3.15), we get

$$\Psi^*(\dot\ell) = \lim_{n\to\infty} \Psi^*(\dot\ell_n) = \lim_{n\to\infty} \int_\Omega \gamma^* \circ \gamma'(\langle z_n,\varphi\rangle)\, dR \geq \int_\Omega \gamma^*\circ\gamma'\circ z_{\dot\ell}\circ\varphi\, dR = \Phi^*(\gamma'(z_{\dot\ell}\circ\varphi)\cdot R)$$

where the last equality is Proposition 2.4. The above inequality is due to the the decomposition $\Phi^*(\ell) = \Phi_+^*(\ell_+) + \Phi_-^*(\ell_-)$ (see [Lé1], Proposition 4.4) together with the lower semicontinuities of $\Phi_\pm^*$ and the convergences: $\lim_{n\to\infty} \gamma'_\pm(\langle z_n,\varphi\rangle_\pm) = \gamma'_\pm((z_{\dot\ell}\circ\varphi)_\pm)$, with respect to the topologies $\sigma(L_{\gamma_\pm^*}, M_{\gamma_\pm})$. But, by Proposition 2.2, we have: $\Psi^*(\dot\ell) \leq \Phi^*(\gamma'(z_{\dot\ell}\circ\varphi)\cdot R)$. This proves the statement (b) of the proposition and completes its proof. ■

An example. We now give an example of a non-subgradient constraint without an infinite force representation. Consider the strongest constraint specified by $\ell = x \in L_{\gamma_o^*}\cdot R$ which corresponds to $V = U = \mathcal{Y} = M_{\gamma_o}$ and $\varphi(\omega) = \delta_\omega$, $\omega\in\Omega$. Take $\gamma(s) = (|s|+1)\log(|s|+1) - |s|$, $\gamma^*(t) = e^{|t|} - |t| - 1$ and $R(d\omega) = e^{-\omega}/(1+\omega^2)\, d\omega$ on $\Omega = [0,\infty[$. The measure $\ell(\omega) = \omega\, R(d\omega)$ corresponds to $\ell = \gamma'(\zeta)\cdot R$ with $\zeta(\omega) = e^\omega - 1$. We have:
$\int_\Omega \gamma^*\circ\gamma'(\zeta)\, dR = \int_{[0,\infty[} (e^\omega - \omega - 1)e^{-\omega}/(1+\omega^2)\, d\omega < \infty : x = \ell \in \mathcal{C}$,
$\int_\Omega \zeta\gamma'(\zeta)\, dR = \int_{[0,\infty[} (e^\omega - 1)\omega e^{-\omega}/(1+\omega^2)\, d\omega = \infty : \partial\Psi^*(x)$ is empty (by Lemma 3.2), and
$\int_\Omega \gamma(\zeta)\, dR = \int_{[0,\infty[} (\omega e^\omega - e^\omega + 1)e^{-\omega}/(1+\omega^2)\, d\omega = \infty : \zeta$ doesn't belong to L_{γ_o}.

The minimization result. Let us first note that it is possible to weaken the assumptions of Propositions 3.3 and 3.4.

Lemma 3.5. *The results of Proposition 3.3 and Proposition 3.4 still hold under the only assumption (3.3) (without assuming (2.6)).*

<u>Proof</u>. If (2.6) is satisfied the result is already proved. If $\gamma \equiv 0$, there is nothing to prove.

Let us suppose that $\gamma_- \equiv 0$ and $\lim_{t\to+\infty}\gamma(t) = +\infty$. Thanks to ([Lé1], Propositions 3.3 and 4.4), $\overline{\Psi}$ is the restriction to the $\sigma(V^{**}V^*)$-closure of V in V^{**} of $\overline{\Phi}(\xi) = \overline{\Phi_+}(\xi_+) + \overline{\Phi_-}(\xi_-)$. But, $\overline{\Phi_-} \equiv 0$ since $\gamma_- \equiv 0$. Hence, $\overline{\Psi}^a(\xi) = \overline{\Psi}_+^a(\xi_+)$ where $\overline{\Psi}_+^a$ is a function $\overline{\Psi}^a$ built upon γ_+ instead of γ. Following the proof of Lemma 3.1 with the function σ built with $\rho = \gamma_+$, we obtain that for any ξ such that $\overline{\Psi}^a(\xi) < \infty$, there exists $z \in \mathcal{X}^*$ such that $\xi_+ = \langle z,\varphi\rangle_+$ where $\langle z,\varphi\rangle$ is measurable and $\overline{\Psi}^a(\xi) = \overline{\Psi}^a(\xi_+) = \int_\Omega \gamma(\langle z,\varphi\rangle)\, dR$.

An inspection of the proofs of Propositions 3.3 and 3.4 allows us to state that they still hold with the above $\overline{\Psi}^a$ (note that, thanks to $\gamma'_- \equiv 0$, some statements of these proofs become trivial).

The same proof works when the parts of γ_- and γ_+ are exchanged. ■

We are now ready to state the main result of this section. Denoting

$$\begin{array}{lllll}\Lambda & : & y\in\mathcal{Y} & \mapsto & \int_\Omega \gamma(\langle y,\varphi(\omega)\rangle)\, R(d\omega) & \in [0,\infty[, \\ \Lambda^* & : & x\in\mathcal{X} & \mapsto & \sup_{y\in\mathcal{Y}}\{\langle x,y\rangle - \Lambda(y)\} & \in [0,\infty],\end{array}$$

we get: $\Psi^* = \Lambda^* \circ \dot{\Pi}$.

It will be useful to represent $dom\ \Lambda^*$ as a subset $\mathcal{C}$ of $\mathcal{X}$, image of $dom\ \Psi^*$ by the application $\dot{\ell} \in V^* \mapsto \dot{\Pi}(\dot{\ell})$, defined at (3.2.b):

$$\mathcal{C} = \{\dot{\Pi}(\dot{\ell}) \in \mathcal{X}\,;\, \dot{\ell} \in dom\ \Psi^*\}.$$

As $\dot{\Pi}$ is linear, $\mathcal{C}$ is a convex subset of $\mathcal{X}$ and $\dot{\ell}$ is a boundary point of $dom\ \Psi^*$ if and only if $\dot{\Pi}(\dot{\ell})$ is a boundary point of $\mathcal{C}$.

Recall that the projection Π is defined at (3.2.a) and that (see Proposition 2.4), Φ and U being defined by (2.4) and (2.5), for any $\ell \in U^*$, we have $\Phi^*(\ell) = \begin{cases} \int_\Omega \gamma^* \left(\frac{d\ell}{dR}\right) dR & \text{if } \ell \ll R \\ +\infty & \text{otherwise} \end{cases}$, where $\ell \ll R$ means that there exists a measurable function $\frac{d\ell}{dR} : \Omega \to \mathbb{R}$ such that $\ell = \frac{d\ell}{dR} \cdot R$.

Theorem 3.6. *We assume (3.3). We have:*

$$\Psi^*(\dot{\ell}_o) = \inf\{\Phi^*(\ell)\,;\, \ell \in U^*, \ell \in \dot{\ell}_o\}, \quad \forall \dot{\ell}_o \in V^* \tag{3.20}$$

and

$$\Lambda^*(x_o) = \inf\{\Psi^*(\dot{\ell})\,;\, \dot{\ell} \in V^*, \dot{\Pi}(\dot{\ell}) = x_o\} = \inf\{\Phi^*(\ell)\,;\, \ell \in U^*, \Pi(\ell) = x_o\}, \quad \forall x_o \in \mathcal{X}. \tag{3.21}$$

If the constraint $x_o \in \mathcal{X}$ is such that $\Lambda^*(x_o) < +\infty$, the minimization problem

$$\min \left\{ \Phi^*(\ell)\,;\, \ell \in U^* \textit{such that } \ell \ll R, \int_\Omega \gamma^* \left(\frac{d\ell}{dR}\right) dR < \infty \textit{ and } \Pi(\ell) = x_o \right\}$$

admits a unique solution ℓ_{x_o} in U^*. Moreover, ℓ_{x_o} has the following form

$$\ell_{x_o} = \gamma'(z_{x_o} \circ \varphi) \cdot R \tag{3.22}$$

where z_{x_o} is the pointwise limit of a sequence in $\mathcal{Y} \subset \mathcal{X}^*$. More precisely, if x_o is an interior point of $\mathcal{C}$ ($x_o \in \mathrm{ri}\, \mathcal{C}$), z_{x_o} is a measurable linear form on $\mathcal{X}$ and $\langle z_{x_o}, \varphi(\cdot)\rangle$ belongs to dom $\overline{\Psi}^a$, while if x_o is a boundary point of $\mathcal{C}$ ($x_o \in \mathrm{rb}\, \mathcal{C}$), z_{x_o} may be $[-\infty, +\infty]$-valued.

In this situation ($\Lambda^*(x_o) < \infty$), we also have:

$$\Lambda^*(x_o) = \Psi^*(\dot{\ell}_{x_o}) = \Phi^*(\ell_{x_o}) = \int_\Omega \gamma^* \big[\gamma'(z_{x_o} \circ \varphi)\big]\, dR < \infty. \tag{3.23}$$

Remark. With the notations of Section 1, $\Lambda^*(x_o)$ is the value of the optimization problem (1.2). Hence, (3.21) is the equality of the values of the primal problem (1.1) and the dual problem (1.2).

Proof. As γ is differentiable, γ^* is strictly convex and so is Φ^*. The uniqueness of the solution to the minimization problem follows.

Let us prove (3.20) and (3.22). For any $\dot{\ell}_o \in V^*$, by Proposition 2.2, we have: $\Psi^*(\dot{\ell}_o) \leq \inf\{\Phi^*(\ell)\,;\, \ell \in \dot{\ell}_o\}$. If $\Psi^*(\dot{\ell}_o) = \infty$, (3.20) is clear. While if $\Psi^*(\dot{\ell}_o) < \infty$, by Propositions 3.3, 3.4 and Lemma 3.5, we know that the infimum is achieved at some $\ell_{x_o} \in U^*$, as described in (3.22), with $\Psi^*(\dot{\ell}_{x_o}) = \Phi^*(\ell_{x_o})$. This proves (3.20) and (3.22).

Let us prove (3.21). Applying to our problem the following canonical transformations:

$$\Omega \to \mathcal{X}, \quad R \to R_\varphi \stackrel{\text{def}}{=} R \circ \varphi^{-1}, \quad \varphi \to (x \in \mathcal{X} \mapsto x \in \mathcal{X}),$$
$$U \to S \stackrel{\text{def}}{=} \{f : \mathcal{X} \to \mathbb{R}\,;\, f \circ \varphi \in U\}, \quad V \to \{f : \mathcal{X} \to \mathbb{R}\,;\, f \circ \varphi \in V\} = \mathcal{Y},$$

the analogues of Φ and Φ^* are: $\Gamma(f) = \Phi(f \circ \varphi), f \in S$ and $\Gamma^*(\lambda) = \begin{cases} \int_\Omega \gamma^* \left(\frac{d\lambda}{dR_\varphi}\right) dR_\varphi & \text{if } \lambda \ll R_\varphi \\ +\infty & \text{otherwise} \end{cases}$, $\lambda \in S^*$, and the analogues of Ψ and Ψ^* are Λ and Λ^*. Therefore, (3.20) becomes: $\Lambda^*(x_o) = \inf\{\Gamma^*(\lambda)\,;\, \lambda \in S^*, \tilde{\Pi}(\lambda) = x_o\}$ where $\tilde{\Pi}(\lambda) = x_o$ means: $\forall y \in \mathcal{Y}, \Big\langle \lambda, \langle y, \cdot \rangle \Big\rangle_{S^*,S} = \Big\langle x_o, y \Big\rangle_{\mathcal{X},\mathcal{Y}}$.

