SF _ CM4

Exercice 2 (feuille 2)

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix}$$

(a) Calculer $R_\theta R_{\theta'}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$a = \cos\theta\cos\theta' - \sin\theta\sin\theta' = \cos(\theta+\theta')$

$b = -\cos\theta\sin\theta' - \sin\theta\cos\theta' = -(\cos\theta\sin\theta' + \sin\theta\cos\theta')$

$c = \sin\theta\cos\theta' + \cos\theta\sin\theta' = -b = \sin(\theta+\theta')$

$d = -\sin\theta\sin\theta' + \cos\theta\cos\theta'$

On a déjà vu que $\boxed{R_\theta R_{\theta'} = R_{\theta+\theta'}} = \begin{pmatrix} \cos(\theta+\theta') & -\sin(\theta+\theta') \\ \sin(\theta+\theta') & \cos(\theta+\theta') \end{pmatrix}$

$$R_\theta^n = \overbrace{R_\theta R_\theta \cdots\cdots R_\theta}^{n \text{ fois}} \qquad R_\theta^2 = R_{\theta+\theta} = R_{2\theta}$$

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$(\theta' = \theta)$

$$\boxed{\begin{array}{l}\text{Hyp de récurrence} \\ R_\theta^n = R_{n\theta}\end{array}}$$

preuve $R_\theta^{n+1} = R_\theta^n R_\theta \underset{\text{hyp de réc}}{=} R_{n\theta} R_\theta \underset{\theta' = n\theta}{=} R_{n\theta+\theta} = R_{(n+1)\theta}.$

□

(b) $S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ $\qquad S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$S_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{('identité')}$$

$$\boxed{I\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}}$$

Mostrar que $\boxed{S_\theta = R_\theta S_0 = S_0 R_{-\theta}}$ $\qquad \boxed{S_0^2 = I}$

$$R_\theta S_0 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = S_\theta$$

$$S_0 R_{-\theta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = S_\theta$$

(c) $$S_\theta S_{\theta'} = R_\theta \underbrace{S_0 S_0}_{S_0^2 = I} R_{-\theta'} = R_\theta \overbrace{I R_{-\theta'}}^{R_{-\theta'}} = R_\theta R_{-\theta'}$$

$$\boxed{\begin{aligned} S_\theta S_{\theta'} &= R_{\theta-\theta'} \\ S_{\theta'} S_\theta &= R_{\theta'-\theta} \end{aligned}} \qquad \boxed{\begin{aligned} S_\theta R_{\theta'} &= S_{\theta-\theta'} \\ R_{\theta'} S_\theta &= S_{\theta+\theta'} \end{aligned}}$$

$$\boxed{S_\theta = R_\theta S_0 = S_0 R_{-\theta}} \qquad \boxed{S_0^2 = I}$$

$$S_\theta R_{\theta'} = S_0 R_{-\theta} R_{\theta'} = S_0 R_{\theta'-\theta} = S_{\theta-\theta'}$$

$$R_{\theta'} S_\theta = \underbrace{R_{\theta'} R_\theta}\, S_0 = R_{\theta+\theta'} S_0 = S_{\theta+\theta'}$$

$$\boxed{\begin{array}{l} S_\theta S_{\theta'} = R_{\theta-\theta'} \\ S_{\theta'} S_\theta = R_{\theta'-\theta} \end{array}} \qquad \boxed{\begin{array}{l} S_\theta R_{\theta'} = S_{\theta-\theta'} \\ R_{\theta'} S_\theta = S_{\theta+\theta'} \end{array}}$$

$$R_\theta R_{\theta'} = R_{\theta+\theta'} \qquad R_0 = I$$

(d) $$(S_\theta)^n = \underbrace{S_\theta \cdots\cdots S_\theta}_{n \text{ fois}} \qquad \boxed{S_\theta^2 = I}$$

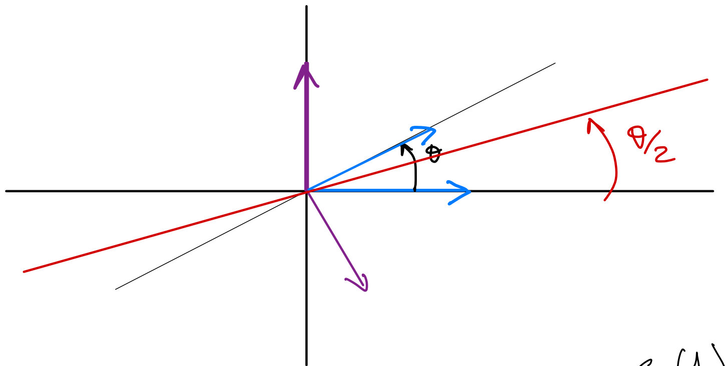
Si n est pair, $n = 2k$ $\quad S_\theta^{2k} = (S_\theta^2)^k = I^k = I$

Si n est impair, $n = 2k+1$, $\quad S_\theta^{2k+1} = S_\theta^{2k} \cdot S_\theta = I S_\theta = S_\theta$

$$S_\theta^n = \begin{cases} I & \text{si } n \text{ pair} \\ S_\theta & \text{si } n \text{ impair.} \end{cases}$$

(e) Montrons que S_θ est une symétrie par rapport à une droite d'angle polaire $\theta/2$

$$\boxed{S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}}$$



$$S_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$S_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

(f) : $\begin{cases} z = x + iy \\ r_\theta(z) = e^{i\theta} z \end{cases}$

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{S_\theta = R_\theta S_0 = S_0 R_{-\theta}} \qquad \boxed{S_0^2 = I}$$

$$S_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} \qquad s_0(z) = x - iy = \overline{z}$$

$$s_\theta(z) = e^{i\theta} s_0(z) = e^{i\theta}\overline{z} \qquad \boxed{s_\theta(z) = e^{i\theta}\overline{z}}$$

$$s_\theta(z) = \overline{e^{-i\theta} z} = \overline{e^{-i\theta}}\,\overline{z} = e^{i\theta}\overline{z}$$

$$S_\theta S_{\theta'} = R_{\theta-\theta'}$$

$$s_\theta\left(s_{\theta'}(z)\right) = s_\theta\left(e^{i\theta'}\bar{z}\right) = e^{i\theta}\,\overline{e^{i\theta'}\bar{z}}$$

$$= e^{i\theta}e^{-i\theta'}z = e^{i(\theta-\theta')}z = r_{\theta-\theta'}(z)$$