# FEYNMAN-KAC FORMULA UNDER A FINITE ENTROPY CONDITION

CHRISTIAN LÉONARD

Abstract. Motivated by entropic optimal transport, we are interested in the Feynman-Kac formula associated to the parabolic equation $(\mathsf{L}+V)g=0$ with a final nonnegative boundary condition and a Markov generator $\mathsf{L}:=\partial_t+\mathsf{b}\cdot\nabla+\Delta_{\mathsf{a}}/2$. It is well-known that when the drift $\mathsf{b}$, the diffusion matrix $\mathsf{a}$ and the scalar potential $V$ are regular enough and not growing too fast, the classical solution $g$ of this PDE, is represented by the Feynman-Kac formula $g_t(x)=E_R[\exp\left(\int_{[t,T]}V(s,X_s)\,ds\right)g(X_T)\mid X_t=x]$ where $R$ is the Markov measure with generator $\mathsf{L}$.

We do not assume that $g$, $\mathsf{b}$ and $V$ are regular, and only require that their growth is controlled by a finite entropy condition. These hypotheses are less restrictive than the standard assumptions of the theory of viscosity solutions, and allow for instance $V$ to belong to some Kato class. We prove that $g$ defined by the Feynman-Kac formula belongs to the domain of the *extended* generator $\mathcal{L}$ of the Markov measure $R$ and satisfies the *trajectorial* identity: $[(\mathcal{L}+V)g](t,X_t)=0$, $dtdP$-*a.e.* where the path measure $P$ is defined by $P:=f(X_0)\exp\left(\int_{[0,T]}V(t,X_t)\,dt\right)g(X_T)\ R$, with $f:\mathbb{R}^n\to[0,\infty)$ another nonnegative function. We also show that the forward drift $\mathsf{b}^P$ of $P$ satisfies $\mathsf{b}^P(t,X_t)=[\mathsf{b}+\mathsf{a}\tilde\nabla\log g](t,X_t)$, $dtdP$-*a.e.*, where $\tilde\nabla$ is some extension of the standard derivative.

Our probabilistic approach relies on stochastic derivatives, semimartingales, Girsanov's theorem and the Hamilton-Jacobi-Bellman equation satisfied by $\log g$.

## Contents



*Date*: June, 2022.

2010 *Mathematics Subject Classification.* 35K20, 60H30, 60J60.

*Key words and phrases.* Diffusion processes, Feynman-Kac formula, Hamilton-Jacobi-Bellman equation, relative entropy, extended generator, stochastic derivative, entropic optimal transport, Kato class.

This research is partially granted by Labex MME-DII (ANR-11-LBX-0023).

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

## 1. Introduction

Let us call for practical use in this article, Feynman-Kac equation the linear parabolic equation

$$\begin{cases} (\partial_t + \mathsf{A} + V)g = 0, & 0 \leqslant t < T, \\ g(T, \bullet) = g_T, & t = T, \end{cases} \tag{FK}$$

where the numerical function $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is the unknown, $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is a scalar potential seen as a multiplicative operator, $g_T : \mathbb{R}^n \to [0, \infty)$ is a given *nonnegative* function, and $\mathsf{A}$ is the Markov diffusion generator

$$\mathsf{A} := \mathsf{b}\cdot\nabla + \Delta_{\mathsf{a}}/2$$

whose coefficients are a velocity field $\mathsf{b} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and a diffusion matrix field $\mathsf{a} : [0, T] \times \mathbb{R}^n \to S_+$ taking its values in the set $S_+$ of *nonnegative* symmetric $n \times n$-matrices. We denote for simplicity $\Delta_{\mathsf{a}} := \sum_{1\leqslant i,j\leqslant n} a_{ij}\partial_i\partial_j$ where $\mathsf{a} = (a_{ij})_{1\leqslant i,j\leqslant n}$.
When the fields $\mathsf{a}, \mathsf{b}$ and $V$ are regular enough and not growing too fast, it is a consequence of Itô formula that a solution to this equation is given by the Feynman-Kac formula

$$g(t, x) = E_R\left[\exp\left(\int_{[t,T]} V(s, X_s)\, ds\right)\, g_T(X_T) \,\middle|\, X_t = x\right], \qquad x \in \mathbb{R}^n, 0 \leqslant t \leqslant T, \tag{1.1}$$

where $X$ is the canonical process, $R$ is the law of a Markov process with generator $\mathsf{A}$, and we denote by $E_R$ the expectation with respect to the measure $R$ and $E_R(\bullet \mid \bullet)$ the corresponding conditional expectation. This formula is named after R. Feynman and M. Kac for their contributions [18, 26, 27] in the late 1940's. Defining

$$S^r_t u(x) := E_R\left[\exp\left(\int_{[r,t]} V(s, X_s)\, ds\right)\, u(X_t) \,\middle|\, X_r = x\right], \qquad 0 \leqslant r \leqslant t \leqslant T, \tag{1.2}$$

for any function $u$ on $\mathbb{R}^n$ such that this expression is meaningful, we see that $g_t = S^t_T g_T$, and the collection of linear operators $(S^r_t)_{0\leqslant r\leqslant t\leqslant T}$ is the Feynman-Kac semigroup, where $g_t(x) := g(t, x)$. The stationary version of equation (FK)

$$(\mathsf{b}\cdot\nabla + \Delta_{\mathsf{a}}/2 + V)g = 0,$$

when $\mathsf{b}, \mathsf{a}, V$ and $g$ do not depend on $t$, is the stationary Schrödinger equation.
The logarithmic transformation

$$\psi := \log g \tag{1.3}$$

links (FK) to the Hamilton-Jacobi-Bellman equation

$$\begin{cases} e^{-\psi}(\partial_t + \mathsf{A})e^{\psi} + V = 0, & 0 \leqslant t < T, \\ \psi(T, \bullet) = \psi_T, & t = T, \end{cases} \tag{HJB}$$

(formally, divide (FK) by $g$ and replace $g$ by $e^{\psi}$, provided that $g > 0$). It was a keystone of Schrödinger's original derivation of his eponym equation[1] because it permits to primarily work with some nonlinear Hamilton-Jacobi-Bellman (HJB) equation which is well-suited to carry both features of particle mechanics and wave evolution, and then to transform it into Schrödinger's linear equation. It is also of importance in the theory of controlled Markov processes, see [20, Ch. 6].

Typical results about classical – i.e. $\mathcal{C}^{1,2}$ – solutions of (HJB) require that $\mathsf{a}$ is uniformly positive definite and that $\mathsf{a}, \mathsf{b}, V$ and $g_T$ are $\mathcal{C}^{1,2}_b$, see [29, 16]. We also know that when $\mathsf{a}, \mathsf{b}, V$ and $g_T$ are continuous, but $\mathsf{a}$ might not be uniformly positive definite, and the

[1] Beware, with our notation the role of the wave function $\Psi$ is played by $g$, not $\psi$, see (1.17) below.

solution $g$ of (FK) is also continuous (the Feller property of $\mathsf{A}$ implies this continuity in several cases), then $\psi := \log g$ where $g$ is given by the Feynman-Kac formula (1.1) is the viscosity solution of (HJB), see [20, thm. II.5.1].

On the other hand, Kac proved in [26] (in one dimension) that if $V$ is an upper and locally bounded measurable function and $\mathsf{A} = \partial_x^2$ is the generator of the Brownian motion, then $g$ given by (1.1) solves (FK) in some weak sense. It was discovered later with $\mathsf{A} = \Delta$ the generator of the Brownian motion in $\mathbb{R}^n$ that when $V$ belongs the Kato class (a set of lowly regular measurable functions which might not be locally bounded but with some integrability properties), see Definition 6.22, that the Feynman-Kac operator $S_t$ defined at (1.2) is a continuous operator from $L^p$ to $L^p$ with $1 \leqslant p \leqslant \infty$ and that $g$ given by (1.1) is continuous, see [8, Ch. 3].

The "Feynman-Kac transform" of $R$ which is the path measure defined by

$$P := f_0(X_0) \exp\left(\int_{[0,T]} V(t, X_t)\, dt\right) g_T(X_T)\ R \tag{1.4}$$

where $f_0 : \mathbb{R}^n \to [0, \infty)$ is another nonnegative function, is a generalization of Doob's $h$-transform of $R$ [13, 14], which is recovered by choosing $f_0 = 1, V = 0$ and taking $g_T = h$. When the solution $g$ of (FK) is $\mathcal{C}^{1,2}$, with standard stochastic calculus arguments one proves [41, 20] that $P$ is the law of a Markov diffusion process with the same matrix field $\mathsf{a}$ as $R$, and drift field

$$\mathsf{b}^P = \mathsf{b} + \mathsf{a}\nabla\psi. \tag{1.5}$$

The path measure $P$ is the solution to the Schrödinger problem (1.14) below, a topic also called *entropic optimal transport* which is tightly related to optimal transport, and is currently an active field of research. More will be said about entropic optimal transport in a moment in this introductory section.

**Main results of the article.** In this article, we prove with probabilistic techniques that $g$ given by the Feynman-Kac formula (1.1) belongs to the domain of the *extended* generator of the Markov measure $R$ and satisfies the trajectorial equation (1.8) below, extending equation (FK) in a probabilistic way. The diffusion matrix field $\mathsf{a}$ is supposed to be regular (typically $\mathsf{a} = \sigma\sigma^*$ with $\sigma$ locally Lipschitz) and invertible, but the coefficients $\mathsf{b}, V$ and the datum $g_T$ are *neither assumed to be regular, nor locally bounded.*

The notion of extended generator of a Markov measures is directly connected to the notion of semimartingale which plays a central role in this article. The main hypothesis of this article is that the *relative entropy* of $P$ with respect to the reference path measure $R$ is finite, i.e.

$$H(P|R) := E_P \log\left(\frac{dP}{dR}\right) < \infty. \tag{1.6}$$

We prove in addition that $g$ admits some generalized spatial derivative $\widetilde{\nabla} g$ and extend (1.5): $P$ is the law of a Markov diffusion process with the same matrix field $\mathsf{a}$ as $R$, and drift field

$$\mathsf{b}^P = \mathsf{b} + \mathsf{a}\widetilde{\nabla}\psi, \tag{1.7}$$

with $\widetilde{\nabla}\psi = \widetilde{\nabla}(g)/g$. The rigorous statement of this formula requires some "almost everywhere" cautions, see Theorem 5.9 for the exact result. The main interest of this result is that it holds even when not much is known a priori about the regularity of $g$. For instance, if $V$ is in the Kato class (a natural assumption in theoretical physics), $g$ might be continuous but one does not know that it is differentiable in general.

The rule of the game in this article is to prohibit regularity hypotheses on $V$ and $g_T$ stronger than the finite entropy condition (1.6). This is suggested by our main motivation which is the entropic optimal transport. But it also appears that this finite entropy assumption is very efficient to derive low regularity results.

*First result.* Theorem 5.24 is our first main result. Its approximate statement is as follows. *Suppose that*

*(i)* $\mathsf{a}$ *and* $\mathsf{b}$ *are such that* $R$ *is the "unique" solution to the martingale problem* $\mathrm{MP}(\mathsf{a},\mathsf{b})$ *(see definition (4.3) and assumption (4.7)),*
*(ii)* $\mathsf{a}$ *is locally bounded and invertible,*
*(iii)* $\mathsf{a},\mathsf{b},V$ *and* $g_T$ *are such that* $H(P|R)<\infty$*, where* $P$ *is defined at (1.4).*

*Then,* $g$ *defined by (1.1) belongs to the domain* $\mathrm{dom}\,\mathcal{L}^{R,P}$ *of the extended generator* $\mathcal{L}^{R,P}$ *of* $R$ *localized by* $P$ *(see Definition 3.6) and*

$$[(\mathcal{L}^{R,P}+V)g](t,X_t)=0,\quad dtdP\text{-}a.e. \tag{1.8}$$

*Remarks* 1.9.

(a) The tricky part of this result is: $g\in\mathrm{dom}\,\mathcal{L}^{R,P}$.
(b) The hypothesis: $H(P|R)<\infty$, is an integrability assumption on the data $\mathsf{a},\mathsf{b},g_T$ and $V$ which requires almost no regularity from $V$ and $g_T$.
(c) The extended generator $\mathcal{L}^{R,P}$ is localized by $P$, see Definition 3.6, and (1.8) is only valid $dtdP$-almost everywhere, rather than $dtdR$-*a.e.* a priori. It is partly because of this self-reference to the observed path measure $P$, that it is possible to get rid of some regularity and growth restrictions on $g_T$ and $V$.
(d) We require that the growths of $\mathsf{b}$ and $\mathsf{a}$ are such that the reference path measure $R$ exists and their regularities are sufficient for $R$ to be the "unique" solution to its martingale problem - typically Lipschitz regularity. But some entropic argument allows us to depart from the regularity of $\mathsf{b}$, see Section 6.
(e) On the other hand, the additional hypothesis that $\mathsf{a}$ is invertible is important for our approach to work: It is there to ensure the Brownian martingale representation theorem which implies that the domain of the extended generators of $R$ and $P$ are algebras which are stable by $C^2$ transformations, see Lemma 4.22.

*Second result.* Theorem 5.9 which extends (HJB) and (1.5) is our second main result. Its approximate statement is as follows.
*Under the same assumptions as before, the function* $\psi=\log g$ *where* $g$ *is given by the Feynman-Kac formula (1.1) solves the following extended HJB equation*

$$\big(\mathcal{L}^{R,P}\psi+|\widetilde{\nabla}^{R,P}\psi|_{\mathsf{a}}^2/2+V\big)(t,X_t)=0,\qquad dtdP\text{-}a.e. \tag{1.10}$$

*with* $\psi_T=\log g_T$*,* $P_T$*-a.e.*
*The Feynman-Kac measure* $P$ *solves the martingale problem* $\mathrm{MP}(\mathsf{a},\mathsf{b}^P)$ *where*

$$\mathsf{b}^P=\mathsf{b}+\mathsf{a}\widetilde{\nabla}^{R,P}\psi. \tag{1.11}$$

Martingale problems are defined at Definition 2.6 – we slightly depart from the standard definition, see Remark 2.7-(c). The existence of $\widetilde{\nabla}^{R,P}\psi$ and its definition are stated at Proposition 4.24 which is an extended Itô formula. When $\psi$ is differentiable, we have: $\widetilde{\nabla}^{R,P}\psi=\nabla\psi$.

In fact, our first result (Theorem 5.24) about equation (FK) is a corollary of the above extended HJB equation.

*Third result.* A sufficient condition on $\mathsf{a}, \mathsf{b}, V, f_0$ and $g_T$ is stated at Theorem 6.26 for $H(P|R)$ to be finite. As it is rather technical, we do not propose in this introduction an approximate description of this set of assumptions. However, let us say that it is large enough to include potentials $V$ in the Kato class. Therefore we are in position to apply our previous results about the FK and HJB equations and the FK-transform $P$. In particular, *the representation* (1.5) *of the drift of $P$ which is inaccessible when $V$ is a generic element of the Kato class ($g$ is only known to be continuous in most favorable situations) admits the extension* (1.11).

*Fourth set of results.* Our last results are of different nature than previous ones. They state that *stochastic derivatives and extended generators are essentially the same.* This already well-known assertion (since Nelson's monograph [39] and its use by Föllmer in a series of works [21, 22, 23]) is the content of Propositions 2.18 and 2.21. The proofs of Proposition 2.18, and also Theorem 5.9, deeply rely on the convolution Lemma 2.11. This technical lemma permits to fill some gaps in the already published literature on the subject. We had to put on a solid ground the use of stochastic derivatives to compute extended generators because it is central in the approach of the present paper.
In addition, the convolution Lemma 2.11 is used in an essential manner when deriving time reversal formulas in the recent article [4].

**Some comments and remaining questions.** It is important to stress that this paper primarily states that $g$ satisfies the identity (1.8) and $\psi$ satisfies the identity (1.10). These statements should not be interpreted as $g$ solves equation (1.8) and $\psi$ solves equation (1.10). We only look at $g$ defined by the Feynman-Kac formula, and $\psi := \log g$, and do not study any new notion of solution to the *equations* (FK) and (HJB): the existence of such a solution would automatically be given by the Feynman-Kac formula. On the other hand, the uniqueness problem is not even clear to state: unique in which regularity class? while we do not even know the minimal regularity of $g$ under our general hypotheses.

We present some problems which are not treated in this article, and make a couple of comments.

(i) As already noticed, if the Feynman-Kac formula defines a function $g$ which is continuous, then $\psi$ is the viscosity solution of (HJB). We do not know whether it remains a viscosity solutions when it is discontinuous, under the only hypothesis that $H(P|R)$ is finite.

(ii) Because (1.8) and (1.10) are trajectorial statements, they are more precise than any solution to a PDE. The specificity of a pathwise representation in a probabilistic context is twofold: one can play with *stopping times* or with *couplings*, and sometimes both. Looking at

$$g(t,x) = E_R\left[ \mathbf{1}_{\{t\leqslant\tau\}} \exp\left( \int_{[t,\tau]} V(s,X_s)\,ds \right)\, g_{\text{final}}(X_\tau) \,\middle|\, X_t = x \right],$$

where $\tau$ is a stopping time, is tempting. Similarly, one may ask whether couplings are of some use when looking at comparison principles or functional inequalities.

(iii) On the other hand, this advantage is balanced by some drawbacks. In particular, the powerful stability properties of viscosity solutions along convergence schemes might not be recovered via (1.10) . Typically, in case of a vanishing viscosity convergence, because the supports of diffusion path measures with different diffusion matrices are disjoint, a trajectorial solution does not permit us to use pointwise convergence.

(iv) Statements (1.8) and (1.10) rely on the existence of some Markov path measure $R$. This is restrictive in comparison to the general definition of viscosity solution which only requires the existence of some semigroup obeying the maximum principle [20, Ch. 2].

**Standard approaches for computing the generator of $P$.** There are three main ways to look at the dynamics of $P$. They rely on (i) Markov semigroups, (ii) Dirichlet forms and (iii) semimartingales.