In addition, if $\Lambda^*(x_o) < \infty$, the infimum is achieved at a unique $\lambda_{x_o} \in S^*$ such that $\tilde{\Pi}(\lambda_{x_o}) = x_o$ and this λ_{x_o} has the special form: $\lambda_{x_o} = (\gamma' \circ z_{x_o}) \cdot R_\varphi$ where z_{x_o} is as in (3.22). We have just shown that: $\Lambda^*(x_o) = \Gamma^*(\lambda_{x_o}) = \int_{\mathcal{X}} \gamma^* \circ \gamma' \circ z_{x_o}\, dR_\varphi$ and $x_o = \tilde{\Pi}(\lambda_{x_o}) = \dot{\Pi}(\dot{\ell}_{x_o})$ with $\ell_{x_o} = \gamma'(z_{x_o} \circ \varphi) \cdot R$. Since we have: $\Psi^* = \Lambda^* \circ \dot{\Pi}$, we obtain: $\Lambda^*(x_o) = \Psi^*(\dot{\ell}_{x_o})$. Together with (3.20), this provides us with (3.21) when $\Lambda^*(x_o) < \infty$. Suppose now that $\Lambda^*(x_o) = \infty$. If there is no $\ell \in U^*$ such that $\Pi(\ell) = x_o$, the equality is clear (with the convention that an infimum on the empty set is equal to infinity). If there is an $\ell \in U^*$ such that $\Pi(\ell) = x_o$, then: $\infty = \Lambda^*(x_o) = \Psi^*(\dot{\ell}_o) \le \inf\{\Phi^*(\ell)\,;\, \ell \in \dot{\ell}_o\}$, which proves the equality (3.21).

The statement (3.23) has been derived during the above argument.

Note that it has been proved in Propositions 3.3 and 3.4 that z_{x_o} is the pointwise limit of a sequence in $\mathcal{Y}$. This completes the proof of the theorem. ■

4. Characterization of the minimizer

Let us start making precise the situation where the force field z_{x_o} appearing in (3.22) assumes infinite values. We begin with some notations.

We introduce now, two vector subspaces of $\mathcal{X}$. As a definition, $\mathcal{X}_o$ is the linear span of $\mathcal{C}$; it is endowed with its relative topology $\sigma(\mathcal{X}_o, \mathcal{Y})$. We denote $\mathcal{X}_\varphi := \Pi(U^*) = \dot{\Pi}(V^*)$. It is endowed with its relative topology $\sigma(\mathcal{X}_\varphi, \mathcal{Y})$ and the corresponding Borel σ-field. Notice that $\varphi(\Omega) \subset \mathcal{X}_\varphi$ since $\langle \varphi(\omega), y \rangle_{\mathcal{X},\mathcal{Y}} = \Big\langle \delta_\omega, \langle y, \varphi(\cdot) \rangle \Big\rangle_{U^*,U}, \forall \omega \in \Omega, y \in \mathcal{Y}$. Clearly, $\mathcal{X}_o \subset \mathcal{X}_\varphi$. Let $R_\varphi = R \circ \varphi^{-1}$ stand for the image measure of R on $\mathcal{X}_\varphi$ by the measurable application $\varphi : \Omega \to \mathcal{X}_\varphi$. If g is a measurable function on $\mathcal{X}_\varphi$ and $(g \circ \varphi) \cdot R$ belongs to *dom* Φ^*, we state

$$x_g \stackrel{\text{def}}{=} \int_{\mathcal{X}_\varphi} x g(x)\, R_\varphi(dx) \stackrel{\text{def}}{=} \Pi\Big((g \circ \varphi) \cdot R\Big).$$

We define

$$\mathcal{G} := \left\{ g : \mathcal{X}_\varphi \to \mathbb{R},\ \text{measurable such that } \int_{\mathcal{X}_\varphi} \gamma^*(g)\, dR_\varphi < \infty \right\},$$

By Theorem 3.6, we have: $\mathcal{C} = \{x_g \,;\, g \in \mathcal{G}\} = dom\ \Lambda^*$. This means that for any $g \in \mathcal{G}$, we have:

$$\langle x_g, y\rangle_{\mathcal{X},\mathcal{Y}} = \int_{\mathcal{X}_\varphi} \langle x, y\rangle g(x)\, R_\varphi(dx), \quad \forall y \in \mathcal{Y}. \tag{4.1}$$

Lemma 4.1. *Let us assume (2.3) and if $R(\Omega) = \infty$: (3.3.a) or (3.3.b). Let x_o be any boundary point of $\mathcal{C}$ ($x_o \in rbdom\ \Lambda^*$), then $\mathcal{C}$ admits at least one support hyperplane at x_o in $\mathcal{X}_o$.*

The direction $n_o \in \mathcal{X}_o^*$ of any support hyperplane of $\mathcal{C}$ at x_o : $\{x \in \mathcal{X}_o \,;\, \langle n_o, x - x_o\rangle = 0\}$, admits a measurable extension $n \in \mathcal{X}_\varphi^*$ to $\mathcal{X}_\varphi$ which satisfies

$$\int_{\mathcal{X}_\varphi} \gamma\Big(\alpha\langle n, x\rangle\Big)\, R_\varphi(dx) < \infty, \text{ for some } \alpha > 0. \tag{4.2}$$

In addition, for $g \in \mathcal{G}$, we have:

$$\langle n_o, x_g\rangle_{\mathcal{X}_o^*,\mathcal{X}_o} = \int_{\mathcal{X}_\varphi} \langle n, x\rangle g(x)\, R_\varphi(dx)$$

where the integral is well defined.

Proof. Let V^* be endowed with the topology $\sigma(V^*, V)$ and the corresponding Borel σ-field. Using the one-one bimeasurable correspondence $\dot{\Pi} : V^* \to \mathcal{X}$, one transports the proof from $(\mathcal{X}, \Lambda^*)$ to (V^*, Ψ^*). We assume for a while that (2.6) holds.

Let us consider an even function σ as in (3.7). If $R(\Omega) < \infty$, $(L_{\sigma^*}, \|\cdot\|_{\sigma^*})$ is $(L_\infty, \|\cdot\|_\infty)$. If $R(\Omega) = \infty$ and (2.6) holds, σ satisfies the Δ_2-condition globally and $\lim_{t\to\infty} \sigma(t)/t < \infty$. If $R(\Omega) = \infty$ and $\gamma_- \equiv 0$ (resp. $\gamma_+ \equiv 0$), σ is defined upon γ_+ (resp. γ_-). If $\gamma \equiv 0$, there is nothing to prove.

The space V is endowed with the Luxemburg norm $\|\cdot\|_\sigma$, V'_σ and V''_σ stand for its strong dual and bidual spaces. Let $\dot{\ell}_o \in V^*$ be a boundary point of $dom\ \Psi^*$ with $\Pi(\dot{\ell}_o) = x_o$. Let us first show that any support hyperplane of $dom\ \Psi^*$ at $\dot{\ell}_o$ is $\sigma(V'_\sigma, V''_\sigma)$-continuous.

The set $A = \{\dot{\ell} \in V^* \,;\, \Psi^*(\dot{\ell}) \le \Psi^*(\dot{\ell}_o) + 1\}$ is convex and closed in V'_σ, as Ψ^* is convex and lower semicontinuous. More, it has a nonempty interior, since by ([Lé1], Lemma 2.1): Ψ^* is continuous on $ridom\ \Psi^*$. But, (see [Bou], Ch. 2, §5, Proposition 3) in a topological vector space, if A is a closed convex set with a nonempty interior, any boundary point of A belongs to at least one support hyperplane of A and any support hyperplane of A is closed. Clearly, $\dot{\ell}_o$ is also a boundary point of A and any support hyperplane of $dom\ \Psi^*$ at $\dot{\ell}_o$ (with direction $\nu_o \in (V'_\sigma)^*$) is a support hyperplane of A at $\dot{\ell}_o$. Hence, ν_o is continuous on V'_σ and therefore it is $\sigma(V'_\sigma, V''_\sigma)$-continuous.

By Hahn-Banach Theorem, ν_o admits a $\sigma(V^*, V''_\sigma)$-continuous extension: ν, on V^*. We shall take $n = \dot{\Pi}(\nu)$. To prove the measurability of n, it remains to check that a continuous linear form ξ on V'_σ ($\xi \in V''_\sigma$) is measurable with respect to the Borel σ-field of the $*$-weak topology $\sigma(V'_\sigma, V)$.

Because of the identifications $V^* \subset U^*$ and $V^{**} \subset U^{**}$, it is enough to prove that any ξ in L'_{σ^*} is measurable with respect to the Borel σ-field of $\sigma(L_{\sigma^*}, M_\sigma)$. By ([Koz], Theorem 2.2; see [Lé1], Theorem 5.1), any $\xi \in L'_{\sigma^*}$ may be decomposed into the sum of an absolutely continuous part ξ^a and a singular part $\xi^s = \xi^s_+ - \xi^s_-$:

$$\xi = \xi^a + \xi^s_+ - \xi^s_-, \tag{4.3}$$

where ξ^s_+ and ξ^s_- are the nonnegative and nonpositive parts of ξ^s. This means that $\xi^a \ll R$ with $\xi^a \in L_\sigma$ and that there exists a nonincreasing sequence of measurable sets $(A_k)_{k\geq 1}$ such that $\lim_{k\to\infty} R(A_k) = 0$ and $\langle \xi^s, \mathbb{1}_{(\Omega\setminus A_k)}\rangle = 0, \forall k \geq 1$.

As σ satisfies the Δ_2-condition, we have: $M_\sigma = L_\sigma$ and $\sigma(L_{\sigma^*}, M_\sigma) = \sigma(L_{\sigma^*}, L_\sigma)$. As a consequence, ξ^a is $\sigma(L_{\sigma^*}, M_\sigma)$-continuous, hence measurable. Thanks to (4.3), it remains to check that any nonnegative singular ξ is measurable.

Let us notice that the norm $h \mapsto \|h\|_{\sigma^*} = \sup\{\langle h, u\rangle\,;\, u \in L_\sigma, \|u\|_\sigma \leq 1\}$ is measurable since L_σ is separable (σ satisfies the Δ_2-condition and $\mathcal{A}$ is assumed to be separable, see [RaR], Theorem 1, p. 87). It follows that the σ-field generated by the $\|\cdot\|_{\sigma^*}$-balls is equal to the Borel σ-field of $\sigma(L_{\sigma^*}, L_\sigma)$. But, in general, L_{σ^*} is not separable and its $\|\cdot\|_{\sigma^*}$-Borel σ-field is strictly larger than the $\|\cdot\|_{\sigma^*}$-ball σ-field (see [Pol] for this discussion). Notice that in the special case where R is a bounded measure, then L_σ is L_1 and L_{σ^*} is L_∞.