*Markov semigroups.* Let $P \in \mathrm{P}(\Omega)$ be a Markov measure (see Definition 2.1 below) and $(T^P_{s,t})_{0\leqslant s\leqslant t\leqslant T}$ be its semigroup on some Banach function space $(U, \|\cdot\|_U)$. For instance $U$ may be the space of all bounded Borel measurable functions equipped with the topology of uniform convergence. Its infinitesimal generator is $A^P = (A^P_t)_{t\in[0,T]}$ with

$$A^P_t u(x) := \|\cdot\|_U\text{-}\lim_{h\to 0^+} \frac{1}{h} E_P[u(X_{t+h}) - u(X_t) \mid X_t = x], \quad u \in \mathrm{dom}\, A^P \tag{1.12}$$

where the domain $\mathrm{dom}\, A^P$ of $A^P$ is precisely the set of all functions $u \in U$ such that the above strong limit exists for all $t \in [0,1)$ and $x \in \mathbb{R}^n$. One can prove rather easily (see [41, Ch. VIII,§3] for instance, in the diffusion case) that when $V$ is zero and $g$ is positive and regular enough, the generator $A^P$ of the Markov semigroup associated with $P$ is given for regular enough functions $u$ on $\mathbb{R}^n$, by

$$A^P_t u(x) = A^R u(x) + \frac{\Gamma(g,u)}{g}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^n \tag{1.13}$$

where $\Gamma$ is the carré du champ operator, defined for all functions $u, v$ such that $u, v$ and the product $uv$ belong to the domain $\mathrm{dom}\, A^R$ of $A^R$, by

$$\Gamma(u,v) = A^R(uv) - uA^R v - vA^R u.$$

For Eq. (1.13) to be meaningful, it is necessary that for all $t \in [0,T]$, $g_t$ and the product $g_t u$ belong to $\mathrm{dom}\, A^R$. But with a non-regular potential $V$, $g$ might be non-regular as well. There is no reason why $g_t$ and $g_t u$ are in $\mathrm{dom}\, A^R$ in general. Clearly, one must drop the semigroup approach and work with Dirichlet forms or semimartingales.

*Dirichlet forms.* The Dirichlet form theory is natural for constructing irregular processes and has been employed in similar contexts, see [1]. It is made-to-measure for reversible processes. Though there exists a theory for non-symmetric Dirichlet forms [36, 40, 44], it is not fully efficient for our purpose.

*Semimartingales.* Working with semimartingales means that instead of the infinitesimal semigroup generators $A^R$ and $A^P$, we consider *extended generators* in the sense of the Strasbourg school [12], see Definition 2.3 below. This natural idea has already been implemented by P.-A. Meyer and W.A. Zheng [37, 38] in the context of stochastic mechanics and also by P. Cattiaux and the author in [5, 6] for solving related entropy minimization problems. But we still had to face the remaining problem of giving some sense to $\Gamma(g_t, u)$ in (1.13). Consequently, restrictive assumptions were imposed: reversibility in [38], and in [6]: the standard hypothesis that the domains of the extended generators of $R$ and $P$ contain "large" sub-algebras. In practice this requirement is uneasy to satisfy, except for standard regular processes. It is all right for the reference measure $R$, but typically when $V$ blows up, $P$ is singular and this large sub-algebra assumption does not seem to be accessible with standard arguments. Moreover, there is no known criterion for this property to be inherited from $R$ by $P$ when $P \ll R$. In contrast, extended generators and considerations about the carré du champ allow us to overcome this obstacle in the present article, see Lemma 4.22 which is based on Lemma A.2.

**Some motivations.** Let us present our motivations for proving the main results of this paper. Although very much related to each other, we have two distinct problems in mind: (i) the first one is coming from theoretical physics in connection with the interpretation of dissipative evolutions and quantum mechanics, (ii) the second one is related to the entropic optimal transport problem, also called the Schrödinger problem.

(i) *A remarkable analogy between dissipative and quantum evolutions.* In the early thirties, Schrödinger [42, 43] addressed the entropy minimization problem

$$\inf\{H(Q|R^V); Q \text{ path measure such that } Q_0 = \mu_0, Q_T = \mu_T\} \tag{1.14}$$

where the couple $(Q_0, Q_T)$ of initial and final marginals of $Q$ is prescribed to be equal to some fixed $(\mu_0, \mu_T)$, and the reference measure is $R^V := \exp\Big(\int_{[0,T]} V(t, X_t)\, dt\Big)\, R$. Its solution $P$ is called a *Schrödinger bridge* whose time-marginal flow $(P_t)_{0\leqslant t\leqslant T}$ is called an *entropic interpolation* between $\mu_0$ and $\mu_T$.

In the specific case where $R$ is the law of a reversible Brownian and $V = 0$, Schrödinger essentially showed that formula (1.4) gives the general shape of the Schrödinger bridge and that the entropic interpolation admits the Radon-Nikodym derivative

$$\frac{dP_t}{dx} = f_t(x) g_t(x) \tag{1.15}$$

where $g$ solves (FK) and $f_t(x) = E_R\Big[f_0(X_0)\exp\Big(\int_{[0,t]} V(s, X_s)\, ds\Big) \Bigm| X_t = x\Big]$ solves a forward-time analogue of (FK)

$$\begin{cases} (-\partial_t + \widetilde{\mathsf{A}} + V) f = 0, & 0 < t \leqslant T, \\ f(0, \bullet) = f_0, & t = 0, \end{cases} \tag{1.16}$$

which again is a Feynman-Kac equation with $\widetilde{\mathsf{A}} = \tilde{\mathsf{b}}\cdot\nabla + \Delta_{\mathsf{a}}/2$ where the vector field $\tilde{\mathsf{b}}$ is the drift of the time reversal of $R$.

He noticed a striking analogy (in his own words) between the solution of the thermodynamical problem (1.14) described by the product formula (1.15) and Born's formula

$$\frac{d\mu_t}{dx} = |\Psi_t|^2(x) = \Psi_t(x)\overline{\Psi}_t(x)$$

where $\mu_t$ is the probability of presence of some quantum system at time $t$ and $\Psi$ is the wave function describing its evolution, solution of his eponym equation:

$$i\hbar\partial_t\Psi = \Big(-\frac{\hbar^2}{2m}\Delta + V\Big)\Psi \quad\iff\quad -i\hbar\partial_t\overline{\Psi} = \Big(-\frac{\hbar^2}{2m}\Delta + V\Big)\overline{\Psi} \tag{1.17}$$

and $\overline{\Psi}$ is the complex conjugate of $\Psi$. To see this analogy, remark that with $V$ instead of $-V$, taking $\tau = it$, we have $-i\partial_t = \partial_\tau$. It follows that equation (1.16) is an analogue of the Schrödinger equation for $\Psi$, while the time-reversed equation (FK) is an analogue of the Schrödinger equation for $\overline{\Psi}$.

Not only does the Feynman-Kac transform $P$ defined at (1.4) provide us with interesting analogies with quantum mechanics, but its family of bridges $P(\bullet \mid X_r = x, X_t = y)$ is the classical thermodynamics analogue of the propagator appearing in Feynman's approach to quantum mechanics [19]: the ill-defined Feynman integral is replaced by a stochastic integral.

A couple of potential applications of the present article are

- the EQM (Euclidean quantum mechanics) program launched a long time ago by Zambrini [49, 7, 50], and
- the closely related theory of Bohmian mechanics [3, 15].

EQM describes the thermal evolution of particle systems, tracking as much as possible the analogies with quantum mechanics, while Bohmian mechanics describes the *pathwise* evolution of quantum systems with the same prediction as standard quantum theory. In these settings, $V$ is a typical scalar potential generating a force field such as the Coulomb potential which is *not locally bounded.* A natural set of such potentials is the Kato class. This is the reason why we prove at Section 6 that scalar potentials in the Kato class satisfy our main finite entropy hypothesis (1.6) under standard additional assumptions on $R$.

(ii) *Entropic optimal transport.* Entropic optimal transport (EOT) is another name for the Schrödinger problem (1.14) emphasizing its connection with (standard) optimal transport (OT). It is an active research topic.

EOT is often considered taking $R$ to be the law of a diffusion process, and seen as an entropic regularization of quadratic OT. In this context, it is known that for "generic" marginal constraints $\mu_0$ and $\mu_T$, the solution of (1.14) is (1.4), i.e. the Schrödinger bridge is the FK-transform $P = f_0(X_0)g_T(X_T)\, R^V$ for some couple of functions $(f_0, g_T)$ solving the *Schrödinger system*

$$\begin{cases} f_0 g_0 &= d\mu_0/dR_0 \\ f_T g_T &= d\mu_T/dR_T \end{cases} \tag{1.18}$$

where $g_0$ is defined by the Feynman-Kac formula (1.1) and $f_T$ is defined in a time-symmetric way, that is for all $x \in \mathbb{R}^n$ and $0 \leqslant t \leqslant T$,

$$g_0(x) := E_R\left[\exp\left(\int_{[0,T]} V(t, X_t)\, dt\right)\ g_T(X_T) \,\middle|\, X_0 = x\right],$$

$$f_T(x) := E_R\left[f_0(X_0)\ \exp\left(\int_{[0,T]} V(t, X_t)\, dt\right) \,\middle|\, X_0 = x\right].$$

The Schrödinger system (1.18) follows from: $dP_t/dR_t = f_t g_t$, $0 \leqslant t \leqslant T$, see Proposition 5.6 below, and extends (1.15). The logarithms $\varphi_0 := \log f_0$ and $\psi_T := \log g_T$ are called the *Schrödinger potentials*, in analogy with the Kantorovich potentials of OT.

The main application to EOT of the results of the present article is the identity (1.11): $\mathsf{b}^P = \mathsf{b} + \mathsf{a}\widetilde{\nabla}^{R,P}\psi$, which characterizes the evolution of the Schrödinger bridge $P$. In this formula $\psi$ satisfies the extended HJB identity (1.10) with the boundary condition $\psi_T$ at time $t = T$ which requires to solve the Schrödinger system (1.18).

This improves already known results in the literature, in particular because $\mathsf{b}$ is allowed to be a singular drift. Indeed, our approach does not require that $\mathsf{a}$, $\mathsf{b}$ and $V$ meet the standard hypotheses of regularity theorems for the Feynman-Kac equation (FK) which are obtained using PDE techniques [16] or stochastic flows [31].

**Future works.** It is the purpose of EQM to transpose well-established results in stochastic analysis of variational processes to standard quantum mechanics, and the other way round. From the start, the EQM program sees the Feynman-Kac equation (FK) as the stochastic deformation of Schrödinger's equation. The associated entropic interpolation: the marginal flow $(P_t)$ of the FK path measure $P$, is governed by some Newton equation in the Otto-Wasserstein space. This was proved by Conforti in [9], assuming that $f_t$ and

$g_t$ are regular enough and not growing too fast. Our aim in a work in progress is to relax these assumptions and to extend this result under the finite entropy condition (1.6).

On the other hand, it is worth expressing Bohmian mechanics as the evolution of a fluid also living in the Otto-Wasserstein space. This is suggested by von Renesse's work [48] where Schrödinger's equation is seen at a heuristic level as a Newton equation in the Otto-Wasserstein space, again. In this perspective, the Newton equation associated to (FK) is the classical rigorous analogue of Newton's equation in Bohm's view of quantum dynamics. This is also a work in progress.

The time-symmetry of formula (1.4) suggests that the forward-in-time equation (1.16) is as important as its backward-in-time analogue (FK). This is crucial in many aspects of EOT and EQM. And indeed this time-symmetry is an important ingredient of these works in progress. It relies on the recent article [4] about time reversal of diffusion processes under a finite entropy condition.

**Further developments.** Let us raise some remaining problems.

- *Jumps*. Replacing the diffusion measure $R$ by the law of a time-continuous Markov process with jumps would lead us to a similar trajectorial identity associated a nonlocal PDE: $(\partial_t + \mathsf{A} + V)g = 0$, with $\mathsf{A}$ a Markov generator expressed with a jump kernel, and to the associated nonlocal HJB equation. However, the conceptual limitation one might face is the martingale representation theorem (MRT), which is crucial to prove the extended Itô formula (Lemma 4.22, Proposition 4.24). Indeed, to our knowledge MRT results are only available for specific classes of jump processes, see [51] for instance.
- *Manifold*. What happens when replacing $\mathbb{R}^n$ by a Riemannian manifold? Unlike the jump setting, there exists a well established Brownian MRT on a complete connected Riemannian manifold, see [17, 24]. It remains to establish a Girsanov theory under a finite entropy condition in the spirit of [32], but this must be straightforward.
- *Viscosity solutions*. The links between the trajectorial representation of the solution to the HJB equation encountered at Theorem 5.9, see (1.10), and its viscosity solution remain to be clarified.
- *Small noise limit*. Suppose that the Feynman-Kac path measure $P^\epsilon$, built on some $f_0^\epsilon, g_T^\epsilon, \mathsf{b}^\epsilon$, $V^\epsilon = V/\epsilon$ and $\mathsf{a}^\epsilon := \epsilon\mathsf{a}$, satisfies the hypotheses of Theorem 5.9 and converges weakly to some $P^0$ in $\mathrm{P}(\Omega)$ as $\epsilon$ tends to zero. This implies that the entropic interpolations converge to the displacement interpolations because taking the marginal flow is a continuous mapping. But one can ask if there is some *almost sure convergence* of the sample paths. This is an open problem whose solution should rely on large deviation results to obtain exponentially small probability of deviations, ensuring some argument based on Borel-Cantelli lemma. In this respect, see the recent article [2] which establishes a large deviation principle in the setting of the static Schrödinger problem. Its dynamical counterpart remains to be proved.

**Literature.** Schrödinger [42, 43] only considered the case where $R$ is the law of a reversible Brownian motion and $V = 0$. His arguments for deriving (1.15) are profound but not rigorous (at this time the axioms of probability theory were unsettled and the Wiener process was unknown), but the impressive strength of the physicist's arguments is sufficient to convince the reader. The extension to a non-zero potential $V$ under the hypothesis that the solutions $f$ and $g$ of the Feynman-Kac equations in both directions of time are regular enough was performed by Zambrini [49]. The actual writing of the Schrödinger problem (1.14) in terms of entropy, as well as its formulation as a large deviation problem for the empirical measure of a system of particles is due to Föllmer [23]. More about this active field of research, in particular its tight connection with optimal transport, can be found in the survey paper [34].

**Outline of the present approach.** A key feature of our approach is the logarithmic transformation (1.3) because it enables us to take advantage of a connection between (HJB) and Girsanov's theory. More precisely, elementary stochastic calculus gives the following expression of the Radon-Nikodym derivative of $P$ defined at (1.4) with respect to $R$:

$$\begin{aligned} \frac{dP}{dR} &:= f_0(X_0) \exp\left( \int_{[0,T]} V(t, X_t)\, dt \right) g_T(X_T) \\ &= f_0(X_0) g_0(X_0) \, \exp\left( \psi_T(X_T) - \psi_0(X_0) - \int_{[0,T]} [e^{-\psi}(\partial_t + \mathsf{A}) e^{\psi}](t, X_t)\, dt \right), \end{aligned} \tag{1.19}$$

where

$$\psi(t,x) = \log g(t,x) = \log\, E_R \left[ \exp\left( \int_{[t,T]} V(s, X_s)\, ds \right) \, g_T(X_T) \,\middle|\, X_t = x \right], \tag{1.20}$$

provided that $\mathsf{a}, \mathsf{b}, g_T$ and $V$ satisfy some growth conditions and that $\psi$ *is regular enough* to apply Itô formula and give sense to $(\partial_t + \mathsf{A})e^{\psi}$. At first sight the identity (1.19) is reminiscent to (HJB), and indeed it establishes a strong link between (FK) and (HJB). The main problem we have to face is to develop this simple idea, when one does not know much about the a priori regularity of $\psi$. In particular, $(\partial_t + \mathsf{A})e^{\psi}$ is a priori undefined.

A good thing to do is to compare the above expression of $dP/dR$ with the one obtained by Girsanov's theory. Indeed, this provides us with valuable informations on $\psi$, and therefore on the solution of (FK). This is possible at the price of working with *extended generators* instead of standard generators of Markov semi-groups, allowing us to extend Itô formula to the domain of the extended generator under the important requirement that (1.6): $H(P|R) < \infty$, holds.

This entropy estimate is in fact a finite energy condition which carries some control of the generalized derivative $\widetilde{\nabla}\psi$ of $\psi$ which takes part of an extended Itô formula (this is the place where it is needed that $\mathsf{a}$ is invertible to make sure that the martingale representation theorem is valid). On the other hand, Girsanov's theory tells us that $\widetilde{\nabla}\psi$ is precisely the additional drift which "translates" $R$ to $P$, see (1.7). For a better understanding of the key point of the present strategy which takes advantage of Girsanov's formula to allow us to get rid of an a priori regularity of the solution $g$, see Remark 5.25.

**Outline of the paper.** The specific features of the present extension of the Feynman-Kac equation (FK) are introduced at Section 4 which contains both standard results about finite entropy diffusion path measures, and a bit of new material designed for our purpose (especially the extended Itô formula at Proposition 4.24). This material is based on extended generators, a standard notion which is revisited at Section 2, using Nelson stochastic derivatives, very much in the spirit of the seminal paper [22] by H. Föllmer (see also [23] and more recently [4]), and generalized at Section 3. Using the preliminary material established in the first four sections, Section 5 is dedicated to the proofs of our main results. At Section 6, sufficient conditions are established on the coefficients of equation (FK) for $H(P|R)$ to be finite.

## 2. Stochastic derivatives. Main results

After reviewing basic notions of semimartingale theory in a general setting (càdlàg paths in a Polish space), we prove at Propositions 2.18 and 2.21 that under some integrability condition, the extended generator and the stochastic derivative of a Markov measure coincide. This is Nelson's way of looking at diffusion generators [39].

The aim of the present section is to provide rigorous proofs of these general results. To our knowledge, although these notions, in connection with the notion of martingale problem, were introduced in the late sixties [39, 30, 45], such detailed proofs do not appear in the literature. However, the guideline they provide and the recognition of the relevance of these notions for our purpose are fully credited to Föllmer [22]. Our main technical tool is the convolution Lemma 2.11.

**Notation and setting.** The set of all probability measures on a measurable set $A$ is denoted by $\mathrm{P}(A)$ and the set of all nonnegative $\sigma$-finite measures on $A$ is $\mathrm{M}(A)$. The push-forward of a measure $\mathsf{q} \in \mathrm{M}(A)$ by the measurable map $f : A \to B$ is $f_\# \mathsf{q}(\bullet) := \mathsf{q}(f \in \bullet) \in \mathrm{M}(B)$.

The state space $\mathcal{X}$ is assumed to be Polish and is equipped with its Borel $\sigma$-field. The path space is the set

$$\Omega = D([0,T], \mathcal{X})$$

of all càdlàg $\mathcal{X}$-valued trajectories $\omega = (\omega_t)_{t\in[0,T]} \in \Omega$. It is equipped with the canonical $\sigma$-field: $\sigma(X_t; t \in [0,T])$ which is generated by the canonical process $X = (X_t)_{t\in[0,T]}$ defined for each $t \in [0,T]$ and $\omega \in \Omega$ by

$$X_t(\omega) = \omega_t \in \mathcal{X}.$$

We denote $\overline{\Omega} := [0,T] \times \Omega$, $\overline{\mathcal{X}} := [0,T] \times \mathcal{X}$, and for any $t \in [0,T]$,

$$\overline{X}_t := (t, X_t) \in \overline{\mathcal{X}},$$

and any function $u : [0,T] \times \mathcal{X} \to \mathbb{R}$,

$$u(\overline{X}) : (t,\omega) \in \overline{\Omega} \mapsto u(t,\omega_t) \in \mathbb{R}.$$

We call any positive measure $Q \in \mathrm{M}(\Omega)$ on $\Omega$ a *path measure*. For any $\mathcal{T} \subset [0,T]$, we denote $X_{\mathcal{T}} = (X_t)_{t\in\mathcal{T}}$ and the push-forward measure $Q_{\mathcal{T}} = (X_{\mathcal{T}})_\# Q$. In particular, for any $0 \leqslant r \leqslant s \leqslant T$, $X_{[r,s]} = (X_t)_{r\leqslant t\leqslant s}$, $Q_{[r,s]} = (X_{[r,s]})_\# Q$, and $Q_t = (X_t)_\# Q \in \mathrm{M}(\mathcal{X})$ denotes the law of the position $X_t$ at time $t$. If $Q \in \mathrm{P}(\Omega)$ is a probability measure, then $Q_t \in \mathrm{P}(\mathcal{X})$.

For any $0 \leqslant t \leqslant T$, $\overline{Q} := \mathrm{Leb}_{[0,T]} \otimes Q$ is the product measure

$$\overline{Q}(dtd\omega) := dtQ(d\omega), \qquad dtd\omega \subset \overline{\Omega}$$

and we denote

$$\bar{\mathsf{q}}(dtdx) := dtQ_t(dx), \qquad dtdx \subset \overline{\mathcal{X}}.$$

For any $Q \in \mathrm{P}(\Omega)$, we denote

$$[Q] := (Q_t; 0 \leqslant t \leqslant T) \in \mathrm{P}(\mathcal{X})^{[0,T]}$$

its time marginal flow.

For any random time $\tau$, we denote $Y_t^\tau := Y_{t\wedge\tau}$ and $\overline{X}_t^\tau := (t \wedge \tau, X_{t\wedge\tau})$.

**Filtration.** *The forward filtration associated with $Q \in \mathrm{M}(\Omega)$ is the $Q$-completion of the canonical filtration.* It fulfills the so-called "usual hypotheses": it is right continuous and contains the $Q$-null sets. Under this hypothesis it is known that any $Q$-martingale admits a càdlàg version, see for instance [35]. We shall choose this version in all cases.

**Basic notions.** We recall the definitions of Markov measure, extended generator and stochastic derivative.

**Definition 2.1** (Markov measure)**.** *A path measure $Q$ such that $Q_t$ is $\sigma$-finite for all $t$ is called a conditionable path measure. A path measure $Q \in \mathrm{M}(\Omega)$ is said to be Markov if it is conditionable and for any $0 \leqslant t \leqslant T$, $Q(X_{[t,T]} \in \bullet \mid X_{[0,t]}) = Q(X_{[t,T]} \in \bullet \mid X_t)$.*

The reason for requiring $Q$ to be conditionable is that it allows for defining the conditional expectations $E_Q(\bullet \mid X_{\mathcal{T}})$ for any $\mathcal{T} \subset [0,T]$ even in the case where $Q$ is an unbounded measure, see [33, Def. 1.10]. Remark that the definition of a $Q$-martingale remains unchanged when $Q$ is unbounded because wether $Q$ is bounded or not any conditioning $Q(\bullet \mid \mathcal{C})$ of $Q$ is a *probability* measure.

Let $Q$ be a path measure. Recall that a process $M$ is called a *local $Q$-martingale* if there exists a sequence $(\tau_k)_{k\geqslant 1}$ of $[0,T] \cup \{\infty\}$-valued stopping times such that $\lim_{k\to\infty} \tau_k = \infty$, $Q$-*a.e.* and for each $k \geqslant 1$, the stopped process $M^{\tau_k}$ is a uniformly integrable $Q$-martingale. A process $Y$ is called a *special $Q$-semimartingale* if $Y = B + M$, $Q$-*a.e.* where $B$ is a predictable bounded variation process and $M$ is a local $Q$-martingale.

**Definition 2.2** (Nice semimartingale)**.** *A process $Y$ is called a nice*[2] *$Q$-semimartingale if $Y = B + M$ where $M$ is a local $Q$-martingale and the bounded variation process $B$ has* absolutely continuous *sample paths $Q$-a.e.*

The notion of extended generator was introduced by H. Kunita [30] and extensively used by P. A. Meyer and his collaborators, see [12]. Here is a variant of his definition.

**Definition 2.3** (Extended generator of a path measure)**.** *Let $Q \in \mathrm{M}(\Omega)$ be a conditionable path measure. A measurable function $u$ on $\overline{\mathcal{X}}$ is said to be in the domain of the extended generator of $Q$ if there exists an adapted process $\big(v(t, X_{[0,t]});\ 0 \leqslant t \leqslant T\big)$ such that $\int_{[0,T]} |v(t, X_{[0,t]})|\, dt < \infty$, $Q$-a.e. and the process*

$$M_t^u := u(\overline{X}_t) - u(\overline{X}_0) - \int_{[0,t]} v(s, X_{[0,s]})\, ds, \quad 0 \leqslant t \leqslant T,$$

*is a local $Q$-martingale. We denote*

$$\mathcal{L}^Q u(t,\omega) := v(t, \omega_{[0,t]})$$

*and call $\mathcal{L}^Q$ the extended generator of $Q$. The domain of the extended generator of $Q$ is denoted by* $\mathrm{dom}\,\mathcal{L}^Q$.

*Remark* 2.4 (Special case where $Q$ is Markov). It is proved at Corollary 3.16 that when $Q$ is Markov, $\mathcal{L}^Q u$ only depends on the current position: $\mathcal{L}^Q u(t,\omega) = \mathcal{L}^Q u(t, \omega_t)$. It is also shown at Corollary 2.23 that under some hypotheses

$$\mathcal{L}^Q = \partial_t + \mathcal{G}_t,$$

if $(\mathcal{G}_t)_{0\leqslant t\leqslant T}$ is the generator of the semigroup associated to the Markov measure $Q$.

We go on with technical considerations.

*Remarks* 2.5.

(a) For any measurable function $u$ on $\overline{\mathcal{X}}$ in $\mathrm{dom}\,\mathcal{L}^Q$, the process $u(\overline{X})$ is a nice $Q$-semimartingale.
(b) The adapted process $t \mapsto \int_{[0,t]} v(s, X_{[0,s]})\, ds$ is predictable since it is continuous.

---

[2] This is a "local" definition in the sense that this notion probably appears somewhere else with another name.

(c) $M^u$ admits a càdlàg $Q$-version as a local $Q$-martingale (we always choose this regular version).
(d) The notation $v = \mathcal{L}u$ almost rightly suggests that $v$ is a function of $u$. Indeed, when $u$ is in $\operatorname{dom} \mathcal{L}^Q$, the Doob-Meyer decomposition of the special semimartingale $u(\overline{X}_t)$ into its predictable bounded variation part $\int v_s\, ds$ and its local martingale part is unique. But one can modify $v = \mathcal{L}^Q u$ on a zero-potential set without breaking the martingale property. As a consequence, $u \mapsto \mathcal{L}^Q u$ is a single-valued linear operator once $\mathcal{L}^Q u$ is identified with a class of functions modulo zero-potential sets. In the present context, a measurable set $A \subset [0,T] \times \mathcal{X}$ is a zero-potential set if $\int_A dQ_t dt = 0$.
(e) This definition of the extended generator is suggested in the comment XV-(21) of [12]. It slightly differs from the definition XV-(20) of [12] which is expressed in terms of resolvents.

Extended generators are connected with martingale problems which were introduced by Stroock and Varadhan [45, 46, 47].

**Definition 2.6** (Martingale problem)**.** *Let $\mathcal{C}$ be a class of measurable real functions $u$ on $\overline{\mathcal{X}}$ and for each $u \in \mathcal{C}$, let $\mathcal{L}u : \overline{\Omega} \to \mathbb{R}$ be some adapted process. Take also a positive $\sigma$-finite measure $\mu_0 \in \mathrm{M}(\mathcal{X})$.*
*One says that the conditionable path measure $Q \in \mathrm{M}(\Omega)$ is a solution to the martingale problem*

$$\mathrm{MP}(\mathcal{C}, \mathcal{L}; \mu_0)$$

*if $Q_0 = \mu_0 \in \mathrm{M}(\mathcal{X})$ and for all $u \in \mathcal{C}$, $Q(\int_{[0,T]} |\mathcal{L}u(t, \omega_{[0,t]})|\, dt = \infty) = 0$ and the process $M^u_t := u(\overline{X}_t) - u(\overline{X}_0) - \int_{[0,t]} \mathcal{L}u(s, X_{[0,s]})\, ds$ is a local $Q$-martingale.*

*Remarks* 2.7.
(a) As in Definition 2.3, the local martingale $M^u$ admits a càdlàg $Q$-version.
(b) Playing with the definitions, it is clear that any path measure $Q \in \mathrm{M}(\Omega)$ is a solution to $\mathrm{MP}(\mathcal{C}, \mathcal{L}^Q; Q_0)$ where $\mathcal{L}^Q$ is the extended generator of $Q$ and $\mathcal{C}$ is any nonempty subset of $\operatorname{dom} \mathcal{L}^Q$.
(c) In any standard definition of a martingale problem, it is assumed that for any $u \in \mathcal{C}$ and *all* $\omega \in \Omega$, we have: $\int_{[0,T]} |\mathcal{L}u(t, \omega_{[0,t]})|\, dt < \infty$ (and not only $Q$-*a.e.* as above). This will not be convenient for our purpose because when looking at $Q \in \mathrm{P}(\Omega)$ such that the relative entropy $H(Q|R)$ with respect to some reference path measure $R$ is finite, the extended generator $\mathcal{L}^Q$ of $Q$ is only defined $\overline{Q}$-*a.e.*, see (4.10) below for instance.

Our aim is to show at Proposition 2.18 that the extended generator can be computed by means of a stochastic derivative. Nelson's definition [39] of the stochastic derivative is the following.

**Definition 2.8** (Stochastic derivative)**.** *Let $Q \in \mathrm{M}(\Omega)$ be a conditionable path measure and $u$ be a measurable real function on $\overline{\mathcal{X}}$ such that $E_Q|u(\overline{X}_s)| < \infty$ for all $0 \leqslant s \leqslant T$.*

*(1) We say that $u$ admits a stochastic derivative under $Q$ at time $t \in [0,T)$ if the following limit*

$$L^Q u(t, X_{[0,t]}) := \lim_{h \to 0^+} E_Q \left( \frac{1}{h} [u(\overline{X}_{t+h}) - u(\overline{X}_t)] \,\middle|\, X_{[0,t]} \right) \tag{2.9}$$

*exists in $L^1(Q)$.*
*In this case, $L^Q u(t, \bullet)$ is called the stochastic derivative of $u$ at time $t$.*

*(2) If $u$ admits a stochastic derivative for almost all $t$, we say that $u$ belongs to the domain* $\operatorname{dom} L^Q$ *of the stochastic derivative $L^Q$ of $Q$.*

*Remark* 2.10 (Special case where $Q$ is Markov). If $Q \in \mathrm{M}(\Omega)$ is Markov, it is immediate to see that for any function $u \in \operatorname{dom} L^Q$,

$$L^Q u(t, X_{[0,t]}) = L^Q u(t, X_t), \quad Q\text{-}a.e., \quad \text{for all } t,$$

with an obvious abuse of notation. The right-hand side of this identity defines a function on $[0,T] \times \mathcal{X}$ because for every $t$, $L^Q u(t, X_t)$ is a conditional expectation (with respect to $X_t$) and on a Polish space any conditional expectation admits a regular version. In the present situation, the function $(t,x) \mapsto \mathcal{L}^Q u(t,x)$ is defined $\mathsf{q}_t$-*a.e.* for every $t$.
While for all $t$, $\mathcal{L}^Q u(t, \bullet)$ is measurable, it does not follow from Definition 2.8 that $L^Q u$ is jointly measurable on $[0,T] \times \mathcal{X}$. However, when $u$ fulfils the hypotheses of Proposition 2.18 below, that is: $u$ is in $\operatorname{dom} \mathcal{L}^Q$ and satisfies $E_Q \int_{[0,T]} \left| \mathcal{L}^Q u(t, X_{[0,t]}) \right|^p dt < \infty$ for some $p \geqslant 1$, if follows from this proposition that $L^Q u = \mathcal{L}^Q u$, implying that $L^Q u$ is jointly measurable.

**A convolution lemma.** Next technical result will be used at several places in the rest of the article.

**Lemma 2.11.** *For all $h > 0$, let $k^h$ be a measurable nonnegative convolution kernel such that* $\operatorname{supp} k^h \subset [-h, h]$ *and $\int_{\mathbb{R}} k^h(s)\, ds = 1$. Let $Q$ be a $\sigma$-finite positive measure on $\Omega$ and $v$ be a function in $L^p(\overline{Q})$ with $1 \leqslant p < \infty$.*
*Define for all $h > 0, t \in [0,T]$ and $\omega \in \Omega$, $k^h * v(t, \omega) := \int_{[0,T]} k^h(t-s) v_s(\omega)\, ds$.*
*Then, $k^h * v$ is in $L^p(\overline{Q})$ and $\lim_{h \to 0^+} k^h * v = v$ in $L^p(\overline{Q})$.*

We see that $k^h(s)\, ds$ is a probability measure on $\mathbb{R}$ which converges weakly to the Dirac measure $\delta_0$ as $h$ tends down to zero.
We shall use this lemma with $p = 1$ or 2, and with $k^h = \frac{1}{h} \mathbf{1}_{[-h,0]}$ or $\frac{1}{h} \mathbf{1}_{[0,h]}$.

*Proof.* In this lemma, we endow $\Omega$ with the Skorokhod topology which turns it into a Polish space and has the interesting property that its Borel $\sigma$-field matches with the canonical $\sigma$-field.

We start the proof by showing that $k^h * v \in L^p(\overline{Q})$ and more precisely

$$\|k^h * v\|_{L^p(\overline{Q})} \leqslant \|v\|_{L^p(\overline{Q})} < \infty. \tag{2.12}$$

Since $v \in L^p(\overline{Q})$, for $Q$-almost all $\omega$, $v(\bullet, \omega) \in L^p([0,T])$. By Jensen's inequality applied with the probability measure $k^h(t)dt$, and the standard $L^1$ estimate of convolution

$$\begin{aligned}\int_{[0,T]} |k^h * v(t,\omega)|^p\, dt &\leqslant \int_{[0,T]} dt \int_{[0,T]} |v(s,\omega)|^p k^h(t-s)\, ds = \|k^h * |v(\bullet,\omega)|^p\|_1 \\ &\leqslant \|k^h\|_1 \| |v(\bullet,\omega)|^p\|_1 = \int_{[0,T]} |v(t,\omega)|^p\, dt.\end{aligned}$$

Letting $h \to 0^+$,

$$\|k^h * v(\bullet, \omega)\|^p_{L^p([0,T])} \leqslant \|v(\bullet,\omega)\|^p_{L^p([0,T])}. \tag{2.13}$$

Integrating with respect to $Q(d\omega)$ leads to (2.12).