Let $(\Omega_p)_{p\geq 1}$ be a localizing sequence for the σ-finite measure $R : (\Omega_p)_{p\geq 1}$ is a nondecreasing sequence of measurable sets such that $R(\Omega_p) < \infty$, for any $p \geq 1$ and $\bigcup_{p\geq 1}\Omega_p = \Omega$. Let $\xi^s \in L'_{\sigma^*}$ be a nonnegative singular form and $(A_k)_{k\geq 1}$ satisfy $\lim_{k\to\infty} R(A_k) = 0$ and $\langle \xi^s, \mathbb{1}_{(\Omega\setminus A_k)}\rangle = 0, \forall k \geq 1$. As

$$\{h \in L_{\sigma^*}\,;\, \langle \xi^s, h\rangle \geq 0\} = \bigcup_{n,k\geq 1}\bigcap_{p\geq 1}\Big\{h \in L_{\sigma^*}\,;\, \big\|\mathbb{1}_{(\Omega_p\cap A_k)}[h - n\mathbb{1}_{A_k}]\big\|_\infty \leq n\langle \xi^s, \mathbb{1}_{A_k}\rangle\Big\}$$

and $\|\cdot\|_\infty$ is measurable in restriction to any Ω_p, we have just proved that ξ^s is measurable. This completes the proof of the measurability of n when (2.6) is satisfied.

Let us assume now that $\gamma_- \equiv 0$ (resp. $\gamma_+ \equiv 0$). Since $\Phi^*(\xi) = \Phi^*_+(\xi_+) + \Phi^*_-(\xi_-)$ (see [Lé1], Proposition 4.4), one obtains that: $\Phi^*(\xi) = \Phi^*_+(\xi)$ if ξ is nonnegative (resp. nonpositive) and $+\infty$ otherwise. By Theorem 3.6, it follows that z_{x_o} appearing in (3.22) is nonnegative (resp. nonpositive) R_φ-a.e. Therefore, reproducing the above proof with γ_+ (resp. γ_-) instead of γ, we obtain a similar result. This completes the proof of the existence and measurability of n.

Let us assume for a while that γ is an even function. Applying ([Lé1], Lemma 2.1) to Λ^* and reasonning with γ instead of σ, n is continuous on $\mathcal{Y}'$: the strong dual of $(\mathcal{Y}, \|\cdot\|_\gamma)$, that is: $n \in \mathcal{Y}''$. By Goldstine's lemma ([Bou], Ch. 4, §3, Proposition 5), n is a pointwise limit (along a filter) of elements of $\mathcal{Y}$: $n = \lim_\alpha y_\alpha$ such that $\|y_\alpha\|_{\mathcal{Y}''} \leq \|n\|_{\mathcal{Y}''}$:

$$\langle n, x_g\rangle = \lim_\alpha \langle y_\alpha, x_g\rangle = \lim_\alpha \int_{\mathcal{X}_\varphi} \langle y_\alpha, x\rangle g(x)\, R_\varphi(dx), \quad \forall g \in \mathcal{G}$$

where the last equality is given by (4.1). As $y_\alpha \in \mathcal{Y}$, $\|y_\alpha\|_{\mathcal{Y}''} = \|y_\alpha\|_\gamma$ so that $\|y_\alpha\|_{\mathcal{Y}''} \leq \|n\|_{\mathcal{Y}''} < \infty$ implies the uniform integrability of $\{\langle y_\alpha, \cdot\rangle g\}_\alpha$ for any $g \in L_{\gamma^*}(R_\varphi)$. It comes out that $\langle n, x_g\rangle = \int_{\mathcal{X}_\varphi} \langle n, x\rangle g(x)\, R_\varphi(dx)$, for any $g \in \mathcal{G}$ and also that: $|\int_{\mathcal{X}_\varphi} \langle n, x\rangle f(x)\, R_\varphi(dx)| < \infty$, for any $f \in L_{\gamma^*}(R_\varphi)$. But the last estimate implies that n belongs to $L_\gamma(R_\varphi)$ (see [RaR], Proposition 4.1.1 for this argument).

One extends the argument to noneven functions γ, noticing that n_o is continuous on the strong duals of $(\mathcal{Y}, \|\cdot\|_{\gamma_+})$ and $(\mathcal{Y}, \|\cdot\|_{\gamma_-})$. This gives: $n \in L_{\gamma_+}(\mathcal{X}_\varphi, R_\varphi) \cap L_{\gamma_-}(\mathcal{X}_\varphi, R_\varphi)$ which implies (4.2), and completes the proof of the lemma. ■

In order to describe properly the boundary of $\mathcal{C}$, we still have to set some definitions up. Let $n \in \mathcal{X}_\varphi^*$ be the outer direction of a support hyperplane. This means that $n \neq 0$ and there exists a boundary point x_o of $\mathcal{C}$ such that $\mathcal{C} \subset \{x \in \mathcal{X}_\varphi\,;\, \langle n, x - x_o\rangle \leq 0\}$. For such an n, we call the <u>n-facet of $\mathcal{C}$</u>, the subset F^n of $\mathcal{C}$:

$$F^n = \mathcal{C} \cap \{x \in \mathcal{X}_o\,;\, \langle n, x - x_o\rangle = 0\}.$$

We also extend this definition to the case where $n = 0$: the 0-facet of $\mathcal{C}$ is $F^0 = \mathcal{C}$ itself.

As a definition, for any $x_o \in \mathcal{C}$, the <u>facet of $\mathcal{C}$ at x_o</u> is the largest convex subset: F_{x_o}, of $\mathcal{C}$ such that x_o belongs to the relative interior of F_{x_o}.

For instance, x_o stands in the relative interior of $\mathcal{C}$ if and only if $F_{x_o} = \mathcal{C}$ and x_o is an <u>extreme point</u> of $\mathcal{C}$ if and only if $F_{x_o} = \{x_o\}$.

If $n_{x_o} \in \mathcal{X}_\varphi^*$ belongs to the relative interior of the cone of the outer normals of $\mathcal{C}$ at x_o, it is called a <u>strict outer normal</u>. In the sequel, the notation n_{x_o} will always stand for a strict outer normal at x_o.

If x_o is an interior point of $\mathcal{C}$, then $F_{x_o} = F^0 = \mathcal{C}$. In this case, we state $n_{x_o} = 0$.

An extreme point x_o is an <u>exposed point</u> of $\mathcal{C}$ if and only if $\{x_o\} = F_{x_o} = F^{n_{x_o}}$.

(4.4) <u>Remark</u>. In the following picture, $(0,0)$ is an extreme point which is not exposed.

$$\mathcal{C} = \mathbb{R}\times]0,\infty[\, \cup\, [0,\infty[\times\{0\} \qquad F_{(0,0)} = \{(0,0)\} \qquad F^{n_0} = [0,\infty[\times\{0\}$$

We denote, for any $A \subset \mathcal{C}$ and any $x_o \in \mathcal{C}$,

$$\mathcal{G}(A) = \{g \in \mathcal{G}\,;\, x_g \in A\},\ \mathcal{G}(x_o) = \{g \in \mathcal{G}\,;\, x_g = x_o\}.$$

We are now ready to give a precise description of the boundary of $\mathcal{C}$.

Proposition 4.2. *Let us assume (3.3). For any $n \in \mathcal{X}_\varphi^*$ such that*

(4.5.a) $$\int_{\mathcal{X}_\varphi} \gamma\Big(\alpha\langle n, x\rangle\Big)\, R_\varphi(dx) < \infty, \text{ for some } \alpha > 0 \quad \text{and}$$

(4.5.b) $$\gamma^*(\kappa_+) R_\varphi(\langle n, \cdot\rangle > 0) + \gamma^*(\kappa_-) R_\varphi(\langle n, \cdot\rangle < 0) < \infty$$

there exists an n-facet F^n of $\mathcal{C}$.

Let us suppose that in addition,

(4.6) $[\gamma^*(\kappa_+) < \infty \text{ or } rb\ \{z \in \mathcal{X}_\varphi^*; \int_\Omega \gamma_+^* \circ \gamma_+' \circ z\, dR_\varphi < \infty\} = \emptyset]$

and $[\gamma^*(\kappa_-) < \infty$ *or rb* $\{z \in \mathcal{X}_\varphi^*; \int_\Omega \gamma_-^* \circ \gamma_-' \circ z \, dR_\varphi < \infty\} = \emptyset]$

then, (4.5) *is also a sufficient condition of existence of* F^n *and*

$$\mathcal{G}(F^n) = \left\{ \kappa_+ 1\!\!1_{\{\langle n,\cdot\rangle > 0\}} + \kappa_- 1\!\!1_{\{\langle n,\cdot\rangle < 0\}} + g_o 1\!\!1_{\{\langle n,\cdot\rangle = 0\}} \, ; \, g_o \text{ s.t. } \int_{\{\langle n,\cdot\rangle = 0\}} \gamma^*[g_o(x)] \, R_\varphi(dx) < \infty \right\}$$

Convention. In the estimate (4.5.b), it is understood that $0\cdot\infty = \infty\cdot 0 = 0$.

Proof. In the following, we shall work in $\mathcal{X}_\varphi$. The result is clear when $n = 0$, let us consider now that $n \neq 0$. Let n be an outer direction of a support hyperplane of $\mathcal{C}$ in $\mathcal{X}_\varphi$; we have: $R_\varphi(\langle n, \cdot\rangle \neq 0) > 0$. A point x_o belongs to F^n if and only if $x_o \in \mathcal{C}$ and x_o is a solution to the optimization problem: $\max\{\langle n, x\rangle \, ; \, x \in \mathcal{C}\}$. Thanks to Lemma 4.1, F^n is the set of the x_g's where g is a solution to

$$\max \left\{ \int_{\mathcal{X}_\varphi} \langle n, x\rangle g(x) \, R_\varphi(dx) \, ; \, g \in \mathcal{G} \right\}. \tag{4.7}$$

If we have (4.5), any x_g with $g(x) = \begin{cases} \kappa_+ & \text{if } \langle n, x\rangle > 0 \\ \kappa_- & \text{if } \langle n, x\rangle < 0 \\ g_o(x) & \text{if } \langle n, x\rangle = 0 \end{cases}$, for some g_o such that $\int_{\{n=0\}} \gamma^*[g_o(x)] \, R_\varphi(dx) < \infty$, is a solution to the optimization problem (4.7). Notice that (4.5.a) is necessary to state (4.7): thanks to Hölder's inequality for the dual pairings of Orlicz spaces: $(L_{\gamma_+}, L_{\gamma_+^*})$ and $(L_{\gamma_-}, L_{\gamma_-^*})$ the integral $\int_{\mathcal{X}_\varphi} \langle n, x\rangle g(x) \, R_\varphi(dx)$ is well defined for any $g \in \mathcal{G}$. Hence, such an x_g belongs to F^n.