Now, we prove the convergence. We first show that the proof can be reduced to the case where $Q$ is a bounded measure. As $Q$ is $\sigma$-finite, there is an increasing sequence $(\Omega_m)$

of measurable subsets of $\Omega$ such that $\cup_m \Omega_m = \Omega$ and $Q(\Omega_m) < \infty$ for all $m$. With (2.12) for $\mathbf{1}_{\overline{\Omega}\setminus\overline{\Omega}_m}\, \overline{Q} = \overline{\mathbf{1}_{\Omega\setminus\Omega_m}\, Q}$ instead of $\overline{Q}$ and the dominated convergence theorem, we see that

$$\begin{aligned} 0 \leqslant \|\mathbf{1}_{\overline{\Omega}\setminus\overline{\Omega}_m} \times (k^h * v)\|_{L^p(\overline{Q})} &= \|k^h * v\|_{L^p(\mathbf{1}_{\overline{\Omega}\setminus\overline{\Omega}_m}\, \overline{Q})} \\ &\leqslant \|v\|_{L^p(\mathbf{1}_{\overline{\Omega}\setminus\overline{\Omega}_m}\, \overline{Q})} = \|\mathbf{1}_{\overline{\Omega}\setminus\overline{\Omega}_m} v\|_{L^p(\overline{Q})} \underset{m\to\infty}{\longrightarrow} 0 \end{aligned} \tag{2.14}$$

where $\overline{\Omega}_m := [0,T] \times \Omega_m$. Considering $\mathbf{1}_{\Omega_m} Q$ for arbitrarily large $m$ instead of $Q$, one can assume without loss of generality that $Q$ is bounded. By the same token, since $Q$ is a bounded nonnegative measure on a Polish space, it is tight: there exists a compact subset containing an arbitrarily close to 1 proportion of the mass of $Q$. Hence, one can assume without loss of generality that $Q$ is a bounded nonnegative measure with a compact support $\Omega_o \subset \Omega$.
As $\overline{\Omega}_o := [0,T] \times \Omega_o$ is Polish, the space $C(\overline{\Omega}_o)$ of all continuous functions on the compact set $\overline{\Omega}_o$ is dense in $L^p(\overline{Q})$, remember that $1 \leqslant p < \infty$. Therefore we can approximate $v$ in $L^p(\overline{Q})$ by a sequence $(v_n)_{n\geqslant 1}$ in $C(\overline{\Omega}_o)$. For all $h$ and $n$

$$\begin{aligned} \|k^h * v - v\|_{L^p(\overline{Q})} &\leqslant \|k^h * (v - v_n)\|_{L^p(\overline{Q})} + \|k^h * v_n - v_n\|_{L^p(\overline{Q})} + \|v_n - v\|_{L^p(\overline{Q})} \\ &\leqslant \|k^h * v_n - v_n\|_{L^p(\overline{Q})} + 2\|v - v_n\|_{L^p(\overline{Q})} \end{aligned}$$

where we used (2.12). Take an arbitrary small $\eta > 0$ and choose $n$ large enough for $\|v - v_n\|_{L^p(\overline{Q})} \leqslant \eta$ to hold. Then,

$$\|k^h * v - v\|_{L^p(\overline{Q})} \leqslant \|k^h * v_n - v_n\|_{L^p(\overline{Q})} + 2\eta. \tag{2.15}$$

Fix this $n$. Since $\overline{\Omega}_o$ is compact, $v_n \in C(\overline{\Omega}_o)$ is a uniformly continuous function. Therefore, for all $\eta > 0$, there exists $h(\eta) > 0$ such that for any $t, t', \omega, \omega'$ satisfying $|t - t'| + d_{\Omega_o}(\omega, \omega') \leqslant h(\eta)$ (with $d_{\Omega_o}$ the Skorokhod distance), we have $|v_n(t', \omega') - v_n(t, \omega)| \leqslant \eta$. In particular, with $\omega = \omega'$, we see that

$$|t' - t| \leqslant h(\eta) \Rightarrow \sup_{\omega\in\Omega_o} |v_n(t', \omega) - v_n(t, \omega)| \leqslant \eta.$$

Because of the property: $\operatorname{supp} k^h \subset [-h, h]$, we deduce from this that for any $\omega \in \Omega_o$, $|k^h * v_n(t, \omega) - v_n(t, \omega)| \leqslant \int_{[0,T]} |v_n(t-s, \omega) - v_n(t, \omega)| k^h(s)\, ds \leqslant \eta$ as soon as $h \leqslant h(\eta)/2$. We used the convention that $v_n(t, \omega)$ vanishes when $t$ is outside $[0, T]$. Consequently $\|k^h * v_n - v_n\|_{L^p(\overline{Q})} \leqslant \eta\, Q(\Omega)^{1/p}$. Finally, with (2.15) this leads us to $\|k^h * v - v\|_{L^p(\overline{Q})} \leqslant [2 + Q(\Omega)^{1/p}]\, \eta$. Since $\eta$ is arbitrary and $Q(\Omega)$ is finite, this shows that $\lim_{h\to 0} \|k^h * v - v\|_{L^p(\overline{Q})} = 0$, which is the desired result. □

**Corollary 2.16.** *Assume that in addition to the hypotheses of Lemma 2.11, for any* $0 \leqslant t \leqslant T$, *the random variable* $v(t, \bullet)$ *is* $\sigma(X_{[0,t]})$*-measurable, resp.* $\sigma(X_t)$*-measurable. Then, the process* $v^h$ *defined by* $v^h(t, \omega) := E_Q[k^h * v(t) \mid X_{[0,t]} = \omega_{[0,t]}]$, *resp.* $v^h(t, \omega) := E_Q[k^h * v(t) \mid X_t = \omega_t]$, *is in* $L^p(\overline{Q})$ *and* $\lim_{h\to 0^+} v^h = v$ *in* $L^p(\overline{Q})$.

*Proof.* By Jensen's inequality

$$\begin{aligned} \|v^h - v\|^p_{L^p(\overline{Q})} &= \int_{\overline{\mathcal{X}}} |E_Q\, [k^h * v(t) \mid X_{[0,t]}] - v(t)|^p\, d\overline{Q} = \int_{\overline{\mathcal{X}}} |E_Q\, [k^h * v(t) - v(t) \mid X_{[0,t]}]|^p\, d\overline{Q} \\ &\leqslant \int_{\overline{\mathcal{X}}} E_Q[|k^h * v(t) - v(t)|^p \mid X_{[0,t]}]\, d\overline{Q} = \int_{\overline{\mathcal{X}}} |k^h * v(t) - v(t)|^p\, d\overline{Q} \\ &= \|k^h * v - v\|^p_{L^p(\overline{Q})} \underset{h\to 0^+}{\longrightarrow} 0, \end{aligned}$$

where the vanishing limit is the content of Lemma 2.11. Replace $X_{[0,t]}$ by $X_t$ for the other statement. $\square$

**Extended generators and stochastic derivatives coincide.** The main result of this section is the following Proposition 2.18 which states that whenever $u$ is in the domain of the extended generator $\mathcal{L}^Q$, one can compute $\mathcal{L}^Q u$ using the stochastic derivative:

$$\mathcal{L}^Q u = L^Q u, \quad \overline{Q}\text{-}a.e. \tag{2.17}$$

On the other hand, it will be proved later at Proposition 2.21 that whenever the limit (2.9) defining the stochastic derivative $L^Q u$ exists in $L^1(\overline{Q})$, $u$ is also in the domain of the extended generator and (2.17) is satisfied.

**Proposition 2.18.** *For any $Q \in \mathrm{M}(\Omega)$, if $u$ is in $\operatorname{dom}\mathcal{L}^Q$ and satisfies $E_Q \int_{[0,T]} \left|\mathcal{L}^Q u(t, X_{[0,t]})\right|^p dt < \infty$ for some $p \geqslant 1$, then*

$$\lim_{h\to 0^+} E_Q \int_{[0,T-h]} \left| \frac{1}{h} E_Q \left[ u(\overline{X}_{t+h}) - u(\overline{X}_t) \,\middle|\, X_{[0,t]} \right] - \mathcal{L}^Q u(t, X_{[0,t]}) \right|^p dt = 0. \tag{2.19}$$

*In particular, this implies that $u \in \operatorname{dom} L^Q$, and the limit*

$$\mathcal{L}^Q u(t, X_{[0,t]}) = L^Q u(t, X_{[0,t]}) := \lim_{h\to 0^+} \frac{1}{h} E_Q \left[ u(\overline{X}_{t+h}) - u(\overline{X}_t) \,\middle|\, X_{[0,t]} \right]$$

*takes place in $L^p(\overline{Q})$.*

*Proof.* The specific convolution kernel $k^h = \frac{1}{h}\mathbf{1}_{[-h,0]}$ gives

$$k^h * v_t = \frac{1}{h} \int_{[t,(t+h)\wedge T]} v_s \, ds.$$

Denoting $v_t := \mathcal{L}^Q u(t, X_{[0,t]})$, with the definition of the extended generator, we see that $h \mapsto [u(\overline{X}_{(t+h)\wedge T}) - u(\overline{X}_t)] - hk^h * v_t$ is a local martingale with zero expectation. It follows that there exists a sequence $(\tau_k)_{k\geqslant 1}$ of $[0,T] \cup \{\infty\}$-valued stopping times such that $\lim_{k\to\infty} \tau_k = \infty$, $Q$-$a.e.$, and for any $0 \leqslant t < T$, $0 < h \leqslant T$ and $k \geqslant 1$,

$$\frac{1}{h} E_Q \left[ u(\overline{X}_{(t+h)\wedge\tau_k\wedge T}) - u(\overline{X}_{t\wedge\tau_k}) \mid X_{[0,t]} \right] = E_Q[k^h * v_{t\wedge\tau_k} \mid X_{[0,t]}].$$

Since it is assumed that $E_Q \int_{[0,T]} |v_t|^p \, dt < \infty$, by Jensen's inequality, (2.13) and dominated convergence, we obtain that the right-hand side tends to $E_Q[k^h * v_t \mid X_{[0,t]}]$ as $k$ tends to infinity (along a subsequence), for almost all $t$, leading to

$$\frac{1}{h} E_Q[u(\overline{X}_{(t+h)\wedge T}) - u(\overline{X}_t) \mid X_{[0,t]}] = E_Q[k^h * v_t \mid X_{[0,t]}].$$

On the other hand, as $v_t$ is $\sigma(X_{[0,t]})$-measurable, we see with Corollary 2.16 that $\lim_{h\to 0^+} E_Q \int_{[0,T]} |k^h * v_t - v_t|^p \, dt = 0$. Gathering these considerations leads to (2.19). $\square$

We provide a result which is complementary to Proposition 2.18, below at Proposition 2.21. Its proof relies on the following easy analytic result.

**Lemma 2.20.** *Let $a, b$ be two measurable functions on $[0,T]$ such that $a$ is right continuous, $b$ is Lebesgue-integrable and $\lim_{h\to 0^+} \int_{[0,T-h]} \left| \frac{1}{h}\{a(t+h) - a(t)\} - b(t) \right| dt = 0$. Then, $a$ is absolutely continuous and its distributional derivative is $\dot{a} = b$.*

*Proof.* Remark first that $t \mapsto \mathbf{1}_{\{0\leqslant t\leqslant T-h\}}\frac{1}{h}\{a(t+h)-a(t)\}$ is integrable for any small enough $0<h\leqslant T$. Take any $0\leqslant r\leqslant s<T$. On one hand, we have $\lim_{h\to 0^+}\int_{[r,s]}\frac{1}{h}\{a(t+h)-a(t)\}\,dt=\int_{[r,s]}b(t)\,dt$ and on the other one: $\int_{[r,s]}\frac{1}{h}\{a(t+h)-a(t)\}\,dt=\frac{1}{h}\int_{[s,s+h]}a(t)\,dt-\frac{1}{h}\int_{[r,r+h]}a(t)\,dt$, so that with the assumed right continuity of $a$ the integrals $\int_{[s,s+h]}a(t)\,dt$ and $\int_{[r,r+h]}a(t)\,dt$ are well defined for any small enough $h$ and we have $\lim_{h\to 0^+}\int_{[r,s]}\frac{1}{h}\{a(t+h)-a(t)\}\,dt=a(s)-a(r)$. Therefore $a(s)-a(r)=\int_{[r,s]}b(t)\,dt$ which is the claimed property. □

**Proposition 2.21.** *Let $u$ be a measurable real function on $\overline{\mathcal{X}}$, and $v$ be an adapted process such that $u(\overline{X})$ and $v$ are $\overline{Q}$-integrable, $t\mapsto u(\overline{X}_t)$ is right continuous (for instance $u$ might be continuous) and*

$$\lim_{h\to 0^+}E_Q\int_{[0,T-h]}\left|\frac{1}{h}E_Q\left[u(\overline{X}_{t+h})-u(\overline{X}_t)\mid X_{[0,t]}\right]-v_t\right|dt=0. \tag{2.22}$$

*Then, $u$ belongs to $\operatorname{dom}\mathcal{L}^Q$ and $\operatorname{dom}L^Q$, and $\mathcal{L}^Qu=L^Qu=v$, $\overline{Q}$-a.e.*

*Proof.* We write $E=E_Q$ and $u_t=u(\overline{X}_t)$ to simplify the notation. Fix $0\leqslant r<T$. We have

$$\begin{aligned}\left|E\left[\int_{[r,T-h]}\left(\frac{1}{h}\{u_{t+h}-u_t\}-v_t\right)dt\,\middle|\,X_{[0,r]}\right]\right|&\\ \leqslant E\left[\int_{[r,T-h]}E\left(\left|\frac{1}{h}\{u_{t+h}-u_t\}-v_t\right|\,\middle|\,X_{[0,t]}\right)dt\,\middle|\,X_{[0,r]}\right].&\end{aligned}$$

With (2.22) and Fatou's lemma, we obtain

$$\begin{aligned}E\left(\liminf_{h\downarrow 0}\left|E\left[\int_{[r,T-h]}\left(\frac{1}{h}\{u_{t+h}-u_t\}-v_t\right)dt\,\middle|\,X_{[0,r]}\right]\right|\right)&\\ \leqslant \lim_{h\to 0^+}E\int_{[r,T-h]}E\left(\left|\frac{1}{h}\{u_{t+h}-u_t\}-v_t\right|\,\middle|\,X_{[0,t]}\right)dt=0.&\end{aligned}$$

Hence, there exists a sequence $(h_n)_{n\geqslant 1}$ of positive numbers such that $\lim_{n\to\infty}h_n=0$ and

$$\lim_{n\to\infty}\int_{[r,T-h_n]}\left|E\left[\left(\frac{1}{h_n}\{u_{t+h_n}-u_t\}-v_t\right)\,\middle|\,X_r\right]\right|dt=0,\quad Q\text{-a.e.}$$

It remains to apply Lemma 2.20 with $a(t)=E\left[u_t\mid X_{[0,r]}\right]$ and $b(t)=E\left[v_t\mid X_{[0,r]}\right]$ to see that for all $0\leqslant r\leqslant s<T$, $E\left[u_s-u_r-\int_{[r,s]}v_t\,dt\mid X_{[0,r]}\right]=0$. As $u_t$ is $Q$-integrable since it is assumed that $u(\overline{X})$ is $\overline{Q}$-integrable and $t\mapsto u(\overline{X}_t)$ is right-continuous, this proves that $M$ is a $Q$-martingale where $M_s:=u(\overline{X}_s)-u(\overline{X}_0)-\int_{[0,s]}v_t\,dt$. Therefore, $u$ belongs to $\operatorname{dom}\mathcal{L}^Q$ and $\mathcal{L}^Qu=v$. To obtain the remaining identity $\mathcal{L}^Qu=L^Qu$, apply Proposition 2.18 with $p=1$. □

**Corollary 2.23.** *Let $(\mathcal{G}_t)_{0\leqslant t\leqslant T}$ be the generator of the semigroup $(T_s^t)_{0\leqslant s\leqslant t\leqslant T}$ associated to the Markov measure $Q$, and let $u:\overline{\mathcal{X}}\to\mathbb{R}$ be an $x$-continuous and $t$-differentiable function such that for each $t$, $u(t,\bullet)\in\operatorname{dom}\mathcal{G}_t$, and $\lim_{h\to 0^+}\int_{[0,T]}\sup_{\mathcal{X}}|h^{-1}(T_t^{t+h}-\mathrm{Id})u-(\partial_t+\mathcal{G}_t)u|\,dt=0$. Then $u$ belongs to $\operatorname{dom}\mathcal{L}^Q$ and $\mathcal{L}^Qu=(\partial_t+\mathcal{G}_t)u$.*

*Proof.* Immediate consequence of Proposition 2.21. □

## 3. Stochastic derivatives. Extensions

The results of previous section are extended by means of the notions of $P$-locality and integration times. The main result of this section is Proposition 3.14.

Before this, we start introducing the backward in time analogues of the already defined (forward) extended generators and stochastic derivatives.

**Reversing time.** Let $Q \in \mathrm{M}(\Omega)$ be any path measure. Its time reversal is

$$Q^* := (X^*)_\# Q \in \mathrm{M}(\Omega),$$

where

$$\begin{cases} X_t^* := \lim_{h\to 0^+} X_{T-t+h}, & 0 \leqslant t < T, \\ X_T^* := X_0, & t = T, \end{cases}$$

is the reversed canonical process. We assume that $Q$ is such that $Q(X_{T^-} \neq X_T) = 0$, i.e. its sample paths are left-continuous at $t = T$. This implies that the time reversal mapping $X^*$ is (almost surely) one-one on $\Omega$. Similarly, we define

$$\overline{X}^*(t, \omega) := (T - t, X_t^*(\omega)), \quad 0 \leqslant t \leqslant T.$$

We introduce the backward extended generator and the backward stochastic derivative

$$\begin{aligned} \overleftarrow{\mathcal{L}}^Q u(t, X_{[t,T]}) &:= \overrightarrow{\mathcal{L}}^{Q^*} u^*(t^*, X^*_{[0,t^*]}), \\ \overleftarrow{L}^Q u(t, X_{[t,T]}) &:= \overrightarrow{L}^{Q^*} u^*(t^*, X^*_{[0,t^*]}), \end{aligned} \tag{3.1}$$

where $u^* : \overline{\mathcal{X}} \to \mathbb{R}$ is defined by $u^*(t^*, \omega^*_{[0,t^*]}) := u(t, \omega_{[t,T]})$, with $t^* := (T-t)^+$, $\omega^*(t) := \omega(t^*)$, and $\overrightarrow{\mathcal{L}}^{Q^*}$ and $\overrightarrow{L}^{Q^*}$ stand respectively for the standard (forward) generator and derivative of $Q^*$ as introduced at Definitions 2.3 and 2.8. Definitions (3.1) match with the following ones.

As a notation, the $\sigma$-field generated by $X_{[t^-,T]}$ is $\sigma(X_{[t^-,T]}) := \cap_{h>0} \sigma(X_{[t-h,T]}) = \sigma(X_{t^-}) \vee \sigma(X_{[t,T]})$.

**Definition 3.2** (Extended backward generator)**.** *Let $Q$ be a conditionable path measure. A process $u$ adapted to the predictable backward filtration $(\sigma(X_{[t^-,T]}); 0 \leqslant t \leqslant T)$ is said to be in the domain of the extended backward generator of $Q$ if there exists a process $v$ also adapted to the predictable backward filtration such that $\int_{[0,T]} |v(t, X_{[t^-,T]})|\, dt < \infty$, $Q$-a.e. and the process*

$$u(t, X_{[t^-,T]}) - u(T, X_T) - \int_{[t,T]} v(s, X_{[s^-,T]})\, ds, \quad 0 \leqslant t \leqslant T,$$

*is a local backward $Q$-martingale. We denote*

$$\overleftarrow{\mathcal{L}}^Q u(t, X_{[t^-,T]}) := v(t, X_{[t^-,T]})$$

*and call $\overleftarrow{\mathcal{L}}^Q$ the extended backward generator of $Q$. Its domain is denoted by* $\mathrm{dom}\, \overleftarrow{\mathcal{L}}^Q$.

Remark that denoting $\overleftarrow{\mathcal{L}}^Q = v$ is consistent with (3.1).

**Definition 3.3** (Stochastic backward derivative)**.** *Let $Q$ be a conditionable path measure and a measurable function $u$ on $[0,T] \times \mathcal{X}$ such that $E_Q|u(s, X_s)| < \infty$ for all $0 \leqslant s \leqslant T$.*

(1) *We say that $u$ admits a stochastic backward derivative under $Q$ at time $t \in (0,T]$ if the following limit*

$$\overleftarrow{L}^Q u(t, X_{[t^-,T]}) := \lim_{h\to 0^+} E_Q \left( \frac{1}{h}[u(\overline{X}_{t-h}) - u(\overline{X}_t)] \,\Big|\, X_{[t^-,T]} \right)$$

*exists in $L^1(Q)$.*
*In this case, $\overleftarrow{L}^Q u(t, \bullet)$ is called the stochastic backward derivative of $u$ at time $t$.*

(2) *If $u$ admits a stochastic backward derivative for almost all $t$, we say that $u$ belongs to the domain* $\mathrm{dom}\, \overleftarrow{L}^Q$ *of the stochastic backward derivative $\overleftarrow{L}^Q$ of $Q$.*

Remark that this definition is consistent with (3.1).

Mimicking almost verbatim the proofs of Propositions 2.18 and 2.21, (consider the convolution kernel $k^{-h} := \frac{1}{h}\mathbf{1}_{[0,h]}$ instead of $k^h = \frac{1}{h}\mathbf{1}_{[-h,0]}$), we arrive at

**Proposition 3.4.** *Let $Q$ be a conditionable path measure.*

(a) *If $u$ is in* $\mathrm{dom}\, \overleftarrow{\mathcal{L}}^Q$ *and such that $E_Q \int_{[0,T]} \left|\overleftarrow{\mathcal{L}}^Q u(t, X_{[t^-,T]})\right|^p dt < \infty$ for some $p \geqslant 1$, then*

$$\lim_{h\to 0^+} E_Q \int_{[h,T]} \left| \frac{1}{h} E_Q \left[ u(\overline{X}_{t-h}) - u(\overline{X}_t) \,\Big|\, X_{[t^-,T]} \right] - \overleftarrow{\mathcal{L}}^Q u(t, X_{[t^-,T]}) \right|^p dt = 0.$$

*In particular, this implies that $u \in$* $\mathrm{dom}\, \overleftarrow{L}^Q$, *and the limit*

$$\overleftarrow{\mathcal{L}}^Q u(t, X_{[t^-,T]}) = \overleftarrow{L}^Q u(t, X_{[t^-,T]}) := \lim_{h\to 0^+} \frac{1}{h} E_Q\big[u(\overline{X}_{t-h}) - u(\overline{X}_t) \mid X_{[t^-,T]}\big]$$

*takes place in $L^p(\overline{Q})$.*

(b) *Let $u$ be a measurable real function on $\overline{\mathcal{X}}$ and $v$ a process adapted to the predictable backward filtration, such that $u(\overline{X}), v$ are $\overline{Q}$-integrable, $t \mapsto u(\overline{X}^*_t)$ is right continuous (for instance $u$ might be continuous) and*

$$\lim_{h\to 0^+} E_Q \int_{[h,T]} \left| \frac{1}{h} E_Q[u(\overline{X}_{t-h}) - u(\overline{X}_t) \mid X_{[t^-,T]}] - v(t, X_{[t^-,T]}) \right| dt = 0.$$

*Then, $u$ belongs to* $\mathrm{dom}\, \overleftarrow{\mathcal{L}}^Q$ *and* $\mathrm{dom}\, \overleftarrow{L}^Q$, *and $\overleftarrow{\mathcal{L}}^Q u = \overleftarrow{L}^Q u = v$, $\overline{Q}$-a.e.*

When working with forward generators and derivatives in contexts where time reversal is not mentioned, we often omit the forward arrow, denoting: $\overrightarrow{\mathcal{L}} = \mathcal{L}$ and $\overrightarrow{L} = L$.

**$P$-locality.** We present some notions which will be useful when looking at the extended HJB equation at Section 5. Let us recall a slight modification of the standard definitions of stochastic integral and local martingale, introduced in [32].

**Definition 3.5** ($P$-locality)**.** *Let $P \in \mathrm{P}(\Omega)$, $Q \in \mathrm{M}(\Omega)$ such that $P \ll Q$.*

(1) *A process $M$ is said to be a $P$-local $Q$-martingale if there exists an increasing sequence of $[0,T] \cup \{\infty\}$-valued stopping times $(\tau_k)_{k\geqslant 1}$ such that $\lim_{k\to\infty} \tau_k = \infty$, $P$-a.e. (and not necessarily $Q$-a.e.) such that the stopped processes $M^{\tau_k}$ are $Q$-martingales, for all $k \geqslant 1$.*

(2) *A process $Y$ is said to be a $P$-local $Q$-stochastic integral if there exists an increasing sequence of $[0,T] \cup \{\infty\}$-valued stopping times $(\tau_k)_{k\geqslant 1}$ such that $\lim_{k\to\infty} \tau_k = \infty$, $P$-a.e. (and not necessarily $Q$-a.e.) such that the stopped processes $Y^{\tau_k}$ are $L^2(Q)$-stochastic integrals, for all $k \geqslant 1$.*

The filtration is the $Q$-completion of the canonical filtration. Since any local $Q$-martingale admits a càdlàg $Q$-version and $P \ll Q$, any $P$-local $Q$-martingale admits a càdlàg $P$-version.

In connection with the notion of $P$-locality, we introduce a modification of the notion of extended generator.

**Definition 3.6** ($P$-local extended generator of the path measure $Q$)**.** *Let $P \in \mathrm{P}(\Omega)$, $Q \in \mathrm{M}(\Omega)$ such that $P \ll Q$.*
*A measurable function $u$ on $\overline{\mathcal{X}}$ is said to be in the domain of the $P$-local extended forward generator of the path measure $Q$ if there exists a $\overline{P}$-almost everywhere defined adapted process $v$ such that $\int_{[0,T]} |v_t|\,dt < \infty$, $P$-a.e. and the process $u(\overline{X}_t) - u(\overline{X}_0) - \int_{[0,t]} v_s\,ds$, $0 \leqslant t \leqslant T$, is a $P$-local $Q$-martingale. We denote*

$$\overrightarrow{\mathcal{L}}^{Q,P} u := v,$$

*and call $\overrightarrow{\mathcal{L}}^{Q,P}$ the $P$-local extended forward generator of $Q$. Its domain is denoted by* $\mathrm{dom}\,\overrightarrow{\mathcal{L}}^{Q,P}$.

*A similar definition holds for the $P$-local extended* backward *generator: $\overleftarrow{\mathcal{L}}^{Q,P}$, of $Q$.*

The $P$-local extended generator of $Q$ only consists in requiring that the local $Q$-martingales entering the Definitions 2.3 and 3.6 of $\overrightarrow{\mathcal{L}}^{Q}$ and $\overleftarrow{\mathcal{L}}^{Q}$ are replaced by $P$-local $Q$-martingales. This notion will allow us to extend in a natural way Itô formula in a diffusion setting at Proposition 4.24.

**Integration times.** In order to motivate Definition 3.7 below, let us start with a remark. Since $u(\overline{X})$ might not be $Q$-integrable, the conditional increment $E_Q[u(\overline{X}_{t+h}) - u(\overline{X}_t) \mid X_{[0,t]}]$ appearing in the expression (2.9) of the stochastic derivative, might not be meaningful. To take this trouble into account, let us introduce the following notion.

**Definition 3.7** (Integration time)**.** *Let $u$ be in* $\mathrm{dom}\,\mathcal{L}^Q$ *and $\tau$ be a stopping time. We say that $\tau$ is a $Q$-integration time of $u$ if the stopped local martingale $M^{u,\tau}$ (recall Definition 2.3) and the stopped process $u(\overline{X}^{\tau})$ are $\overline{Q}$-integrable.*

Clearly, for any $Q$-integration time $\tau$ of $u$, $E_Q[u(\overline{X}^{\tau}_{t+h}) - u(\overline{X}^{\tau}_t) \mid X_{[0,t]}]$ is well defined. Of course, for the stopping time $\tau$ to be a $Q$-integration time of $u$ it is necessary and sufficient that two of the following properties hold:

- $u(\overline{X}^{\tau})$ is $\overline{Q}$-integrable;
- $M^{u,\tau}$ is $\overline{Q}$-integrable;
- $(t,\omega) \mapsto \mathbf{1}_{\{\tau(\omega)>t\}}\mathcal{L}^Q u(t,\omega_{[0,t]})$ is $\overline{Q}$-integrable.

In which case the three properties are satisfied.

**Lemma 3.8.** *Let us take a conditionable path measure $Q$, a function $u$ in* $\mathrm{dom}\,\mathcal{L}^Q$ *and $\tau$ any $Q$-integration time of $u$. Then,*

$$\lim_{h\to 0^+} E_Q \int_{[0,T-h]} \mathbf{1}_{\{\tau>t\}} \left| \frac{1}{h} E_Q\left[u(\overline{X}^{\tau}_{t+h}) - u(\overline{X}^{\tau}_t) \,\middle|\, X_{[0,t]}\right] - \mathcal{L}^Q u(t, X_{[0,t]}) \right| dt = 0. \tag{3.9}$$

*Proof.* Denoting $Q^\tau := (X^\tau)_\# Q$ the law of the stopped canonical process $X^\tau$ at the random time $\tau$, for any $0 \leqslant t \leqslant T$ and any bounded measurable function $U : X^\tau(\Omega) \to \mathbb{R}$, as a consequence of general considerations in measure theory, we have

$$E_{Q^\tau}(U \mid X_{[0,t]}) \circ X^\tau = E_Q(U(X^\tau) \mid X^\tau_{[0,t]}), \qquad Q\text{-}a.e.$$

It follows that

$$
\begin{aligned}
& E_{Q^\tau} \int_{[0,T-h]} \left| \frac{1}{h} E_{Q^\tau}[u(\overline{X}_{t+h}) - u(\overline{X}_t) \mid X_{[0,t]}] - \mathcal{L}^{Q^\tau} u(t, X_{[0,t]}) \right| dt \\
&= E_Q \int_{[0,T-h]} \left| \frac{1}{h} E_{Q^\tau}[u(\overline{X}_{t+h}) - u(\overline{X}_t) \mid X_{[0,t]}] \circ X^\tau - \mathcal{L}^{Q^\tau} u(t, X_{[0,t]}) \circ X^\tau \right| dt \\
&= E_Q \int_{[0,T-h]} \left| \frac{1}{h} E_Q[u(\overline{X}^\tau_{t+h}) - u(\overline{X}^\tau_t) \mid X^\tau_{[0,t]}] - \mathcal{L}^{Q^\tau} u(t, X^\tau_{[0,t]}) \right| dt \\
&= E_Q \int_{[0,T-h]} \left| \frac{1}{h} E_Q[u(\overline{X}^\tau_{t+h}) - u(\overline{X}^\tau_t) \mid X^\tau_{[0,t]}] - \mathbf{1}_{\{\tau > t\}} \, \mathcal{L}^Q u(t, X_{[0,t]}) \right| dt,
\end{aligned}
$$

where at last equality we use

$$
\mathcal{L}^{Q^\tau} u(t, X^\tau_{[0,t]}) = \mathbf{1}_{\{\tau > t\}} \, \mathcal{L}^Q u(t, X_{[0,t]}), \quad \text{for almost all } t,
$$

which is a direct consequence of the very definitions of $\mathcal{L}^Q u$ and $\mathcal{L}^{Q^\tau} u$.
Applying Proposition 2.18 to $Q^\tau$ with $p = 1$, this identity gives us

$$
\lim_{h \to 0^+} E_Q \int_{[0,T-h]} \left| \frac{1}{h} E_Q[u(\overline{X}^\tau_{t+h}) - u(\overline{X}^\tau_t) \mid X^\tau_{[0,t]}] - \mathbf{1}_{\{\tau > t\}} \mathcal{L}^Q u(t, X_{[0,t]}) \right| dt = 0,
$$

which is equivalent to (3.9). $\square$

*Remarks* 3.10.

(a) Writing $\lim_{h \to 0^+} E_Q \left( \frac{1}{h} [u(\overline{X}^\tau_{t+h}) - u(\overline{X}^\tau_t)] \mid X^\tau_{[0,t]} \right)$ is a slight abuse of notation. It should be written $\lim_{h \to 0^+} E_Q \left( \frac{1}{h} [u(\overline{X}^\tau_{(t+h) \wedge T}) - u(\overline{X}^\tau_t)] \mid X^\tau_{[0,t]} \right)$, with $(t+h) \wedge T$ instead of $t + h$. This simplification will be kept at several places in the sequel.

(b) It is necessary that $Q(\tau > t \mid X_{[0,t]}) > 0$ for (3.9) to be a nontrivial assertion. Lemma 3.12 below tells us that such stopping times $\tau$ exist.

*Remark* 3.11 (The Markov case). Denoting $DU^\tau_t := u(\overline{X}^\tau_{t+h}) - u(\overline{X}^\tau_t)$, it is tempting to infer from Lemma 3.8 that for any Markov measure $Q$,

$$
E_Q[DU^\tau_t \mid X_{[0,t]}] = \mathbf{1}_{\{\tau > t\}} E_Q[DU^\tau_t \mid X_t].
$$

But *this is false in general*, unless $\tau$ is a "Markov stopping time", i.e. unless $X^\tau_\# Q$ is a Markov measure. More precisely, in general last equality in: $E_Q(DU^\tau_t \mid X_{[0,t]}) = \mathbf{1}_{\{\tau > t\}} E_Q(DU^\tau_t \mid X_{[0,t]}) = \mathbf{1}_{\{\tau > t\}} E_Q(DU^\tau_t \mid X_t)$, fails.
Note in passing that (3.9) is a statement about the stopped path measure $X^\tau_\# Q$.

Lemma 3.8 suggests a way to compute $\mathcal{L}^Q u$ when integrability is lacking. It is stated below at Proposition 3.14. Let us start with a simple remark stated as a lemma.

**Lemma 3.12.** *Any function in* $\operatorname{dom} \mathcal{L}^Q$ *admits a sequence of $Q$-integration times tending $Q$-almost everywhere to infinity.*

*Proof.* For any $u \in \operatorname{dom} \mathcal{L}^Q$, we denote $M^u_t := u(\overline{X}_t) - u(\overline{X}_0) - \int_{[0,t]} \mathcal{L}^Q u(s, X_{[0,s]}) \, ds$, the local $Q$-martingale which appears at Definition 2.3. By the very definition of a local $Q$-martingale, there exists a sequence $(\sigma_k)$ of stopping times such that $\lim_{k \to \infty} \sigma_k = \infty$, $Q$-*a.e.* and the stopped local martingales $M^{u,\sigma_k}$ are genuine uniformly integrable $Q$-martingales. Furthermore, since it is assumed that $\int_{[0,T]} |\mathcal{L}^Q u(t, X_{[0,t]})| \, dt < \infty$, $Q$-*a.e.*, the stopping times $\theta_k := \inf\{t : \int_{[0,t]} |\mathcal{L}^Q u(s, X_{[0,s]})| \, ds \geqslant k\}$ satisfy $\lim_{k \to \infty} \theta_k = \infty, Q$-*a.e.* Therefore, the sequence $(\tau_k := \sigma_k \wedge \theta_k)$ tends almost surely to infinity. We also see that for any $k$, $M^{u,\tau_k}$ and $u(\overline{X}^{\tau_k})$ are uniformly $Q$-integrable. $\square$

*Remark* 3.13 (Integration time trick). Note that if $(\tau_k)$ is a sequence of $Q$-integration times tending $Q$-*a.e.* to infinity, and $(\tau'_k)$ is any sequence of stopping times tending $Q$-*a.e.* to infinity, then as a consequence of the stopped martingale theorem, $\tau''_k := \tau_k \wedge \tau'_k$, defines a sequence of $Q$-integration times tending $Q$-*a.e.* to infinity. To keep notation easy, in this situation we still write $\tau_k$ instead of $\tau''_k$. Let us call this operation the *integration time trick*.

**Proposition 3.14.** *Let $Q$ be a conditionable measure, take a function $u$ in* $\mathrm{dom}\,\mathcal{L}^Q$ *and fix $t \in [0,T)$. There exist an increasing sequence $(\tau_k)$ of $Q$-integration times of $u$ and a sequence $(h_n)$ of positive numbers such that $\lim_{k\to\infty} \tau_k = \infty$, $Q$-a.e., $\lim_{n\to\infty} h_n = 0$ and for each $k$ we have*

$$\mathcal{L}^Q u(t, X_{[0,t]}) = \lim_{n\to\infty} \frac{1}{h_n} E_Q\left[u(\overline{X}^{\tau_k}_{t+h_n}) - u(\overline{X}^{\tau_k}_t) \,\Big|\, X_{[0,t]}\right], \quad (\mathbf{1}_{\{\tau^k > t\}}\overline{Q})\text{-}a.e. \tag{3.15}$$

*Moreover, by taking a further subsequence*

$$\mathcal{L}^Q u(t, X_{[0,t]}) = \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{h_n} E_Q\left[u(\overline{X}^{\tau_k}_{t+h_n}) - u(\overline{X}^{\tau_k}_t) \,\Big|\, X_{[0,t]}\right], \quad \overline{Q}\text{-}a.e.$$

*Proof.* We have already seen at Lemma 3.12 that there exists a sequence $(\tau'_k)$ of $Q$-integration times which tends almost surely to infinity. By means of the integration time trick (see Remark 3.13), it can be chosen as an increasing sequence by considering $\tau_k = \max_{1\leqslant i\leqslant k} \tau'_i$. The almost everywhere convergence (3.15) is a direct consequence of the $L^1$-limit (3.9). The remaining statement is an easy consequence of (3.15), $\lim_{k\to\infty} Q(\tau_k > t) = 1$, and a diagonal subsequence argument to extract a sequence $(h_n)$ from the array $(h_{k,n})$. □

Next result is intuitively obvious. Nevertheless, its proof is not as direct as one could wish. As a corollary of Proposition 3.14, its proof relies on Proposition 2.18, again.