Now, let us prove the second statement. Let $n \in \mathcal{X}_\varphi^*$ be an outer direction of a support hyperplane of $\mathcal{C}$. The estimate (4.5.a) has already been obtained in Lemma 4.1.

Let us prove (4.5.b). If *rb* $\{z \in \mathcal{X}_\varphi^*; \int_\Omega \gamma_\pm^* \circ \gamma_\pm' \circ z \, dR_\varphi < \infty\} = \emptyset$, by Proposition 3.4, *dom* $\Psi_\pm^*$ has no boundary. Under (4.6), one has only to consider the case where $\gamma_\pm^*(\kappa_\pm) < \infty$.

Supposing that $R_\varphi(\langle n, \cdot\rangle > 0) > 0$, let us maximize $\int_{\{\langle n,\cdot\rangle > 0\}} \langle n, x\rangle g(x) R_\varphi(dx)$ on $\mathcal{G}$. Let $A \subset \{\langle n, \cdot\rangle > 0\}$ be such that $R_\varphi(A) < \infty$. As F^n is nonempty, the maximum in (4.7) is attained, and so is the maximum of

$$\max \left\{ \int_A \langle n, x\rangle g(x) \, R_\varphi(dx) \, ; \, g : A \to \mathbb{R} \text{ such that } \int_A \gamma^*[g(x)] \, R_\varphi(dx) < \infty \right\}. \tag{4.8}$$

But, if there exists $B \subset A$ such that $R_\varphi(B) > 0$ and $\operatorname{ess\,sup}_{x \in B} g_1(x) < \kappa_+$, the maximum in (4.8) is not attained at g_1, since the function $g_2 = g_1 1\!\!1_{A\setminus B} + \frac{1}{2}(\operatorname{ess\,sup}_{x\in B} g_1(x) + \kappa_+) 1\!\!1_B$ gives a strictly greater $\int_A \langle n, x\rangle g(x) \, R_\varphi(dx)$. Consequently, as they are attained, the solutions of (4.8) are the functions g which satisfy $g 1\!\!1_A = \kappa_+ 1\!\!1_A$. This implies that $\gamma^*(\kappa_+) R_\varphi(A) < \infty$.

As R_φ is σ-finite, there exists a nondecreasing sequence $(A_k)_{k\geq 1}$ such that $\bigcup_{k\geq 1} A_k = \{\langle n, \cdot\rangle > 0\}$ and $R_\varphi(A_k) < \infty, \forall k \geq 1$. Therefore, any solution g_* of

$$\max \left\{ \int_{\{\langle n,\cdot\rangle > 0\}} \langle n, x\rangle g(x) \, R_\varphi(dx) \, ; \, g \text{ such that } \int_{\{\langle n,\cdot\rangle > 0\}} \gamma^*[g(x)] \, R_\varphi(dx) < \infty \right\}$$

satisfies $g_* \mathbb{1}_{A_k} = \kappa_+ \mathbb{1}_{A_k}, \forall k \geq 1$. It follows that $g_* \mathbb{1}_{\{\langle n,\cdot\rangle>0\}} = \kappa_+ \mathbb{1}_{\{\langle n,\cdot\rangle>0\}}$ and $\gamma^*(\kappa_+) R_\varphi(\langle n,\cdot\rangle > 0) < \infty$. One concludes with

$$\int_{\mathcal{X}_\varphi} \langle n,x\rangle g(x) R_\varphi(dx) = \int_{\{\langle n,\cdot\rangle>0\}} \langle n,x\rangle g(x) R_\varphi(dx) + \int_{\{\langle n,\cdot\rangle<0\}} \langle n,x\rangle g(x) R_\varphi(dx).$$

■

Remark. An extension of Proposition 4.2 when (4.6) fails, would require the knowledge of the solutions to (4.7) in this situation.

Corollary 4.3. *Let us assume (3.3) and (4.6). Let x_o be an exposed boundary point of $\mathcal{C}$ (this means that $F^{n_{x_o}} = \{x_o\}$) and n_{x_o} be a strict outer normal of $\mathcal{C}$ at x_o. Then,*

$$\mathcal{G}(x_o) = \{g_{x_o}\}$$

where $g_{x_o} = \kappa_+ \mathbb{1}_{\{\langle n_{x_o},\cdot\rangle>0\}} + \kappa_- \mathbb{1}_{\{\langle n_{x_o},\cdot\rangle<0\}}$.
The equation $\Pi(\ell) = x_o, \ell \in U^*$ admits a unique solution ℓ_{x_o} in dom Φ^* :

$$\ell_{x_o} = \left(\kappa_+ \mathbb{1}_{\{\langle n_{x_o},\varphi\rangle>0\}} + \kappa_- \mathbb{1}_{\{\langle n_{x_o},\varphi\rangle<0\}}\right) \cdot R$$

In addition, we have

$$\Lambda^*(x_o) = \Phi^*(\ell_{x_o}) = \gamma^*(\kappa_+) R(\langle n_{x_o},\varphi\rangle > 0) + \gamma^*(\kappa_-) R(\langle n_{x_o},\varphi\rangle < 0).$$

Remark. Taking the convention stated under Proposition 4.2 into account, the functions being defined up to negligible sets: if $\kappa_+ = +\infty$, then $g_{x_o} = \kappa_- \mathbb{1}_{\{\langle n_{x_o},\cdot\rangle<0\}}$ and if $\kappa_- = -\infty$, then $g_{x_o} = \kappa_+ \mathbb{1}_{\{\langle n_{x_o},\cdot\rangle>0\}}$.

Proof. As x_o is an exposed boundary point of $\mathcal{C}$, we have $F^{n_{x_o}} = \{x_o\}$. This means that $\{\langle n_{x_o},\cdot\rangle = 0\} = \{0\}, R_\varphi$-a.e. in $\mathcal{X}_\varphi$. One obtains the uniqueness of the solutions with Proposition 4.2. The last identity follows from this uniqueness together with Theorem 3.6. ■

From now on, the restriction (4.6) will be assumed. Let γ be an even function. If γ^* satisfies the Δ_2-condition (globally if R is unbounded), then (4.6) is satisfied. On the other hand, $\gamma^*(\kappa) < \infty$ is equivalent to the existence of an asymptotic line at infinity with slope $\kappa < \infty$. The Boltzmann-Shannon entropy (see (5.3)) corresponds to $\gamma^*(t) = (t+1)\log(t+1) - t$. That is $\gamma(s) = e^s - s - 1$ which satisfies (4.6) since $\gamma^*(\kappa_-) = \gamma^*(-1) = 1$ and γ^*_+ is globally Δ_2.

Let us describe the relevant force fieds. Let J be a totally ordered countable index set which admits a smaller element: $\flat$. We consider a family $(n) = (n^j)_{j\in J}$ of measurable linear forms on $\mathcal{X}_\varphi$. For any $j \in J$, let us denote

$$\mathcal{T}^j_+ = \{n^j > 0\} \cap \bigcap_{i<j} \{n^i = 0\} \quad \text{and} \quad \mathcal{T}^j_- = \{n^j < 0\} \cap \bigcap_{i<j} \{n^i = 0\}$$

with the convention: $\bigcap_{i<\flat} \{n^i = 0\} = \mathcal{X}_\varphi$, so that $\mathcal{T}^\flat_+ = \{n^\flat > 0\}$ and $\mathcal{T}^\flat_- = \{n^\flat < 0\}$. We define

$$\mathcal{S} = \bigcap_{j\in J} \{n^j = 0\}, \quad \mathcal{T}_+ = \bigcup_{j\in J} \mathcal{T}^j_+, \quad \mathcal{T}_- = \bigcup_{j\in J} \mathcal{T}^j_- \quad \text{and} \quad \mathcal{T} = \mathcal{T}_+ \cup \mathcal{T}_-$$

($\mathcal{S}, \mathcal{T}_+$ and $\mathcal{T}_-$ form a measurable partition of $\mathcal{X}_\varphi$).

Let us introduce a new notation for the force fields. Let $\bar{z}$ be a measurable linear form on $\mathcal{X}_\varphi$ and $(n) = (n^j)_{j\in J}$ as above. We define the application $\bar{z} + \infty\cdot(n) : \mathcal{X}_\varphi \to [-\infty, +\infty]$, for any $x \in \mathcal{X}_\varphi$, by

$$\langle \bar{z} + \infty\cdot(n), x\rangle = \begin{cases} +\infty & \text{if } x \in \mathcal{T}_+ \\ -\infty & \text{if } x \in \mathcal{T}_- \\ \langle \bar{z}, x\rangle & \text{if } x \in \mathcal{S}. \end{cases}$$

It is a measurable application. In the special case where (n) is reduced to a unique element ($J = \{\flat\}$), we denote $(n) = n$. And if $(n) = 0$, $\bar{z} + \infty\cdot(n) = \bar{z}$ belongs to $\mathcal{X}_\varphi^*$.

One says that $\bar{z} + \infty\cdot(n)$ is an admissible force field if

(4.9.a) $\displaystyle\int_{\mathcal{S}} \gamma(\langle \bar{z}, x\rangle)\, R_\varphi(dx) < \infty$ and
for any $\varepsilon > 0, K \geq 1, f_1, \dots, f_K \in L_{\gamma_+^*}, g_1, \dots, g_K \in L_{\gamma_-^*}$, there exists $y \in \mathcal{Y}$ such that
$$\sum_{k\leq K} \left| \int_\Omega \langle \bar{z} - y, \varphi(\cdot)\rangle (\mathbb{1}_{\{\langle \bar{z},\varphi(\cdot)\rangle\cdot \geq 0\}} f_k + \mathbb{1}_{\{\langle \bar{z},\varphi(\cdot)\rangle\cdot \leq 0\}} g_k)\, dR \right| \leq \varepsilon.$$

(4.9.b) $\displaystyle\int_{\mathcal{S}} \gamma^*[\gamma'(\langle \bar{z}, x\rangle)]\, R_\varphi(dx) < \infty$

(4.9.c) for any $j \in J$, there exists $\alpha > 0$ such that
$$\int_{\cap_{i<j}\{n^i=0\}} \gamma\big(\alpha\langle n^j, x\rangle\big)\, R_\varphi(dx) < \infty$$

(4.9.d) $\gamma^*(\kappa_+) R_\varphi(\mathcal{T}_+) + \gamma^*(\kappa_-) R_\varphi(\mathcal{T}_-) < \infty.$

By Lemma 3.1, the property (4.9.a) is satisfied if and only if $\langle \bar{z}, \varphi(\cdot)\rangle$ belongs to *dom* $\overline{\Psi}^a$.
Notice that, because of (4.9.b) and (4.9.d), we have: $\gamma'(\langle \bar{z} + \infty\cdot(n), \cdot\rangle) \in \mathcal{G}$.

Because of (4.9.d), in order that the force field admits a true infinite component, it is necessary that $\gamma^*(\kappa_+) < \infty$ or $\gamma^*(\kappa_-) < \infty$.