**Corollary 3.16.** *For any Markov measure $Q \in \mathrm{M}(\Omega)$ and any $u \in \mathrm{dom}\,\mathcal{L}^Q$*

$$\mathcal{L}^Q u(t, X_{[0,t]}) = \mathcal{L}^Q u(t, X_t), \qquad \overline{Q}\text{-}a.e.$$

*(with some obvious abuse of notation).*

*Proof.* Assuming that for some $t$ the limit in the first term of next series of identities exists $Q$-*a.e.*, we have

$$\begin{aligned}
&\lim_{n\to\infty} h_n^{-1} E_Q\left[u(\overline{X}^{\tau}_{t+h_n}) - u(\overline{X}^{\tau}_t) \,\Big|\, X_{[0,t]}\right] \\
&= \lim_{n\to\infty} h_n^{-1}\Big(E_Q\left[\mathbf{1}_{\{t<\tau\}}\{u(\overline{X}^{\tau}_{t+h_n}) - u(\overline{X}^{\tau}_t)\} \,\Big|\, X_{[0,t]}\right] \\
&\qquad\qquad + E_Q\left[\mathbf{1}_{\{t\geqslant\tau\}}\{u(\overline{X}^{\tau}_{t+h_n}) - u(\overline{X}^{\tau}_t)\} \,\Big|\, X_{[0,t]}\right]\Big) \\
&= \mathbf{1}_{\{t<\tau\}} \lim_{n\to\infty} h_n^{-1} E_Q\left[u(\overline{X}^{\tau}_{t+h_n}) - u(\overline{X}^{\tau}_t) \,\Big|\, X_{[0,t]}\right] \\
&= \mathbf{1}_{\{t<\tau\}} \lim_{n\to\infty} h_n^{-1} E_Q\left[u(\overline{X}_{t+h_n}) - u(\overline{X}_t) \,\Big|\, X_{[0,t]}\right] \\
&= \mathbf{1}_{\{t<\tau\}} \lim_{n\to\infty} h_n^{-1} E_Q\left[u(\overline{X}_{t+h_n}) - u(\overline{X}_t) \,\Big|\, X_t\right],
\end{aligned}$$

where all these equalities hold $Q$-*a.e.* The first equality is an obvious decomposition. The second one follows from $t \geqslant \tau \implies \overline{X}^{\tau}_{t+h_n} = \overline{X}^{\tau}_t = (\tau, X_\tau) \implies u(\overline{X}^{\tau}_{t+h_n}) - u(\overline{X}^{\tau}_t) = 0$ for the rightmost term, and the fact that the event $\{t < \tau\}$ is $\sigma(X_{[0,t]})$-measurable because $\tau$ is a stopping time, for the leftmost term. The third equality is obvious because the limit is pointwise. Last equality follows from the assumed Markov property of $Q$. We conclude with Proposition 3.14. □

Note that although in presence of a stopping time, this proof does not rely on a strong Markov property argument.

*Remarks* 3.17.

(a) *About càdlàg versions of local martingales.* In the previous two sections, we took some care making precise the conditions for the filtration to fulfill the "usual hypotheses" to imply that martingales admit càdlàg versions. This will not be used in what follows because it will be assumed later that the diffusion field $\mathsf{a}$ is invertible to assure the Brownian martingale representation (hence any local martingale is continuous as a Brownian stochastic integral). These precautions about the filtration are necessary if one wishes to extend our results to a more general setting, including jumps for instance, as a careful reading of the proof of Theorem 5.9 indicates.

(b) *About time reversal.* Similarly, time reversal will not play any role in the present article. However, we decided to include considerations about it because it plays a major role in entropic optimal transport theory. In particular, our previous results about backward generators and derivatives are used in the recent article [4] and will be utilized in future research of the author.

(c) *About $P$-locality.* Of course, if $P$ and $R$ are equivalent path measures, then the notion of $P$-locality is useless. However, there are situations where the scalar potential $V$ is irregular enough for $R$ not to be absolutely continuous with respect to $P$. For instance this happens when $V$ is strong enough for the sample paths of $P$ not to reach the nodal set $\{(t,x) : dP_t/dx = 0\}$, see [11, Remark. 4.1] and [52] in a similar but different context.

## 4. Some more preliminary results

**Relative entropy.** Let us start with the definition. Let $A$ be any measurable space. The relative entropy of $\mathsf{p} \in \mathrm{P}(A)$ with respect to the reference measure $\mathsf{r} \in \mathrm{M}(A)$ is

$$H(\mathsf{p}|\mathsf{r}) := \int_A \log(d\mathsf{p}/d\mathsf{r})\, d\mathsf{p} \in (-\infty, \infty]$$

if $\mathsf{p}$ is absolutely continuous with respect to $\mathsf{r}$ and $\log_-(d\mathsf{p}/d\mathsf{r})$ is $\mathsf{p}$-integrable. We set $H(\mathsf{p}|\mathsf{r}) = \infty$ otherwise.
We set $\log_- x := \max(-\log x, 0)$ and $\log_+ x := \max(\log x, 0)$.

*Remark* 4.1. As an immediate consequence of this definition, any $\mathsf{p} \in \mathrm{P}(A)$ such that $H(\mathsf{p}|\mathsf{r}) < \infty$ verifies $\log(d\mathsf{p}/d\mathsf{r}) \in L^1(\mathsf{p})$. However it is useful to define $H(\bullet \mid \mathsf{r})$ on the following subset of $\mathrm{P}(A)$ :

$$D_{H(\cdot|\mathsf{r})} := \left\{ \mathsf{p} \in \mathrm{P}(A);  \int_A W\, d\mathsf{p} < \infty, \text{ for some } W : A \to [0,\infty) \text{ such that } \int_A e^{-W}\, d\mathsf{r} < \infty \right\}.$$

Note that when $\mathsf{r}$ is a bounded measure, then $W = 0$ does the work and $D_{H(\cdot|\mathsf{r})} = \mathrm{P}(A)$.

The reason for this choice is guided by the next result which provides us with a criterion for $\log_-(d\mathsf{p}/d\mathsf{r}) \in L^1(\mathsf{p})$.

**Proposition 4.2.** *Let $\mathsf{r}$ be a $\sigma$-finite measure on $A$.*

*(a) (i) There exists a measurable function $W : A \to [0, \infty)$ such that $\int_A e^{-W}\, d\mathsf{r} < \infty$.*

*(ii) For any such function $W$ and any $\mathsf{p} \in \mathrm{P}(A)$ verifying $\mathsf{p} \ll \mathsf{r}$ and $\int_A W\, d\mathsf{p} < \infty$, we have:* $\log_-(d\mathsf{p}/d\mathsf{r}) \in L^1(\mathsf{p})$.

*(b) Moreover, $D_{H(\cdot|\mathsf{r})}$ is a convex subset of the vector space of all bounded signed measures and $H(\bullet|\mathsf{r})$ is a $(-\infty, \infty]$-valued convex function on this set.*

*Proof.* Take $\mathsf{p} \in \mathrm{P}(A)$ such that $\mathsf{p} \ll \mathsf{r}$ and denote $\rho := d\mathsf{p}/d\mathsf{r}$ for simplicity.
Remark that when $\mathsf{r}(A) < \infty$, because $\sup_{0 \leqslant z \leqslant 1} z|\log z| = e^{-1}$, we always have

$$\int_A \log_- \rho\, d\mathsf{p} = \int_{\{\rho \leqslant 1\}} |\log \rho|\, d\mathsf{p} = \int_{\{\rho \leqslant 1\}} \rho |\log \rho|\, d\mathsf{r} \leqslant e^{-1}\mathsf{r}(A) < \infty.$$

This corresponds to $W = 0$.

• Proof of (a). Now, we only assume that $\mathsf{r}$ is $\sigma$-finite. Statement (i) is a direct consequence of this hypothesis. Therefore, let us prove (ii). We set $\mathsf{r}_W := e^{-W}\,\mathsf{r} \in \mathrm{M}(A)$ and denote $\rho_W := d\mathsf{p}/d\mathsf{r}_W = e^W \rho$. For any $\mathsf{p} \in \mathrm{P}(A)$, we see that

$$\begin{aligned}
\int_{\{\rho \leqslant 1\}} |\log \rho|\, d\mathsf{p} &= \int_{\{\rho \leqslant 1\}} |\log \rho_W - W|\, d\mathsf{p} \\
&\leqslant \int_{\{\log \rho_W \leqslant W\}} |\log \rho_W|\, d\mathsf{p} + \int_A W\, d\mathsf{p} \\
&\leqslant \int_A \log_- \rho_W\, d\mathsf{p} + 2\int_A W\, d\mathsf{p}.
\end{aligned}$$

But we know by our preliminary remark that $\int_A \log_- \rho_W\, d\mathsf{p}$ is finite because $\mathsf{r}_W(A) = \int_A e^{-W}\, d\mathsf{r} < \infty$.

• Proof of (b). Finally, $D_{H(\cdot|\mathsf{r})}$ is a convex set because for any $\mathsf{p}, \mathsf{p}'$ in the convex set $\mathrm{P}(A)$ such that $\int_A W\, d\mathsf{p}, \int_A W'\, d\mathsf{p}' < \infty$ with $W, W' : A \to [0, \infty)$, $\int_A (e^W + e^{W'})\, d\mathsf{r} < \infty$, we have $\int_A W''\, d(\theta \mathsf{p} + (1-\theta)\mathsf{p}') < \infty$ where $0 \leqslant \theta \leqslant 1$ and $W'' := \min(W, W')$ satisfies $\int_A e^{W''}\, d\mathsf{r} < \infty$. To prove that $H(\bullet|\mathsf{r})$ is convex, write it as $H(\mathsf{p}|\mathsf{r}) = \int_A [\log \rho_W - W]\, d\mathsf{p} = \int_A h(\rho_W)\, d\mathsf{r}_W - \int_A W\, d\mathsf{p}$ with $h(\rho) := \rho \log \rho$ a convex function. □

**Reference diffusion measure.** Let $\mathsf{a}$ be a diffusion matrix field on $[0, T] \times \mathbb{R}^n$, $\mathsf{c}$ be some $\mathbb{R}^n$-valued predictable process and $\nu \in \mathrm{M}(\mathbb{R}^n)$. We say that the path measure $Q \in \mathrm{M}(\Omega)$ solves the martingale problem with initial measure $\nu$ and characteristics $(\mathsf{c}, \mathsf{a})$ if $Q_0 = \nu$ and

$$\begin{array}{l} dX_t = \mathsf{c}_t\, dt + dM_t^Q, \\ d\langle X \rangle_t = d\langle M^Q \rangle_t = \mathsf{a}(\overline{X}_t)\, dt, \end{array} \qquad \overline{Q}\text{-}a.e.,$$

where $M^Q$ is some local $Q$-martingale. We denote this property by

$$Q \in \mathrm{MP}(\mathsf{a}, \mathsf{c}; \nu). \tag{4.3}$$

This is equivalent to: $Q$ solves $\mathrm{MP}(\mathcal{C}, \mathcal{L}; Q_0)$ using Definition 2.6 with $\mathcal{C} = \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^n)$ and $\mathcal{L} = \mathsf{c} \cdot \nabla + \Delta_{\mathsf{a}}/2$. We also write shortly $Q \in \mathrm{MP}(\mathsf{a}, \mathsf{c})$ instead of $Q \in \mathrm{MP}(\mathsf{a}, \mathsf{c}; Q_0)$.
It is implicitly assumed that $\int_{[0,T]} |\mathsf{c}_t|\, dt < \infty$ and $\int_{[0,T]} \|\mathsf{a}(\overline{X}_t)\|\, dt < \infty$, $Q$-$a.e.$ This last property is satisfied for instance if $\mathsf{a}$ is locally bounded:

$$\sup_{t \in [0,T], x \in K} \|\mathsf{a}(t, x)\| < \infty, \quad \text{for any compact set } K \subset \mathbb{R}^n. \tag{4.4}$$

**Definition 4.5** (The reference Markov measure $R$)**.** *Let $R \in \mathrm{M}(\Omega)$ be a Markov measure solution to*

$$R \in \mathrm{MP}(\mathsf{a}, \mathsf{b}), \tag{4.6}$$

*for some vector field $\mathsf{b} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$. In addition to (4.4), we assume that $R$ fulfils the uniqueness condition:*

$$\forall R' \in \mathrm{M}(\Omega),\ [R' \in \mathrm{MP}(\mathsf{a}, \mathsf{b}; R_0) \textit{ and } R' \ll R] \implies R' = R. \tag{4.7}$$

Assumption (4.7) is necessary to write explicit formulas for relative entropies and Radon-Nikodym derivatives of path measures with respect to $R$. It is proved in [25, Thm. 12.21] that (4.7) holds if and only if $R$ is an extremal point of the (convex) set of solutions to its own martingale problem: $\mathrm{MP}(\mathsf{a}, \mathsf{b}; R_0)$. To emphasize this uniqueness property, we shall sometimes write

$$R = \mathrm{MP}(\mathsf{a}, \mathsf{b})$$

with an equality.
Clearly $R(\bullet) = \int_{\mathbb{R}^n} R^{x_o}(\bullet)\, R_0(dx_o)$, with $R^{x_o}$ the law of the Markov process with initial position $x_o$ and generator

$$\partial_t u + \mathsf{b}\cdot\nabla u + \Delta_{\mathsf{a}} u/2, \qquad u \in C_c^{1,2}([0,T] \times \mathbb{R}^n),$$

see Lemma 4.14 below for a precise statement. The measure $R$ is possibly an unbounded $\sigma$-finite positive measure. This occurs for instance when $R$ is the reversible Wiener measure: its reversing measure is Lebesgue measure, and we take $\mathsf{b} = 0$, $\mathsf{a} = \mathrm{Id}$ and $R_0 = \mathrm{Leb}$.
We note for future reference that it is assumed implicitly that

$$\int_{[0,T]} |\mathsf{b}(\overline{X}_t)|\, dt < \infty, \quad R\text{-}a.e. \tag{4.8}$$

**Girsanov theory.** Take $Q \in \mathrm{P}(\Omega)$ such that

$$H(Q|R) < \infty. \tag{4.9}$$

We know by the Girsanov theory under a finite entropy condition [32] that there exists some $\mathbb{R}^n$-valued predictable process $\beta^{Q|R}$ which is defined $\overline{Q}$-*a.e.* such that $\beta^{Q|R} \in \mathrm{range}[\mathsf{a}(\overline{X})]$, $\overline{Q}$-*a.e.*, $Q$ solves the martingale problem

$$Q = \mathrm{MP}(\mathsf{a}, \mathsf{b} + \mathsf{a}\beta^{Q|R}), \tag{4.10}$$

and $Q$ inherits the uniqueness property (4.7) from $R$. Furthermore, because of (4.7), we know that

$$\begin{aligned}\frac{dQ}{dR} &= \mathbf{1}_{\{dQ/dR>0\}}\, \frac{dQ_0}{dR_0}(X_0) \exp\left( \int_{[0,T]} \beta_t^{Q|R} \cdot dM_t^R - \int_{[0,T]} |\beta_t^{Q|R}|^2_{\mathsf{a}(\overline{X}_t)}/2\ dt \right)\\ &= \mathbf{1}_{\{dQ/dR>0\}}\, \frac{dQ_0}{dR_0}(X_0) \exp\left( \int_{[0,T]} \beta_t^{Q|R} \cdot dM_t^Q + \int_{[0,T]} |\beta_t^{Q|R}|^2_{\mathsf{a}(\overline{X}_t)}/2\ dt \right),\end{aligned} \tag{4.11}$$

where we denote $|\beta|^2_{\mathsf{a}} := \beta \cdot \mathsf{a}\beta$,

$$dM_t^R = dX_t - \mathsf{b}_t\, dt \quad \text{and} \quad dM^Q = dX_t - (\mathsf{b}_t + \mathsf{a}(\overline{X}_t)\beta_t^{Q|R})\, dt.$$

Here $M^Q$ is a local $Q$-martingale and the local $R$-martingale $M^R$ is seen as a $Q$-local $R$-martingale to define $\int_{[0,T]} \beta_t^{Q|R} \cdot dM_t^R$ as a $Q$-local $R$-stochastic integral, see Definitions

3.5. Moreover,

$$H(Q|R) = H(Q_0|R_0) + E_Q \int_{[0,T]} |\beta_t^{Q|R}|^2_{\mathsf{a}(\overline{X}_t)}/2 \; dt. \tag{4.12}$$

Of course, the requirement $H(Q|R) < \infty$ implies that

$$E_Q \int_{[0,T]} |\beta_t^{Q|R}|^2_{\mathsf{a}(\overline{X}_t)} \, dt < \infty. \tag{4.13}$$

**Lemma 4.14.** *Under the assumptions (4.4), (4.7) and (4.9), $C_c^{1,2}([0,T] \times \mathbb{R}^n)$ is included in* $\operatorname{dom} \mathcal{L}^Q$*, and for any $u \in C_c^{1,2}([0,T] \times \mathbb{R}^n)$,*

$$\mathcal{L}^Q u = (\partial_t + \mathsf{v}^Q \cdot \nabla + \Delta_{\mathsf{a}}/2)u,$$

*with $\mathsf{v}^Q := \mathsf{b} + \mathsf{a}\beta^{Q|R}$.*
*If in addition $Q$ is Markov, then the process $\beta^{Q|R}$ turns out to be a vector field:*

$$\beta_t^{Q|R} = \beta^{Q|R}(\overline{X}_t), \;\; \bar{Q}\text{-a.e.}, \tag{4.15}$$

*for some measurable $\beta^{Q|R} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ which is defined $\bar{\mathsf{q}}$-a.e.*

Note that unlike Lemma 4.22 below, next assumption (4.18) is not required for this lemma to hold.

*Proof.* For any $u \in C_c^{1,2}([0,T] \times \mathbb{R}^n)$, identifying Itô formula and the basic identity attached to $Q \in \mathrm{MP}(\mathsf{a}, \mathsf{v}^Q)$

$$\begin{aligned} du(\overline{X}_t) &= \partial_t u(\overline{X}_t) \, dt + \nabla u(\overline{X}_t) \cdot dX_t + \Delta_{\mathsf{a}} u(\overline{X}_t)/2 \; dt \\ &= \underbrace{\left(\partial_t u(\overline{X}_t) + \mathsf{v}^Q(t, X_{[0,t]}) \cdot \nabla u(\overline{X}_t) + \Delta_{\mathsf{a}} u(\overline{X}_t)/2\right)}_{\text{candidate to be } \mathcal{L}^Q u(t, X_{[0,t]})} dt + dM_t^{u,Q}, \end{aligned}$$

(rely on the uniqueness of the Doob-Meyer decomposition), we see that the increment of the local $Q$-martingale $M^{u,Q}$ is

$$dM_t^{u,Q} = \nabla u(\overline{X}_t) \cdot dM_t^Q \tag{4.16}$$

with

$$dM_t^Q = dX_t - \mathsf{v}^Q(t, X_{[0,t]}) \, dt,$$

the increment of the canonical local $Q$-martingale. By assumption (4.4), $M^{u,Q}$ is a square integrable martingale because $E_Q |M_T^{u,Q}|^2 = E_Q \int_{[0,T]} |\nabla u|^2_{\mathsf{a}}(\overline{X}_t) \, dt \leqslant T \sup |\nabla u|^2_{\mathsf{a}} < \infty$. To prove the first part of the lemma, it remains to verify

$$\int_{[0,T]} |(\partial_t + [\mathsf{b}(\overline{X}_t) + \mathsf{a}(\overline{X}_t)\beta^{Q|R}(t, X_{[0,t]})] \cdot \nabla + \Delta_{\mathsf{a}(\overline{X}_t)}/2)u(\overline{X}_t)| \, dt < \infty, \quad Q\text{-a.e.}$$

We already know with (4.4) that $\sup_{t,x} |(\partial_t + \Delta_{\mathsf{a}}/2)u(t,x)| < \infty$, and with (4.8) and $Q \ll R$ that $\int_{[0,T]} |\mathsf{b}(\overline{X}_t)| \, dt < \infty$, $Q$-a.e. On the other hand, by Cauchy-Schwarz inequality,

$$\begin{aligned} E_Q \int_{[0,T]} |(\mathsf{a}\beta^{Q|R}) \cdot \nabla u|^2(t, X_{[0,t]}) \, dt &\leqslant E_Q \int_{[0,T]} |\beta^{Q|R}|^2_{\mathsf{a}} \; |\nabla u|^2_{\mathsf{a}}(t, X_{[0,t]}) \, dt \\ &\leqslant \sup |\nabla u|^2_{\mathsf{a}} \; E_Q \int_{[0,T]} |\beta^{Q|R}|^2_{\mathsf{a}} \; (t, X_{[0,t]}) \, dt < \infty, \end{aligned}$$

where we use assumption (4.4) to control $\sup |\nabla u|_{\mathsf{a}}$, and assumption (4.9) to obtain (4.13). Therefore, $\int_{[0,T]} |(\mathsf{a}\beta^{Q|R})\cdot\nabla u|(t, X_{[0,t]})\, dt < \infty,\ Q\text{-}a.e.$, and the proof of the first statement is done.

Now, suppose that in addition $Q$ is Markov. By Corollary 3.16, $\mathcal{L}^Q u(t, X_{[0,t]}) = v(\overline{X}_t),\ \overline{Q}\text{-}a.e.$ for some function $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$. Hence, for any $u \in C_c^{1,2}([0,T] \times \mathbb{R}^n)$, $\mathsf{a}(\overline{X}_t)\beta_t^{Q|R}\cdot\nabla u(\overline{X}_t) = v(\overline{X}_t) - \partial_t u(\overline{X}_t) - \mathsf{b}\cdot\nabla u(\overline{X}_t) - \Delta_{\mathsf{a}(\overline{X}_t)} u(\overline{X}_t)/2$, $\overline{Q}$-$a.e.$ This implies that $\mathsf{a}_{\overline{X}}\beta^{Q|R} = \mathsf{v}(\overline{X})$, $\overline{Q}$-$a.e.$, for some vector field $\mathsf{v} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$. Of course, $\mathsf{v} \in \mathrm{range}(\mathsf{a})$, $\bar{\mathsf{q}}$-$a.e.$ As $\beta^{Q|R}$ also satisfies $\beta^{Q|R} \in \mathrm{range}(\mathsf{a}_{\overline{X}})$, without assuming that $\mathsf{a}$ is invertible, one can solve uniquely this equation to obtain $\beta^{Q|R} = \mathsf{a}^{-1}\mathsf{v}(\overline{X})$, $\overline{Q}$-$a.e.$ where $\mathsf{a}^{-1}$ is the generalized inverse of $\mathsf{a}$ defined by $\lim_{n\to\infty}(\mathsf{a} + \mathrm{Id}/n)^{-1}$. We conclude taking $\beta^{Q|R} = \mathsf{a}^{-1}\mathsf{v}$. □

**Kinetic action.** We observe that

$$H(Q|R) - H(Q_0|R_0) = H(Q|R^{Q_0}) = E_Q \int_{[0,T]} \frac{1}{2} |\mathsf{v}_t^{Q|R}|^2_{\mathsf{g}(\overline{X}_t)}\, dt \tag{4.17}$$

is an average kinetic action, where

$$\mathsf{v}^{Q|R} := \mathsf{a}\beta^{Q|R}$$

should be interpreted as a stochastic relative velocity between $Q$ and $R$, and

$$|\mathsf{v}|^2_{\mathsf{g}} := \begin{cases} \mathsf{v}\cdot\mathsf{a}^{-1}\mathsf{v}, & \text{if } \mathsf{v} \in \mathrm{range}(\mathsf{a}), \\ +\infty, & \text{otherwise.} \end{cases}$$

**Martingale representation.** The martingale representation theorem will play an important role. We assume that $R$ is the law of a diffusion process solution to the stochastic differential equation

$$d\mathsf{Y}_t = \mathsf{b}(t, \mathsf{Y}_t)\, dt + \sigma(t, \mathsf{Y}_t)\, d\mathsf{W}_t$$

where $\mathsf{W}$ is an $n$-dimensional Brownian motion built on some unspecified filtered triple $(\Xi, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathcal{F}, \mathbb{P})$ where the $n \times n$-matrix field $\sigma$ satisfies $\sigma\sigma^{\mathsf{t}} = \mathsf{a}$. In this setting, the martingale representation theorem states that if

$$\mathsf{a}(t,x) \text{ is invertible for all } (t,x) \in [0,T] \times \mathbb{R}^n, \tag{4.18}$$

for any local $(\mathcal{F}^{\mathsf{Y}}, \mathbb{P})$-martingale $\mathsf{M}$, where $\mathcal{F}^{\mathsf{Y}} := (\sigma(\mathsf{Y}_{[0,t]}))_{0\leqslant t\leqslant T}$ is the $\mathbb{P}$-complete natural filtration of the process $\mathsf{Y}$, there exists a predictable process $\Phi^{\mathsf{M}}$ such that $\sigma\Phi^{\mathsf{M}}$ is locally square integrable and

$$\mathsf{M}_t = \mathsf{M}_0 + \int_0^t \Phi_s^{\mathsf{M}}\cdot\sigma(s, \mathsf{Y}_s)\, d\mathsf{W}_s, \quad 0 \leqslant t \leqslant T, \quad \mathbb{P}\text{-}a.e.$$

Now, let us go back to the canonical setting by taking the image by $\mathsf{Y} : \Xi \to \Omega$ of $(\Xi, \mathcal{F}^{\mathsf{Y}}, \mathbb{P})$ to obtain $R = \mathsf{Y}_{\#}\mathbb{P}$ and some sub-filtration of the canonical one. After completing it by the $R$-negligible sets, this filtration coincides with the $R$-completion of the canonical filtration. We also see that under the law $R$, $dM_t^R := dX_t - \mathsf{b}(\overline{X}_t)\, dt \overset{\text{Law}}{=} \sigma(t, \mathsf{Y}_t)\, d\mathsf{W}_t$. More generally, Girsanov's theory also tells us that for any $Q$ such that $H(Q|R) < \infty$, under the law $Q$ we have $dM_t^Q := dX_t - \mathsf{v}^Q(t, X_{[0,t]})\, dt \overset{\text{Law}}{=} \sigma(t, \mathsf{Y}_t)\, d\mathsf{W}_t$. Otherwise stated,

$$dM_t^Q := dX_t - \mathsf{v}^Q(t, X_{[0,t]})\, dt = \sigma(\overline{X}_t)\, dW_t^Q, \quad \overline{Q}\text{-}a.e., \tag{4.19}$$

where $W^Q$ is a $Q$-Brownian motion on the canonical space equipped with the $Q$-completion of the canonical filtration. Moreover, for any function $u$ in $\operatorname{dom}\mathcal{L}^Q$, by the martingale representation theorem which holds because of the assumed invertibility of $\mathsf{a}$,

$$dM_t^{u,Q} := du(\overline{X}_t) - \mathcal{L}^Q u(t, X_{[0,t]})\,dt = \alpha_t^{u,Q} \cdot dM_t^Q, \quad \overline{Q}\text{-a.e.}, \tag{4.20}$$

for some predictable process $\alpha^{u,Q}$. This implies in particular

$$d\langle u(\overline{X}), v(\overline{X})\rangle_t^Q = \alpha_t^{u,Q}\cdot \mathsf{a}(\overline{X}_t)\alpha_t^{v,Q}\,dt. \tag{4.21}$$

**Lemma 4.22.** *We assume* (4.18)*, i.e.* $\mathsf{a}$ *is invertible.*

*(a) Then* $\operatorname{dom}\mathcal{L}^Q$ *is an algebra, meaning that for any* $u, v \in \operatorname{dom}\mathcal{L}^Q$*, the product* $uv$ *is still in* $\operatorname{dom}\mathcal{L}^Q$*.*

*(b) For any* $u \in \operatorname{dom}\mathcal{L}^Q$ *and any function* $F \in C^2(\mathbb{R})$*,* $F(u)$ *is also in* $\operatorname{dom}\mathcal{L}^Q$*.*

*Proof.* Under the ellipticity assumption (4.18), we have (4.21). This is the key of the proof.

• Proof of (a). As a definition of the forward generator

$$du(\overline{X}_t) = \mathcal{L}_t^Q u(t, X_{[0,t]})\,dt + dM_t^u, \qquad dv(\overline{X}_t) = \mathcal{L}_t^Q v(t, X_{[0,t]})\,dt + dM_t^v,$$

and applying Itô formula in the forward sense of time

$$\begin{aligned}
d(uv)(\overline{X}_t) &= u(\overline{X}_t)dv(\overline{X}_t) + v(\overline{X}_t)du(\overline{X}_t) + d[u(\overline{X}), v(\overline{X})]_t\\
&= u(\overline{X}_t)dv(\overline{X}_t) + v(\overline{X}_t)du(\overline{X}_t) + d\langle u(\overline{X}), v(\overline{X})\rangle_t\\
&= [u\mathcal{L}_t^Q v(t, X_{[0,t]}) + v\mathcal{L}_t^Q u(t, X_{[0,t]})]\,dt + d\langle u(\overline{X}), v(\overline{X})\rangle_t\\
&\qquad + u(\overline{X}_t)dM_t^v + v(\overline{X}_t)dM_t^u.
\end{aligned}$$

The bounded variation part of this semimartingale is

$$[u\mathcal{L}_t^Q v(t, X_{[0,t]}) + v\mathcal{L}_t^Q u(t, X_{[0,t]})]\,dt + d\langle u(\overline{X}), v(\overline{X})\rangle_t^Q. \tag{4.23}$$

With (4.21), this shows that $uv(\overline{X})$ is a *nice* $Q$-semimartingale, which is the announced result.

• Proof of (b). By Itô formula and (4.21):

$$\begin{aligned}
dF(u)(\overline{X}_t) &= F'(u(\overline{X}_t))du(\overline{X}_t) + F''(u(\overline{X}_t))d\langle u(X), u(X)\rangle_t/2\\
&= [F'(u(\overline{X}_t))\mathcal{L}_t^Q u(t, X_{[0,t]}) + F''(u(\overline{X}_t))\alpha_t^{u,Q}\cdot \mathsf{a}(\overline{X}_t)\alpha_t^{u,Q}/2]\,dt + F'(u(\overline{X}_t))\,dM_t^u,
\end{aligned}$$

we see that the bounded variation part of $F(u(\overline{X}))$ is absolutely continuous, which is the announced result. □

This simple consequence of the martingale representation theorem is crucial for the remainder of the article.

**Extended Itô formula.** We already saw at (4.16) while proving Lemma 4.14 that for any regular function $u \in C_c^{1,2}([0,T]\times\mathbb{R}^n)$, $dM_t^{u,Q} = \nabla u(\overline{X}_t)\cdot dM_t^Q$. Let us look at the extension of this identity to the case where $u$ belongs to $\operatorname{dom}\mathcal{L}^Q$.

**Proposition 4.24** (Extended Itô formula)**.** *We assume that* $R$ *satisfies* (4.4)*,* (4.7)*,* (4.18)*, and that* $Q$ *is Markov and satisfies* (4.9)*.*
*Remember that by Lemma 4.14,* $Q = \mathrm{MP}(\mathsf{a}, \mathsf{v}^Q)$ *with*

$$\mathsf{v}^Q = \mathsf{b} + \mathsf{a}\beta^{Q|R}. \tag{4.25}$$

*The following statements are verified.*

(a) *For any $u$ in* $\operatorname{dom}\mathcal{L}^Q$, *there exists a* $\bar{\mathsf{q}}$-*a.e. defined vector field* $\widetilde{\nabla}^Q u$ *such that*

$$\begin{aligned} du(\overline{X}_t) &= \mathcal{L}^Q u(\overline{X}_t)\,dt + \widetilde{\nabla}^Q u(\overline{X}_t)\cdot dM^Q_t \\ &= [\mathcal{L}^Q - \mathsf{v}^Q\cdot\widetilde{\nabla}^Q]u(\overline{X}_t)\,dt + \widetilde{\nabla}^Q u(\overline{X}_t)\cdot dX_t, \quad \overline{Q}\text{-}a.e. \end{aligned} \tag{4.26}$$

(b) *Let $P \ll Q$. For any $u$ in* $\operatorname{dom}\mathcal{L}^{Q,P}$ *(recall Definition 3.6), there exists a* $\bar{\mathsf{p}}$-*a.e.-defined vector field* $\widetilde{\nabla}^{Q,P}u$ *such that*

$$\begin{aligned} du(\overline{X}_t) &= \mathcal{L}^{Q,P} u(\overline{X}_t)\,dt + \widetilde{\nabla}^{Q,P} u(\overline{X}_t)\cdot dM^Q_t \\ &= [\mathcal{L}^{Q,P} - \mathsf{v}^Q\cdot\widetilde{\nabla}^{Q,P}]u(\overline{X}_t)\,dt + \widetilde{\nabla}^{Q,P} u(\overline{X}_t)\cdot dX_t, \quad \overline{P}\text{-}a.e., \end{aligned} \tag{4.27}$$

*where* $\widetilde{\nabla}^{Q,P}u(\overline{X}_t)\cdot dM^Q_t$ *is the increment of a $P$-local $Q$-stochastic integral (recall Definition 3.5).*

*Remark* 4.28. Note that $\widetilde{\nabla}^{Q,P}u$ is only defined $\bar{\mathsf{p}}$-*a.e.* (and not $\bar{\mathsf{q}}$-*a.e.*), and unlike (4.26), the identity (4.27) only holds $\overline{P}$-*a.e.*, but is meaningless $\overline{Q}$-*a.e.* in general.

*Proof.* The proof of statement (b) is an almost verbatim modification of the proof of statement (a). It is left to the reader.
In what follows, the identities hold $\overline{Q}$-*a.e.* Let $u \in \operatorname{dom}\mathcal{L}^Q$. By the very definition of the extended generator:

$$du(\overline{X}_t) = \mathcal{L}^Q u(\overline{X}_t)\,dt + dM^{u,Q}_t, \tag{4.29}$$

with $M^{u,Q}$ a local $Q$-martingale. And by (4.20):

$$dM^{u,Q}_t = \alpha^{u,Q}_t \cdot dM^Q_t$$

for some predictable process $\alpha^{u,Q}$. Taking next Lemma 4.30 for granted, it follows that $dM^{u,Q}_t = \widetilde{\nabla}^Q u(\overline{X}_t)\cdot dM^Q_t$, and we complete the proof with (4.29). □

It remains to prove

**Lemma 4.30.** *Under the ellipticity condition (4.18), the Markov measure $Q$ is such that the process $\alpha^{u,Q}$ only depends on the current position: $\alpha^{u,Q}_t = \widetilde{\nabla}^Q u(\overline{X}_t)$, $0 \leqslant t \leqslant T$, for some* $\bar{\mathsf{q}}$-*almost everywhere defined vector field* $\widetilde{\nabla}^Q u : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$.

*Proof.* We have just seen that

$$dM^{u,Q}_t = \alpha^{u,Q}_t \cdot dM^Q_t = du(\overline{X}_t) - \mathcal{L}^Q u(\overline{X}_t)\,dt,$$

and for any function $v$ in $C^{1,2}_c([0,T]\times\mathbb{R}^n)$, Itô formula is

$$dv(\overline{X}_t) = \partial_t v(\overline{X}_t)\,dt + \nabla v(\overline{X}_t)\cdot dX_t + \Delta_{\mathsf{a}} v(\overline{X}_t)/2\ dt.$$

Hence, the quadratic covariation of $u(\overline{X})$ and $v(\overline{X})$ satisfies

$$d\big\langle u(\overline{X}), v(\overline{X})\big\rangle_t = \alpha^{u,Q}_t \cdot \mathsf{a}\nabla v(\overline{X}_t)\,dt.$$

Therefore, with intuitive arguments we see that

$$\begin{aligned} \alpha^{u,Q}_t\cdot\mathsf{a}\nabla v(\overline{X}_t)\,dt &= E_Q\left(du(\overline{X}_t)dv(\overline{X}_t)\,\middle|\, X_{[0,t]}\right) = E_Q\left(du(\overline{X}_t)dv(\overline{X}_t)\,\middle|\, X_t\right) \\ &= E_Q\big(\alpha^{u,Q}_t \mid X_t\big)\cdot\mathsf{a}\nabla v(\overline{X}_t)\,dt, \end{aligned}$$

where we used the Markov property of $Q$ at second equality. Since $v$ is arbitrary and $\mathsf{a}$ is invertible, we see that $\alpha^{u,Q}_t = E_Q\big(\alpha^{u,Q}_t \mid X_t\big)$, proving the lemma.