Theorem 4.4. *We assume (3.3) and (4.6).*
With any $x_o \in \mathcal{C}$, one can associate an admissible force field $z_{x_o} = \bar{z}_{x_o} + \infty\cdot(n)_{x_o}$ such that

$$x_o = \int_{\mathcal{X}_\varphi} x\gamma'\big(\langle z_{x_o}, x\rangle\big)\, dR_\varphi \tag{4.10}$$

and

$$\Lambda^*(x_o) = \Phi^*\Big(\gamma'\big(\langle z_{x_o}, \varphi\rangle\big) \cdot R\Big) < \infty. \tag{4.11}$$

Conversely, if $x_o \in \mathcal{X}$ is related to an admissible force field z_{x_o} by (4.10), then (4.11) is satisfied.

Remark. If there exists $t_o \in \mathbb{R}$ such that $\gamma'(t_o) \in \{\kappa_+, \kappa_-\}$, then there may be several different admissible force fields associated with one boundary constraint.
For instance, with $\kappa_- = 0$ and $\kappa_+ > 0$, $\bar{z} = 0$ is an admissible force field associated with $x_o = 0$ which is a boundary point of $\mathcal{C}$.

An important consequence of Theorem 3.6 and Theorem 4.4 is the following result.

Theorem 4.5. *We assume (3.3) and (4.6).*

If $x_o \in \mathcal{X}$ satisfies $\Lambda^*(x_o) < \infty$, the minimization problem

$$\min \left\{ \Phi^*(\ell)\,;\, \ell \in U^* \text{such that } \ell \ll R, \int_\Omega \gamma^* \left(\frac{d\ell}{dR}\right) dR < \infty \text{ and } \Pi(\ell) = x_o \right\} \tag{4.12}$$

admits a solution ℓ_{x_o} which has the following form

$$\ell_{x_o} = \gamma'\big(\langle z_{x_o}, \varphi\rangle\big) \cdot R$$

where z_{x_o} is an admissible force field.

Conversely, if $\bar{z} + \infty\cdot(n)$ is an admissible force field, putting $x_o = \int_{\mathcal{X}_\varphi} x\gamma'\big(\langle \bar{z} + \infty\cdot(n), x\rangle\big)\, R_\varphi(dx)$, we have $\Lambda^*(x_o) < \infty$ and the minimization problem (4.12) admits $\ell_{x_o} = \gamma'\big(\langle \bar{z} + \infty\cdot(n), \varphi\rangle\big) \cdot R$ as a solution.

In addition, since γ is assumed to be differentiable, the solution of (4.12) is unique when $\Lambda^*(x_o) < \infty$.

Proof of Theorem 4.5. It is a straightforward consequence of Theorems 3.6 and 4.4. ■

Proof of Theorem 4.4. We begin with the direct statement. If x_o is an interior point of $\mathcal{C}$, choosing $(n)_{x_o} = 0$, the result has been proved in Proposition 3.3 and Lemma 3.5.

Let us first suppose that $x_o \in \mathcal{C}$ satisfies

$$F_{x_o} = F^{n_{x_o}} \tag{4.13}$$

where n_{x_o} is a strict outer normal of $\mathcal{C}$ at x_o. We denote $\mathcal{T}_+ = \{\langle n_{x_o}, \cdot\rangle > 0\}, \mathcal{T}_- = \{\langle n_{x_o}, \cdot\rangle < 0\}$, $\mathcal{T} = \{\langle n_{x_o}, \cdot\rangle \neq 0\}$, $\mathcal{S} = \{\langle n_{x_o}, \cdot\rangle = 0\}$ and $g_{\mathcal{T}} := \kappa_+ \mathbb{1}_{\mathcal{T}_+} + \kappa_- \mathbb{1}_{\mathcal{T}_-}$. By Proposition 4.2, we have

$$g \in \mathcal{G}(x_o) \Longrightarrow g = g_{\mathcal{T}} + \mathbb{1}_{\mathcal{S}} g_o \text{ for some } g_o \text{ such that } \int_{\mathcal{S}} \gamma^*[g_o(x)]\, R_\varphi(dx) < \infty. \tag{4.14}$$

Using the convention stated under Proposition 4.2, we denote

$$x_{\mathcal{T}} := \kappa_+ \int_{\mathcal{T}_+} x\, R_\varphi(dx) + \kappa_- \int_{\mathcal{T}_-} x\, R_\varphi(dx)$$
$$x_{\mathcal{S}} := x_o - x_{\mathcal{T}} = \int_{\mathcal{S}} x g_o(x)\, R_\varphi(dx).$$

We shall write the subscripts $\mathcal{T}$ and $\mathcal{S}$ to indicate that we consider the measures $\mathbb{1}_{\mathcal{T}} \cdot R_\varphi$ and $\mathbb{1}_{\mathcal{S}} \cdot R_\varphi$ instead of R_φ.

We have $\Lambda^*_{\mathcal{S}}(x_{\mathcal{S}}) \leq \Phi^*_{\mathcal{S}}\Big((g_o \mathbb{1}_{\mathcal{S}})\circ\varphi\cdot R_\varphi\Big) < \infty$. By (4.13), $x_{\mathcal{S}}$ is an interior point of $\mathcal{C}_{\mathcal{S}}$. By Proposition 3.3 and Lemma 3.5, there exists a real valued force field $\bar{z}_{x_o}$ defined on $\mathcal{S}$ such that

$$\bar{z}_{x_o} \in \mathit{dom}\, \overline{\Lambda}^a_{\mathcal{S}} \tag{4.15}$$

and

$$x_{\mathcal{S}} = \int_{\mathcal{S}} x\gamma'(\langle \bar{z}_{x_o}, x\rangle)\, R_\varphi(dx) \tag{4.16}$$

and

$$\Lambda^*_{\mathcal{S}}(x_{\mathcal{S}}) = \Phi^*_{\mathcal{S}}\Big(\gamma'(\langle \bar{z}_{x_o}, \varphi\rangle)\mathbb{1}_{\mathcal{S}}(\varphi)\cdot R\Big) < \infty. \tag{4.17}$$

On the other hand, by Proposition 2.2 and (4.14), we get: $\Lambda^*_{\mathcal{T}}(x_{\mathcal{T}}) \le \Phi^*_{\mathcal{T}}\Big(g_{\mathcal{T}}\circ\varphi\cdot R\Big) \le \Lambda^*(x_o) < \infty$. Hence, applying Proposition 4.2, we obtain

$$\mathcal{G}_{\mathcal{T}}(x_{\mathcal{T}}) = \{g_{\mathcal{T}}\} \tag{4.18}$$

and

$$\Lambda^*_{\mathcal{T}}(x_{\mathcal{T}}) = \Phi^*_{\mathcal{T}}\Big(g_{\mathcal{T}}\circ\varphi\cdot R\Big) = \gamma^*(\kappa_+)R_\varphi(\mathcal{T}_+) + \gamma^*(\kappa_-)R_\varphi(\mathcal{T}_-). \tag{4.19}$$

Taking $z_{x_o} = \bar{z}_{x_o} + \infty\cdot n_{x_o}$ (with $J = \{\flat\}$ and $n^\flat_{x_o} = n_{x_o}$), (4.10) follows from (4.16) and (4.18). We also get:

$$\begin{aligned}
\Lambda^*(x_o) &= \inf\{\Phi^*(\ell)\,;\, \Pi(\ell) = x_o\} && \text{(by Theorem 3.6)}\\
&= \inf\{\Phi^*_{\mathcal{S}}(\ell_1) + \Lambda^*_{\mathcal{T}}(x_{\mathcal{T}})\,;\, \Pi_{\mathcal{S}}(\ell_1) = x_{\mathcal{S}}\} && \text{(by (4.18))}\\
&= \inf\{\Phi^*_{\mathcal{S}}(\ell)\,;\, \Pi_{\mathcal{S}}(\ell) = x_{\mathcal{S}}\} + \Lambda^*_{\mathcal{T}}(x_{\mathcal{T}}) &&\\
&= \Lambda^*_{\mathcal{S}}(x_{\mathcal{S}}) + \Lambda^*_{\mathcal{T}}(x_{\mathcal{T}}) && \text{(by Theorem 3.6, with (4.17)).}
\end{aligned}$$

Together with (4.17) and (4.19), this yields (4.11) and the admissibility properties (4.9.b) and (4.9.d) for z_{x_o}. Finally, thanks to (4.15) and Lemma 3.1, z_{x_o} satisfies the admissibility properties (4.9.a) and (4.9.c).

Let us consider the general situation where (4.13) may not be satisfied (for an example of such a situation, see (4.4)). We choose for $n^0_{x_o}$ a strict outer normal of $\mathcal{C}$ at x_o. By Lemma 4.1, $n^0_{x_o}$ satisfies (4.2). If x_o is a boundary point of $E^0 = \{x \in \mathcal{C}\,;\, \langle n^0_{x_o}, x - x_o\rangle = 0\}$, we go on and choose for $n^1_{x_o}$ a strict outer normal of E^0 at x_o such that $(n^0_{x_o}, n^1_{x_o})$ is linearly independent. By Lemma 4.1, $n^1_{x_o}$ satisfies: $\exists\alpha > 0, \int_{\{n^0_{x_o}=0\}} \gamma(\alpha\langle n^1_{x_o}, x\rangle)\, R_\varphi(dx) < \infty$. Recursively, if for $k \in \mathbb{N}, k \ge 1$, x_o is a boundary point of $E^k = \{x \in \mathcal{C}\,;\, \langle n^0_{x_o}, x - x_o\rangle = \cdots = \langle n^k_{x_o}, x - x_o\rangle = 0\}$, we choose for $n^{k+1}_{x_o}$ a strict outer normal of E^k at x_o such that $(n^0_{x_o}, \dots, n^{k+1}_{x_o})$ is linearly independent. By Lemma 4.1, $n^{k+1}_{x_o}$ satisfies: $\exists\alpha > 0, \int_{\bigcap_{0\le i\le k}\{n^i_{x_o}=0\}} \gamma(\alpha\langle n^{k+1}_{x_o}, x\rangle)\, R_\varphi(dx) < \infty$. If x_o is an interior point of E^{k+1}, we choose $(n)_{x_o} = (n^0_{x_o}, \dots, n^{k+1}_{x_o})$. Otherwise, we go on.
If x_o is a boundary point of $\bigcap_{k\ge 0} E^k$, we go on with $\bigcap_{k\ge 0} E^k$ instead of $\mathcal{C}$. This is the reason why a general ordered set J is needed, instead of $\mathbb{N}$ with its natural order. Finally, we choose for $(n^j_{x_o})_{j\in J}$ the maximal element of these recursively built $(n^i_{x_o})_{i\in I}$, i.e.: J is the increasing union of these I's endowed with the induced order structure and $(n^j_{x_o})_{j\in J}$ is defined projectively.
By construction, $(n^j_{x_o})_{j\in J}$ satisfies the admissibility condition (4.9.c).

Let us check that J is countable. By (4.2), the family $(n^j_{x_o})_{j\in J}$ stands in L_σ which is a Hausdorff separarable space (since σ satisfies the Δ_2-condition and $\mathcal{A}$ is a separable σ-field). Therefore, any linearly independent family of L_σ is countable, and so is $(n^j_{x_o})_{j\in J}$.
By construction, we have

$$F_{x_o} = (x_o + \mathcal{S}) \cap \mathcal{C} \tag{4.20}$$

and applying recursively Proposition 4.2, we obtain

$$\mathcal{G}(F_{x_o}) = \left\{\kappa_+ \mathbb{I}_{\mathcal{T}_+} + \kappa_- \mathbb{I}_{\mathcal{T}_-} + g_o \mathbb{I}_{\mathcal{S}}\,;\, g_o \text{ such that } \int_{\mathcal{S}} \gamma^*([g_o(x)]\, R_\varphi(dx) < \infty \right\}.$$

By Theorem 3.6, there exists g_o such that $\gamma^*(\kappa_+)R_\varphi(\mathcal{T}_+) + \gamma^*(\kappa_-)R_\varphi(\mathcal{T}_-) + \int_{\mathcal{S}} \gamma^*[g_o(x)]\, R_\varphi(dx) = \Lambda^*(x_o) < \infty$. This proves the admissibility property (4.9.d) for $(n^j_{x_o})_{j\in J}$. Thanks to (4.20), one concludes the proof of the direct statement as in the situation where (4.13) is satisfied.

Now, let us show the converse statement. An admissible force field $\bar z + \infty\cdot(n)$ is given, we put: $x_o = \int_{\mathcal{X}_\varphi} x\gamma'(\langle \bar z + \infty\cdot(n), x\rangle)\, R_\varphi(dx)$. Taking the admissibility properties (4.9.b) and (4.9.d) into account, with Proposition 2.2, we have

$$\Lambda^*(x_o) \le \int_{\mathcal{X}_\varphi} \gamma^*[\gamma'(\langle \bar z + \infty\cdot(n), x\rangle)]\, R_\varphi(dx) < \infty.$$

The admissibility properties (4.9.b), (4.9.c) and (4.9.d), together with Proposition 4.2 yield $F_{x_o} \subset (x_o + \mathcal{S}) \cap \mathcal{C}$. (The equality (4.20) also holds when γ is not affine at infinity, i.e. $\gamma'(\mathbb{R}) =]\kappa_-, \kappa_+[$, but it may fail otherwise). As for the proof of the direct statement, we denote $g_{\mathcal{T}} := \kappa_+ \mathbb{I}_{\mathcal{T}_+} + \kappa_- \mathbb{I}_{\mathcal{T}_-}$ and

$$\begin{aligned} x_{\mathcal{T}} &:= \int_{\mathcal{T}} x g_{\mathcal{T}}(x)\, R_\varphi(dx) = \kappa_+ \int_{\mathcal{T}_+} x\, R_\varphi(dx) + \kappa_- \int_{\mathcal{T}_-} x\, R_\varphi(dx) \\ x_{\mathcal{S}} &:= x_o - x_{\mathcal{T}} = \int_{\mathcal{S}} x\gamma'(\langle \bar z, x\rangle)\, R_\varphi(dx). \end{aligned}$$

Thanks to (4.9.b) and (4.9.d), we have $\Lambda^*_{\mathcal{S}}(x_{\mathcal{S}}) < \infty$ and $\Lambda^*_{\mathcal{T}}(x_{\mathcal{T}}) < \infty$. As in the proof of the direct statement, we obtain

$$\begin{aligned} \Lambda^*_{\mathcal{T}}(x_{\mathcal{T}}) &= \Phi^*_{\mathcal{T}}\Big((g_{\mathcal{T}} \circ \varphi)\cdot R\Big) \quad \text{and} \\ \Lambda^*(x_o) &= \Lambda^*_{\mathcal{S}}(x_{\mathcal{S}}) + \Lambda^*_{\mathcal{T}}(x_{\mathcal{T}}). \end{aligned}$$

Hence, to prove the converse statement, it remains to show that

$$\Lambda^*_{\mathcal{S}}(x_{\mathcal{S}}) = \int_{\mathcal{S}} \gamma^*[\gamma'(\langle \bar z, x\rangle)]\, R_\varphi(dx). \tag{4.21}$$

We live in $\mathcal{S}$. Because of the admissibility property (4.9.a), $\xi_{\mathcal{S}} := \langle \bar z, \varphi\rangle$ belongs to *dom* $\overline{\Psi}_{\mathcal{S}}$. More, $\dot\ell_{\mathcal{S}}$: the class of $\gamma'(\langle \bar z, \varphi\rangle)\mathbb{I}_{\mathcal{S}}(\varphi)\cdot R$, belongs to $\partial\overline{\Psi}_{\mathcal{S}}(\xi_{\mathcal{S}})$. Indeed, $\gamma(t+s) \ge \gamma(t) + s\gamma'(t), \forall t, s \in \mathbb{R}$, so that

$$\begin{aligned} \overline{\Psi}_{\mathcal{S}}(\xi_{\mathcal{S}} + \eta) &= \overline{\Psi}^a_{\mathcal{S}}(\xi_{\mathcal{S}} + \eta^a) + \overline{\Psi}^s_{\mathcal{S}}(\eta^s) \\ &\ge \overline{\Psi}^a_{\mathcal{S}}(\xi_{\mathcal{S}} + \eta^a) \\ &\ge \overline{\Psi}^a_{\mathcal{S}}(\xi_{\mathcal{S}}) + \int_{\{\varphi\in\mathcal{S}\}} \eta^a \gamma'(\xi_{\mathcal{S}})\, dR. \end{aligned}$$

As η^a stands in the linear span of *dom* $\overline{\Psi}_{\mathcal{S}}$, $\eta^a_\pm \in L_{\gamma_\pm}$. On the other hand, by the admissibility property (4.9.b), $\gamma'(\xi_{\mathcal{S}})_\pm$ stands in $L_{\gamma^*_\pm}$. Therefore, the integral $\int_{\{\varphi\in\mathcal{S}\}} \eta^a\gamma'(\xi_{\mathcal{S}})\, dR$ is well defined, by Hölder's inequality. It comes out that: $\dot{\ell}_{\mathcal{S}} \in \partial\overline{\Psi}_{\mathcal{S}}(\xi_{\mathcal{S}})$.

But $\Psi^*_{\mathcal{S}}$ and $\overline{\Psi}_{\mathcal{S}}$ are convex conjugate to each other for the duality (V^*, V^{**}), so that by Proposition 2.1.a: $\Psi^*_{\mathcal{S}}(\dot{\ell}_{\mathcal{S}}) + \overline{\Psi}_{\mathcal{S}}(\xi_{\mathcal{S}}) = \langle \dot{\ell}_{\mathcal{S}}, \xi_{\mathcal{S}}\rangle = \int_{\{\varphi\in\mathcal{S}\}} \gamma'(\xi_{\mathcal{S}})\xi_{\mathcal{S}}\, dR$. It follows that

$$\Lambda^*_{\mathcal{S}}(x_{\mathcal{S}}) = \Psi^*_{\mathcal{S}}(\dot{\ell}_{\mathcal{S}}) = \int_{\{\varphi\in\mathcal{S}\}} [\gamma'(\xi_{\mathcal{S}})\xi_{\mathcal{S}} - \gamma(\xi_{\mathcal{S}})]\, dR = \int_{\{\varphi\in\mathcal{S}\}} \gamma^*[\gamma'(\xi_{\mathcal{S}})]\, dR,$$

which is (4.21). This completes the proof of the theorem. ■

As a direct corollary of the above proof, one obtains a similar characterization of the minimizers associated with a subgradient constraint, without assuming (4.6).

Theorem 4.6. *We assume (3.3).*
With any $x_o \in \mathcal{C}$ such that $\partial\Lambda^*(x_o) \neq \emptyset$, in particular if x_o stands in the relative interior of $\mathcal{C}$, one can associate an admissible force field $\bar{z}_{x_o}$ with no infinite component such that (4.10) and (4.11) hold.
Moreover, the minimization problem (4.12) admits the unique solution $\ell_{x_o} = \gamma'(\langle \bar{z}_{x_o}, \varphi\rangle) \cdot R$.
Conversely, if $x_o \in \mathcal{X}$ is related to an admissible force field $\bar{z}$ without infinite component, x_o being defined by (4.10), we have $\Lambda^*(x_o) < \infty$.
If in addition, $\partial\Lambda^*(x_o) \neq \emptyset$, the minimization problem (4.12) admits $\ell_{x_o} = \gamma'\big(\langle \bar{z}, \varphi\rangle\big) \cdot R$ as its unique solution.

5. Some examples

We present some illustrations of the previous results. Other examples are developped in [Lé2]. We begin with the traditional marginal problem on a product probability space. At the last subsection, some comments will be given about an interesting counterexample of I. Csiszár.

Marginal problem. The dual equality for the problem (5.1.a) below and partial results of representation of the minimizing solution have already been obtained in [Beu], [Cs1], [BLN], [CG] and [Lé2] (see also the references quoted in [BLN] and [CG]). At Theorem 5.1 below, we give a characterization of its solution.
Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces, $\Omega = \Omega_1 \times \Omega_2$ is endowed with the product σ-field $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Let the probability measures R, ν_1 and ν_2 be given respectively on the spaces Ω, Ω_1 and Ω_2. If P is a probability measure on Ω, P_1 and P_2 stand for its marginals on Ω_1 and Ω_2 : for any $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, $P_1(A_1) = P(A_1 \times \Omega_2)$ and $P_2(A_2) = P(\Omega_1 \times A_2)$. We denote $M_1(\Omega)$ the set of all probability measures on Ω. We consider the minimization problem

$$\inf\{\Phi^*(P + \kappa_- R)\, ;\, P \in M_1(\Omega), P \ll R, P_1 = \nu_1, P_2 = \nu_2\} \tag{5.1.a}$$

with $\Phi^*(P + \kappa_- R) = \begin{cases} \int_\Omega \gamma^*(\frac{dP}{dR} + \kappa_-)\, dR & \text{if } P \ll R \\ +\infty & \text{otherwise} \end{cases}$, assuming $\kappa_- > -\infty$.

Clearly, if $\Phi^*(P+\kappa_- R) < \infty$, then P is a nonnegative measure and since ν_1 has a unit mass, (5.1.a) is equivalent to

$$\text{(5.1.b)} \qquad \inf\{\Phi^*(P+\kappa_- R)\,;\, P \text{ signed measure on } \Omega, P_1 = \nu_1, P_2 = \nu_2\}.$$

In the special case where

$$\text{(5.2)} \qquad \gamma(x) = e^x - x - 1, \quad x \in \mathbb{R},$$

in restriction to $M_1(\Omega)$, $\Phi^*(P+\kappa_- R)$ is the relative entropy of P with respect to R :

$$\text{(5.3)} \qquad I(P \mid R) = \begin{cases} \int_\Omega \log\left(\frac{dP}{dR}\right) dP & \text{if } P \ll R \\ +\infty & \text{otherwise} \end{cases}, \quad P \in M_1(\Omega).$$

Notice that $\nu_1 \times \nu_2$ has the desired marginals, but it may not be absolutely continuous with respect to R. This happens for instance with $\Omega_1 = \Omega_2 = [-1,+1]$, $\nu_1 = \nu_2 = \frac{1}{2}\mathbb{1}_{[-1,+1]}(x)\,dx$ and R : any measure supported by the unit circle.

Let $B(\Omega_1)$ and $B(\Omega_2)$ be the spaces of the bounded measurable functions on Ω_1 and Ω_2. For any measurable functions θ_1 on Ω_1 and θ_2 on Ω_2, we denote $\theta_1 \oplus \theta_2(\omega_1,\omega_2) = \theta_1(\omega_1) + \theta_2(\omega_2), \omega_1 \in \Omega_1, \omega_2 \in \Omega_2$.

Let us describe a general admissible force field in this framework: we transcript the definition of Section 4 with $\mathcal{Y} = B(\Omega_1) \times B(\Omega_2)$ and $\varphi(\omega_1,\omega_2) = (\delta_{\omega_1}, \delta_{\omega_2})$, (see the proof of Theorem 5.1 below). Let J be a totally ordered countable index set which admits a smaller element: $\flat$. We consider two families of functions $(h_1^j)_{j\in J}$ and $(h_2^j)_{j\in J}$ on Ω_1 and Ω_2 respectively, such that for any $j \in J$, $h_1^j \oplus h_2^j$ is measurable on $\Omega_1 \times \Omega_2$. We denote

$$\Omega_o^h = \bigcap_{j\in J} \{h_1^j \oplus h_2^j = 0\},$$
$$\Omega_+^h = \bigcup_{j\in J} \Big[\{h_1^j \oplus h_2^j > 0\} \cap \bigcap_{i<j} \{h_1^i \oplus h_2^i = 0\}\Big],$$
$$\Omega_-^h = \bigcup_{j\in J} \Big[\{h_1^j \oplus h_2^j < 0\} \cap \bigcap_{i<j} \{h_1^i \oplus h_2^i = 0\}\Big],$$

with the convention: $\bigcap_{i<\flat}\{h_1^i \oplus h_2^i = 0\} = \Omega_1 \times \Omega_2$. We also consider f_1 and f_2 two functions on Ω_1 and Ω_2 respectively, such that $f_1 \oplus f_2$ is measurable on $\Omega_1 \times \Omega_2$. The collection $(J, (h_1^j)_{j\in J}, (h_2^j)_{j\in J}, f_1, f_2)$ corresponds to an admissible force field if and only if

(5.4.a) $\int_{\Omega_o^h} \gamma(f_1 \oplus f_2)\, dR < \infty$ and
for any $\varepsilon > 0, F \in L^{\gamma_+^*}(\Omega_1 \times \Omega_2), G \in L^{\gamma_-^*}(\Omega_1 \times \Omega_2)$, there exist $\theta_1 \in B(\Omega_1), \theta_2 \in B(\Omega_2)$ such that

$$\left| \int_{\Omega_1 \times \Omega_2} (f_1 \oplus f_2 - \theta_1 \oplus \theta_2)(\mathbb{1}_{\{f_1 \oplus f_2 \geq 0\}} F + \mathbb{1}_{\{f_1 \oplus f_2 \leq 0\}} G)\, dR \right| \leq \varepsilon.$$

(5.4.b) $\int_{\Omega_o^h} \gamma^*[\gamma'(f_1 \oplus f_2)]\, dR < \infty$

(5.4.c) for any $j \in J$, there exists $\alpha > 0$ such that

$$\int_{\cap_{i<j}\{h_1^i \oplus h_2^i = 0\}} \gamma\big(\alpha h_1^j \oplus h_2^j\big)\, dR < \infty$$

(5.4.d) $\gamma^*(\kappa_+) R(\Omega_+^h) + \gamma^*(\kappa_-) R(\Omega_-^h) < \infty.$

We define $\tilde\gamma(x) := \gamma(x) - \kappa_- x, x \in \mathbb{R}$.

Theorem 5.1. *We assume (2.3), γ is differentiable and $\kappa_- > -\infty$. The constraint (ν_1, ν_2) satisfies*

$$\sup\Big\{ \int_{\Omega_1} \theta_1\, d\nu_1 + \int_{\Omega_2} \theta_2\, d\nu_2 - \int_{\Omega_1\times\Omega_2} \tilde\gamma(\theta_1 \oplus \theta_2)\, dR\,;\, \theta_1 \in B(\Omega_1), \theta_2 \in B(\Omega_2)\Big\} < \infty \tag{5.5}$$

if and only if there exists a measurable function g on $\Omega_1 \times \Omega_2$ such that

$$\nu_1(d\omega_1) = [\int_{\Omega_2} g(\omega_1, \omega_2)\, R(d\omega_2 \mid \omega_1)] \cdot R_1(d\omega_1),$$

$$\nu_2(d\omega_2) = [\int_{\Omega_1} g(\omega_1, \omega_2)\, R(d\omega_1 \mid \omega_2)] \cdot R_2(d\omega_2) \text{ and}$$

$$\int_{\Omega_1\times\Omega_2} \gamma^*(g + \kappa_-)\, dR < \infty.$$

If this condition is satisfied, the minimization problem (5.1) admits a unique solution which has the form

$$P_* = \big[(\kappa_+ - \kappa_-)\mathbb{1}_{\Omega_+^h} + (\gamma'(f_1 \oplus f_2) - \kappa_-)\, \mathbb{1}_{\Omega_o^h}\big] \cdot R \tag{5.6}$$

where Ω_o^h, Ω_+^h, f_1 and f_2 are built upon some $(J, (h_1^j)_{j\in J}, (h_2^j)_{j\in J}, f_1, f_2)$ which satisfies (5.4). Conversely, if $(J, (h_1^j)_{j\in J}, (h_2^j)_{j\in J}, f_1, f_2)$ satisfies (5.4), then (5.5) is satisfied with $\nu_1 = P_{*1}$, $\nu_2 = P_{*2}$ and (5.1) admits P_* as its unique solution.

Proof. It is a direct application of Theorem 4.5 where $\mathcal{Y}$ and φ are correctly chosen. We take $\mathcal{Y} = B(\Omega_1) \times B(\Omega_2)$. The duality bracket between $B(\Omega_1) \times B(\Omega_2)$ and $M_1(\Omega_1) \times M_1(\Omega_2) \subset \mathcal{X}$ is given, for any $\theta_1 \in B(\Omega_1), \theta_2 \in B(\Omega_2), \mu_1 \in M_1(\Omega_1), \mu_2 \in M_1(\Omega_2)$, by

$$\langle(\theta_1, \theta_2), (\mu_1, \mu_2)\rangle = \int_{\Omega_1} \theta_1\, d\mu_1 + \int_{\Omega_2} \theta_2\, d\mu_2.$$

We also take

$$\varphi(\omega) = (\delta_{\omega_1}, \delta_{\omega_2}),\ \omega = (\omega_1, \omega_2) \in \Omega$$

and we note that (3.1) and (4.6) are satisfied: for all $\theta_1 \in B(\Omega_1), \theta_2 \in B(\Omega_2)$

$$\int_{\Omega_1\times\Omega_2} \gamma[\theta_1(\omega_1) + \theta_2(\omega_2)]\, R(d\omega_1 d\omega_2) < \infty, \int_{\Omega_1\times\Omega_2} \gamma^* \circ \gamma'[\theta_1(\omega_1) + \theta_2(\omega_2)]\, R(d\omega_1 d\omega_2) < \infty$$

The constraint energy functional is given, for any $\nu_1 \in M_1(\Omega_1), \nu_2 \in M_1(\Omega_2)$, by

$$\begin{aligned}
&\Lambda^*(\nu_1 + \kappa_- R_1, \nu_2 + \kappa_- R_2)\\
&= \sup\Big\{ \int_{\Omega_1} \theta_1\, d(\nu_1 + \kappa_- R_1) + \int_{\Omega_2} \theta_2\, d(\nu_2 + \kappa_- R_2) - \int_{\Omega_1\times\Omega_2} \gamma[\theta_1(\omega_1) + \theta_2(\omega_2)]\, R(d\omega_1 d\omega_2)\\
&\qquad ;\, \theta_1 \in B(\Omega_1), \theta_2 \in B(\Omega_2)\Big\}\\
&= \sup\Big\{ \int_{\Omega_1} \theta_1\, d\nu_1 + \int_{\Omega_2} \theta_2\, d\nu_2 - \int_{\Omega_1\times\Omega_2} \tilde\gamma(\theta_1 \oplus \theta_2)\, dR\,;\, \theta_1 \in B(\Omega_1), \theta_2 \in B(\Omega_2)\Big\}.
\end{aligned}$$

Therefore, (5.5) is $\Lambda^*(\nu_1+\kappa_- R_1, \nu_2+\kappa_- R_2) < \infty$. Thanks to Theorem 4.5, if the constraint (ν_1, ν_2) satisfies (5.5), the minimization problem (5.1) admits a unique solution P_* which has the form

$$P_*(d\omega_1 d\omega_2) = \left[\gamma'(\langle z_\nu, (\delta_{\omega_1}, \delta_{\omega_2})\rangle - \kappa_-\right] \cdot R(d\omega_1 d\omega_2) \tag{5.7}$$

where z_ν is an admissible force field.

Conversely, any probability measure P_* of the form (5.7) is a solution to (5.1) with $\nu_1 = P_{*1}, \nu_2 = P_{*2}$, provided that (5.5) holds.

Since any linear form z on the set $\{(\delta_{\omega_1}, \delta_{\omega_2})\,;\, \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ has the form: $\langle z, (\delta_{\omega_1}, \delta_{\omega_2})\rangle = z_1(\omega_1)+z_2(\omega_2)$, $(\omega_1, \omega_2) \in \Omega$, the admissible force fields z_ν are in correspondence with the collections $(J, (h_1^j)_{j\in J}, (h_2^j)_{j\in J}, f_1, f_2)$ which satisfy (5.4). This completes the proof of the theorem. ■

An admissible force field z_ν with at most one infinite direction ($(n) = n, n = 0$ corresponding to the interior situation) is such that

$$\langle z_\nu, (\delta_{\omega_1}, \delta_{\omega_2})\rangle = \begin{cases} f_1(\omega_1) + f_2(\omega_2) & \text{if } h_1(\omega_1) + h_2(\omega_2) = 0 \\ -\infty & \text{if } h_1(\omega_1) + h_2(\omega_2) < 0 \\ +\infty & \text{if } h_1(\omega_1) + h_2(\omega_2) > 0 \end{cases}$$

where $f_1 \oplus f_2$ and $h_1 \oplus h_2$ are measurable. In this case, (5.6) is

$$P_* = \left[(\kappa_+ - \kappa_-)\mathbb{1}_{\{h_1\oplus h_2>0\}} + (\gamma'(f_1 \oplus f_2) - \kappa_-)\,\mathbb{1}_{\{h_1\oplus h_2=0\}}\right] \cdot R. \tag{5.8}$$

Notice that it may happen that f_1 and f_2 fail to be separately measurable (see [BL6], [RüT], [FöG], [CG] for this problem). Similarly, h_1^i and h_2^i may not be separately measurable.

<u>Example 1</u>. In the situation (5.2) where the relative entropy is minimized, for a general admissible force field described by $(J, (h_1^j)_{j\in J}, (h_2^j)_{j\in J}, f_1, f_2)$ which satisfies (5.4), (5.6) becomes

$$P_* = \mathbb{1}_{[\cap_{j\in J}\{h_1^j\oplus h_2^j=0\}]}\, e^{f_1\oplus f_2} \cdot R \text{ with } \begin{cases} \sum_{j\in J} R(\{h_1^j \oplus h_2^j > 0\} \cap [\cap_{i<j}\{h_1^i \oplus h_2^i = 0\}]) = 0 \\ \sum_{j\in J} R(\{h_1^j \oplus h_2^j < 0\} \cap [\cap_{i<j}\{h_1^i \oplus h_2^i = 0\}]) < \infty \end{cases} \tag{5.9}$$

If there is at most one infinite direction, we obtain

$$P_* = \mathbb{1}_{\{h_1\oplus h_2=0\}} e^{f_1\oplus f_2} \cdot R$$

with $R(h_1 \oplus h_2 > 0) = 0$ and $R(h_1 \oplus h_2 < 0) < \infty$.

<u>Example 2</u>. Let us consider (5.1) with $\Omega_1 = \Omega_2 = [-1, +1], R(d\omega_1 d\omega_2) = \frac{1}{4}\mathbb{1}_{\{|\omega_1|,|\omega_2|\le 1\}}\, d\omega_1 d\omega_2$, $\nu_1(d\omega_1) = (1 - |\omega_1|)\,\mathbb{1}_{\{|\omega_1|\le 1\}}\, d\omega_1$, $\nu_2(d\omega_2) = (1 - |\omega_2|)\,\mathbb{1}_{\{|\omega_2|\le 1\}}\, d\omega_2$ and

$$\gamma^*(x) = \begin{cases} \frac{1+x}{2}\log(1 + x) + \frac{1-x}{2}\log(1 - x) & \text{if } -1 < x < +1 \\ \log 2 & \text{if } x \in \{-1, +1\} \\ +\infty & \text{otherwise} \end{cases} .$$

This gives $\gamma(x) = \log\left(\frac{e^x+e^{-x}}{2}\right)$, $x \in \mathbb{R}$, $\kappa_- = -1, \kappa_+ = +1$ and $\gamma^*(\kappa_-) = \gamma^*(\kappa_+) = \log 2 < \infty$. One can see that $\overline{P}(d\omega_1 d\omega_2) = 2\mathbb{1}_{(|\omega_1|+|\omega_2|<1)}\, R(d\omega_1 d\omega_2)$ has the desired marginals: $\overline{P}_1 = \nu_1, \overline{P}_2 =$

ν_2 and since $0 \le \frac{d\overline{P}}{dR} \le 2$, it follows that $\Phi^*(\overline{P} + \kappa_- R) < \infty$. Denoting $\overline{h}_1(\omega_1) = \frac{1}{2} - |\omega_1|$, $\overline{h}_2(\omega_2) = \frac{1}{2} - |\omega_2|$, we get

$$\overline{P} + \kappa_- R = \left(\kappa_+ 1\!\!\mathrm{I}_{\{\overline{h}_1 \oplus \overline{h}_2 > 0\}} + \kappa_- 1\!\!\mathrm{I}_{\{\overline{h}_1 \oplus \overline{h}_2 < 0\}}\right) \cdot R$$

with $R(\overline{h}_1 \oplus \overline{h}_2 = 0) = 0$. Thanks to Proposition 4.2, we obtain $\mathcal{G}(F^{\overline{h}}) = \{\frac{d(\overline{P}+\kappa_- R)}{dR}\}$ which implies that $(\nu_1 + \kappa_- R_1, \nu_2 + \kappa_- R_2)$ is an exposed point of $\mathcal{C}$. By Corollary 4.3, we know that $\overline{P}$ is the unique measure P on Ω which satisfies:

$$P_1 = \nu_1, P_2 = \nu_2, P \ll R \quad \text{and} \quad 0 \le \frac{dP}{dR} \le 2. \tag{5.10}$$

On the other hand, since $\Phi^*(\overline{P} + \kappa_- R) = \log 2$ is the maximal finite value of Λ^*, it follows from Proposition 2.2 that $\overline{P}$ is the unique solution to (5.10).

Example 3. Let us consider now the marginal problem in the following simple situation: $\Omega_1 = \{a, b\}, \Omega_2 = \{A, B\}$, $R = \frac{1}{2}\delta_{(a,A)} + \frac{1}{4}\delta_{(b,A)} + \frac{1}{4}\delta_{(b,B)}$, $\nu_1 = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$ and $\nu_2 = \frac{1}{2}\delta_A + \frac{1}{2}\delta_B$. It is clear that the unique solution to this marginal problem (without any minimization) is

$$P_* = \frac{1}{2}\delta_{(a,A)} + \frac{1}{2}\delta_{(b,B)}.$$

We have $\frac{dP_*}{dR}(a, A) = 1$, $\frac{dP_*}{dR}(b, A) = 0$ and $\frac{dP_*}{dR}(b, B) = 2$. This example was proposed by P. Cattiaux and H. Föllmer ([CaF]) to show that in the situation (5.2), although the relative entropy $I(P_* \mid R)$ is finite, $\frac{dP_*}{dR}$ has not the product form suggested by $e^{f_1 \oplus f_2}$ in the interior case.

We suppose, as in the case (5.2), that $\kappa_- > -\infty$ and $\kappa_+ = +\infty$. It is easily checked that P_* has the form (5.8), which is

$$P_* = 1\!\!\mathrm{I}_{\{h_1 \oplus h_2 = 0\}}[\gamma'(f_1 \oplus f_2) - \kappa_-] \cdot R$$

with $R(h_1 \oplus h_2 > 0) = 0$. Indeed, choose $h_1(a) = 1, h_1(b) = -1, h_2(A) = -1, h_2(B) = 1$ and for f_1 and f_2 take any solution of $\begin{cases} \gamma'(f_1(a) + f_2(A)) &= \kappa_- + 1 \\ \gamma'(f_1(b) + f_2(B)) &= \kappa_- + 2 \end{cases}$.

One sees that, parametrizing the constraints by $s(\delta_a, 0) + t(\delta_b, 0) + u(0, \delta_A) + v(0, \delta_B)$, $(s, t, u, v) \in \mathbb{R}^4$, the set $\mathcal{C}$ has the equation $\begin{cases} u + v &= s + t \\ s &\ge \kappa_-/2 \\ v &\ge \kappa_-/4 \\ u - s &\ge \kappa_-/4 \end{cases}$ and that $(h_1(a), h_1(b), h_2(A), h_2(B)) = (1, -1, -1, 1)$ is an outer normal of $\mathcal{C}$ at the boundary point $(\frac{1}{2} + \frac{\kappa_-}{2}, \frac{1}{2} + \frac{\kappa_-}{2}, \frac{1}{2} + \frac{3\kappa_-}{4}, \frac{1}{2} + \frac{\kappa_-}{4})$ which corresponds to $(\nu_1 + \kappa_- R_1, \nu_2 + \kappa_- R_2)$.

As indicated in Theorem 3.6, $\frac{dP_*}{dR}$ is a pointwise limit of the form: $\dfrac{dP_*}{dR} + \kappa_- = \gamma'(\lim_{n\to\infty} f_1^n \oplus f_2^n)$. For instance, one may choose, for any $n \ge 1$, $f_1^n(a) = \gamma'^{-1}(\kappa_- + 1)$, $f_1^n(b) = -n$, $f_2^n(A) = 0$ and $f_2^n(B) = n + \gamma'^{-1}(\kappa_- + 2)$.

A counterexample of I. Csiszár. In [Cs1] (p. 152), I. Csiszár considers the following minimization problem

$$\inf \left\{ I(P \mid R)\,;\, P \in M_1([0, 1]), \int_{[0,1]} f_n \, dP = \frac{1}{4}, n \ge 1 \right\} \tag{5.11}$$

where I is the relative entropy (see (5.3)), $\dfrac{dR}{d\omega} = \dfrac{4}{1+3e}\mathbb{1}_{[0,\frac{1}{4}]}(\omega) + \dfrac{4e}{1+3e}\mathbb{1}_{]\frac{1}{4},1]}(\omega)$ and for any $n \geq 1, \omega \in [0,1]$, $f_n(\omega) = \begin{cases} 1+u_n & \text{if } 0 \leq \omega < \frac{1}{4n} \\ 1 & \text{if } \frac{1}{4n} \leq \omega < \frac{1}{4} \\ -u_n/n & \text{if } \frac{1}{4} \leq \omega < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq \omega \leq 1 \end{cases}$ with $u_n \geq 0$ and $\lim_{n\to\infty} \dfrac{u_n}{n} = 0$. This problem enters the framework of the present paper taking $\gamma(x) = e^x - x - 1$ (see (5.2)), $\Omega = [0,1]$, for $\mathcal{Y}$ let us choose the space of the real sequences $(y_n)_{n\geq 0}$ such that $\sum_{n\geq 1} |y_n|(1+u_n) < \infty$ and for φ let us take $\varphi(\omega) = (f_n(\omega))_{n\geq 0}$ where $f_0 \equiv 1$. The dual bracket is given by $\langle (y_n)_{n\geq 0}, \varphi(\omega)\rangle = \sum_{n\geq 0} y_n f_n(\omega)$. The assumptions (3.3) are clearly satisfied.

Lebesgue's measure on $[0,1]: Q(d\omega) = \mathbb{1}_{[0,1]}(\omega)\, d\omega$ satisfies: $I(Q \mid R) < \infty$, $\int_{[0,1]} f_n\, dQ = \frac{1}{4}, \forall n \geq 1$ and $\dfrac{dQ}{dR}(\omega) = e^{(c+g(\omega))}$ where $g(\omega) = \mathbb{1}_{[0,\frac{1}{4}[}(\omega)$ and e^c is the normalizing constant. Moreover, $(f_n)_{n\geq 1}$ tends to g pointwise (almost everywhere) and in $L^1(R) = L^1(Q)$. Hence, Q is a candidate to the solution of (5.11).

Nevertheless, by means of a geometric type argument, Csiszár shows that with $u_n = \sqrt{n}$, Q is not the solution of (5.11). We recover this result by proving that $c + g(\cdot)$ is not an admissible force field: the second part of condition (4.9.a) fails. Indeed, with $h(\omega) = \omega^{-3/4}$, one gets $\mathbb{1}_{\{c+g(\cdot)\geq 0\}} h \in L^{x\log x}(R)$, $\mathbb{1}_{\{c+g(\cdot)\leq 0\}} h \in L^\infty(R)$ (notice that $c \approx -0,17$) and $\lim_{n\to\infty} \int_{[0,1]} f_n h\, dR = \infty$.

Similarly, if $\inf_{n\geq 1} u_n > 0$ and for some $\delta > 0$, $\liminf_{n\to\infty} \dfrac{u_n}{(\log n)^{2+\delta}} > 0$, taking $h(\omega) = \dfrac{1}{\omega|\log\omega|^{2+\delta}}$, one proves that Q is not the solution of (5.11).

On the other hand, if $u_n = o_{n\to\infty}(\log n)$, then $\lim_{n\to\infty} \|f_n - g\|_{e^x} = 0$ and thanks to Hölder's inequality in Orlicz spaces, (4.9.a) is satisfied. Hence, Q is the solution to (5.11).

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