However, it is necessary to justify rigorously the above string of identities. Firstly, one must localize (by means of stopping times) to give some meaning at the conditional expectation $E_Q\big(\alpha^{u,Q}_t \mid X_t\big)$, and the above identity

$$d\left\langle u(\overline{X}), v(\overline{X})\right\rangle_t = E_Q\big(du(\overline{X}_t)dv(\overline{X}_t) \mid X_{[0,t]}\big), \quad \overline{Q}\text{-}a.e., \tag{4.31}$$

still needs to be carefully established, again with some localization argument.
The first localization argument is standard, so we leave it to the reader.
Finally, (4.31) follows from Lemma A.2 in the appendix. To verify that the hypotheses of this lemma are satisfied note that in the present Brownian setting under the hypothesis (4.18), by Lemma 4.22 we know that $\mathrm{dom}\,\mathcal{L}^Q$ is an algebra. On the other hand the martingale representation theorem implies that any martingale is continuous. Hence the assumption about $M^{Q,[u,v]}$ in Lemma A.2 is trivially satisfied because $M^{Q,[u,v]} = 0$. □

*Remarks* 4.32.

(i) By (4.16), if $u$ is $C^{1,2}$-regular, then $\widetilde{\nabla}^Q u = \nabla u$, $\bar{\mathsf{q}}$-*a.e.*

(ii) It is proved by Cont and Fournié in [10, Thm. 5.9] that under hypotheses slightly more general than those of Proposition 4.24 (no entropy appears), any square integrable $Q$-martingale $N$ is represented as the stochastic integral

$$N_t = N_0 + \int_0^t \nabla_{M^Q} N_s \cdot dM^Q_s$$

where $M^Q$ appears at (4.19) and the *Cont-Fournié non-anticipative derivative* $\nabla_{M^Q} N$ is introduced in [10]. Therefore,

$$\widetilde{\nabla}^Q u(\overline{X}) = \nabla_{M^Q} M^{Q,u}, \qquad \overline{Q}\text{-}a.e.$$

The Cont-Fournié derivative is the predictable projection of the Malliavin (anticipative) derivative.

(iii) A remarkable extension of Itô formula is also obtained in [10], which goes in another direction than Proposition 4.24: stochastic differentials of regular non-anticipative functionals are considered in [10], while Proposition 4.24 gives a result for possibly not regular functions only depending on the current position.

## 5. Feynman-Kac formula

The main character of this section is the Feynman-Kac measure $P$ already encountered in the introduction at (1.4). It will allow us to derive pathwise properties of an extended (HJB) equation at Theorem 5.9 and an extended (FK) equation at Theorem 5.24.

**The Feynman-Kac measure.** It is defined by

$$P := f_0(X_0)\exp\left(\int_{[0,T]} V(\overline{X}_t)\,dt\right) g_T(X_T)\ R \in \mathrm{P}(\Omega), \tag{5.1}$$

where as in (4.6)

$$R = \mathrm{MP}(\mathsf{a}, \mathsf{b})$$

is a reference diffusion measure satisfying (4.7), $f_0$ and $g_T$ are nonnegative measurable functions on $\mathbb{R}^n$, $V : [0,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$, and we use the convention $e^{-\infty} = 0$. It is also assumed that all these quantities are such that $P$ is a probability measure and

$$H(P|R) < \infty.$$

For $P$ to be a probability measure it is necessary that the measurable subsets

$$\begin{aligned}
D_+ &:= \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n; R\Big(\int_{[0,T]} V_+(\overline{X}_t)\,dt < \infty \mid X_0 = x; X_T = y\Big) > 0\right\},\\
D_- &:= \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n; R\Big(\int_{[0,T]} V_-(\overline{X}_t)\,dt < \infty \mid X_0 = x; X_T = y\Big) > 0\right\},\\
D &:= \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n; f_0(x) > 0, g_T(y) > 0\},
\end{aligned}$$

verify $R_{01}$-*a.e.*

$$(a):\ D_+ \cup D_- = \mathbb{R}^n \times \mathbb{R}^n, \qquad (b):\ D \subset D_+, \qquad (c):\ D \cap D_- \neq \varnothing. \tag{5.2}$$

Indeed, (a), (b) and (c) are implied respectively by the properties of $dP/dR$ of being well defined, finite, and not identically equal to zero, up to some $R_{0T}$-negligible set. We denote $R_{0T} := (X_0, X_T)_\# R$ the law of the endpoint position under $R$.

Sufficient conditions for (5.2) and $H(P|R) < \infty$ will be given at Section 6. From now on, when assuming that $P$ is in $\mathrm{P}(\Omega)$ it is supposed implicitly that (5.2) holds.

The measure $P$ will bring us valuable informations about equation (FK).

**Positive integration.** For all $0 \leqslant t \leqslant T$ and $P_t$-almost every $x \in \mathbb{R}^n$, define the measurable functions $f, g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned}
f(t,x) &:= f_t(x) := E_R\left(f_0(X_0)\exp\Big(\int_{[0,t]} V(\overline{X}_s)\,ds\Big)\,\Big|\, X_t = x\right),\\
g(t,x) &:= g_t(x) := E_R\left(\exp\Big(\int_{[t,T]} V(\overline{X}_s)\,ds\Big) g_T(X_T)\,\Big|\, X_t = x\right).
\end{aligned} \tag{5.3}$$

One must be careful with these conditional expectations because it is not assumed that the integrands are $R$-integrable, neither that $R$ is bounded. However, they are well defined $\bar{\mathsf{p}}$-almost everywhere (but not $\bar{\mathsf{r}}$-*a.e.*) as conditional expectations of nonnegative functions with respect to a possibly unbounded Markov measure. This is warranted by next

**Lemma 5.4** ([33, §4]). *Let $R \in \mathrm{M}(\Omega)$ be a Markov measure and $P \in \mathrm{P}(\Omega)$ a probability measure such that $P \ll R$ and $\dfrac{dP}{dR} = \alpha\zeta\beta$ with $\alpha, \zeta, \beta$* nonnegative *functions such that $\alpha \in \sigma(X_{[0,s]})$, $\zeta \in \sigma(X_{[s,t]})$ and $\beta \in \sigma(X_{[t,T]})$ for some $0 \leqslant s \leqslant t \leqslant T$. Then,*

$$\left\{\begin{array}{l} E_R(\alpha \mid X_s), E_R(\beta \mid X_t) \in (0,\infty) \\ E_R(\alpha\beta \mid X_{[s,t]}) = E_R(\alpha \mid X_s) E_R(\beta \mid X_t) \in (0,\infty) \end{array}\right. \qquad P\text{-a.e.}$$

*(and not $R$-a.e. in general). In addition,*

$$\frac{dP_{[s,t]}}{dR_{[s,t]}}(X_{[s,t]}) = E_R(\alpha \mid X_s)\zeta E_R(\beta \mid X_t) \in (0,\infty), \qquad P\text{-a.e.}$$

*Remark* 5.5. Even if the product $\alpha\beta$ is integrable, it is not true in general that the nonnegative factors $\alpha$ and $\beta$ are integrable. Therefore, a priori the conditional expectations $E_R(\alpha \mid X_s)$ and $E_R(\beta \mid X_t)$ might have been infinite.

**Proposition 5.6.** *The Feynman-Kac measure $P$ given at (5.1) is Markov and*

$$\frac{dP_t}{dR_t} = f_t g_t, \quad P_t\text{-a.e.}, \quad \forall 0 \leqslant t \leqslant T. \tag{5.7}$$

*Proof.* For any $0 \leqslant t \leqslant T$, the Radon-Nikodym derivative $Z := dP/dR$ equals $Z = Z_{[0,t]}Z_{[t,T]}$ with $Z_{[0,t]} := f_0(X_0)\exp\Big(\int_{[0,t]} V(\overline{X}_r)\,dr\Big)$, $Z_{[t,T]} := \exp\Big(\int_{[t,T]} V(\overline{X}_r)\,dr\Big) g_T(X_T)$.

The identity (5.7) is a direct consequence of Lemma 5.4 applied with $s = t$, $\zeta = 1$, $\alpha = Z_{[0,t]}$ and $\beta = Z_{[t,T]}$.

The Markov property is proved by showing that for any bounded measurable functions $u(X_{[0,t]})$ and $v(X_{[t,T]})$, we obtain

$$
\begin{aligned}
E_P[u(X_{[0,t]})v(X_{[t,T]}) \mid X_t] &= \frac{E_R[Z\, u(X_{[0,t]})v(X_{[t,T]}) \mid X_t]}{E_R[Z \mid X_t]} \\
&= \frac{E_R[Z_{[0,t]}u(X_{[0,t]})Z_{[t,T]}v(X_{[t,T]}) \mid X_t]}{E_R[Z_{[0,t]}Z_{[t,T]} \mid X_t]} \\
&= \frac{E_R[Z_{[0,t]}u(X_{[0,t]}) \mid X_t]}{E_R[Z_{[0,t]} \mid X_t]}\ \frac{E_R[Z_{[t,T]}v(X_{[t,T]}) \mid X_t]}{E_R[Z_{[t,T]} \mid X_t]} \\
&= E_P[u(X_{[0,t]}) \mid X_t]\, E_P[v(X_{[t,T]}) \mid X_t].
\end{aligned}
$$

We invoked the Markov property of $R$ and Lemma 5.4 at the last but one equality. □

**Extended HJB equation.** We also introduce the measure $\bar{\mathsf{p}}(dtdx) = dtP_t(dx)$ on $[0,T] \times \mathbb{R}^n$ and the notation

$$
\varphi := \log f, \qquad \psi := \log g \tag{5.8}
$$

where $f$ and $g$ are defined at (5.3). We are ready to state the first main result of this section.

**Theorem 5.9.** *Let $R \in \mathrm{M}(\Omega)$ be a Markov measure satisfying (4.6), (4.7) with $\mathsf{a}$ verifying (4.4), (4.18), and let $P \in \mathrm{P}(\Omega)$ be given by (5.1). Then, under the assumption that*

$$
-\infty < H(P|R) < \infty,
$$

*the following statements hold.*

*(a) The $\bar{\mathsf{p}}$-a.e. defined function*

$$
\psi(t,x) := \log g_t(x) = \log E_R\Big(\exp\Big(\int_{[t,T]} V(\overline{X}_s)\,ds\Big) g_T(X_T)\,\Big|\, X_t = x\Big) \in \mathbb{R},
$$

*is in $\operatorname{dom}\mathcal{L}^{R,P}$ and it satisfies the extended HJB equation*

$$
\big(\mathcal{L}^{R,P}\psi + |\widetilde{\nabla}^{R,P}\psi|^2_{\mathsf{a}}/2 + V\big)(\overline{X}) = 0, \qquad \overline{P}\text{-a.e.} \tag{5.10}
$$

*with $\psi_T = \log g_T$, $P_T$-a.e. Remember that the existence of $\widetilde{\nabla}^{R,P}\psi$ and its definition are stated at Proposition 4.24 (extended Itô formula).*

*(b) The Feynman-Kac measure $P$ solves $\mathrm{MP}(\mathsf{a}, \mathsf{v}^P)$ where*

$$
\mathsf{v}^P = \mathsf{b} + \mathsf{a}\widetilde{\nabla}^{R,P}\psi. \tag{5.11}
$$

*(c) In addition, $\psi \in \operatorname{dom}\mathcal{L}^P$,*

$$
\mathcal{L}^P\psi(\overline{X}) = \big(|\widetilde{\nabla}^P\psi|^2_{\mathsf{a}}/2 - V\big)(\overline{X}), \quad \overline{P}\text{-a.e.} \tag{5.12}
$$

*and*

$$
\begin{aligned}
\mathcal{L}^{R,P}\psi(\overline{X}) &= \big(\mathcal{L}^P\psi - |\widetilde{\nabla}^P\psi|^2_{\mathsf{a}}\big)(\overline{X}), \qquad \overline{P}\text{-a.e.} \\
\widetilde{\nabla}^{R,P}\psi(\overline{X}) &= \widetilde{\nabla}^P\psi(\overline{X}), \qquad \overline{P}\text{-a.e.}
\end{aligned} \tag{5.13}
$$

Before proving this theorem, we make some remarks and establish preliminary estimates at Lemma 5.17 under a finite entropy condition.

*Remarks* 5.14.

(a) It follows from (5.10) that

$$\mathcal{L}^{R,P}\psi + |\widetilde{\nabla}^{R,P}\psi|_{\mathsf{a}}^2/2 + V = 0, \qquad \bar{\mathsf{p}}\text{-}a.e. \tag{5.15}$$

and it turns out that when $\psi$ is a finite $C^{1,2}$ function, (5.15) is the standard HJB equation (HJB):

$$(\partial_t + \mathsf{b}\cdot\nabla + \Delta_{\mathsf{a}}/2)\psi + |\nabla\psi|_{\mathsf{a}}^2/2 + V = 0.$$

(b) By (5.7), $f_t g_t > 0$, $P_t$-*a.e.* Consequently, for all $t$, $\psi_t := \log g_t$ is well defined $P_t$-*a.e.* as a real valued function.

For any $0 \leqslant s < t \leqslant T$, the restriction $P_{[s,t]}$ of $P$ to $\sigma(X_{[s,t]})$ satisfies

$$\frac{dP_{[s,t]}}{dR_{[s,t]}} = \frac{dP_s}{dR_s}(X_s)\exp\left(\psi(\overline{X}_t) - \psi(\overline{X}_s) + \int_{[s,t]} V(\overline{X}_r)\,dr\right), \qquad P\text{-}a.e. \tag{5.16}$$

This formula will be used in a while.
Let us prove it. The following identities hold $P$-*a.e.* (and not necessarily $R$-*a.e.*) without further mention:

$$\begin{aligned}
\frac{dP_{[s,t]}}{dR_{[s,t]}} &= E_R\left[\frac{dP}{dR}\,\middle|\, X_{[s,t]}\right] \overset{(i)}{=} E_R\left[f_0(X_0)\exp\left(\int_{[0,T]} V(\overline{X}_r)\,dr\right) g_T(X_T)\,\middle|\, X_{[s,t]}\right]\\
&= \exp\left(\int_{[s,t]} V(\overline{X}_r)\,dr\right) E_R\Big[f_0(X_0)\exp\left(\int_{[0,s]} V(\overline{X}_r)\,dr\right)\\
&\qquad\qquad \times \exp\left(\int_{(t,T]} V(\overline{X}_r)\,dr\right) g_T(X_T)\,\Big|\, X_{[s,t]}\Big]\\
&\overset{(ii)}{=} \exp\left(\int_{[s,t]} V(\overline{X}_r)\,dr\right) E_R\Big[f_0(X_0)\exp\left(\int_{[0,s]} V(\overline{X}_r)\,dr\right)\,\Big|\, X_{[s,t]}\Big]\\
&\qquad\qquad \times E_R\Big[\exp\left(\int_{(t,T]} V(\overline{X}_r)\,dr\right) g_T(X_T)\,\Big|\, X_{[s,t]}\Big]\\
&\overset{(iii)}{=} f_s(X_s)\exp\left(\int_{[s,t]} V(\overline{X}_r)\,dr\right) g_t(X_t) = f_s(X_s)g_s(X_s)\,\frac{g_t(X_t)}{g_s(X_s)}\exp\left(\int_{[s,t]} V(\overline{X}_r)\,dr\right)\\
&\overset{(iv)}{=} \frac{dP_s}{dR_s}(X_s)\exp\left(\psi(\overline{X}_t) - \psi(\overline{X}_s) + \int_{[s,t]} V(\overline{X}_r)\,dr\right)
\end{aligned}$$

where we used (5.1) at (i), the Markov property of $R$ and Lemma 5.4 at (ii), the definitions (5.3) of $f_s$ and $g_t$ at (iii), and (5.7) and the definition $\psi := \log g$ at (iv). Recall the time-symmetric formulation of the Markov property: a path measure $R$ is Markov if and only if for any $0 \leqslant s \leqslant t \leqslant T$, $X_{[0,s]}$ and $X_{[t,T]}$ are $R(\bullet \mid X_{[s,t]})$-independent.

Remark that no division by zero occurs $P$-*a.e.* because $g_s(X_s)$ is positive $P$-*a.e.* by (5.7).

**Lemma 5.17** (Finite entropy estimates). *Suppose that $-\infty < H(P|R) < \infty$. Then,*

*(i)* $\varphi_t, \psi_t \in L^1(P_t)$, *for all* $0 \leqslant t \leqslant T$;
*(ii)* $V(\overline{X}) \in L^1(\overline{P})$; $\quad V_t \in L^1(P_t)$ *for almost every* $0 \leqslant t \leqslant T$;
*(iii)* $t \mapsto \langle V_t, P_t\rangle := \int_{\mathbb{R}^n} V_t\,dP_t$ *is dt-integrable;*
*(iv)* $\psi(\overline{X}_t) - \psi(\overline{X}_s) + \int_{[s,t]} V(\overline{X}_r)\,dr \in L^1(P)$ *for all* $0 \leqslant s \leqslant t \leqslant T$;

*and*

$$H(P|R) = \langle \varphi_0, P_0 \rangle + \int_{[0,T]} \langle V_t, P_t \rangle \, dt + \langle \psi_T, P_T \rangle.$$

*Proof.* For any $0 \leqslant s \leqslant t \leqslant T$, $H(P_s|R_s) \leqslant H(P_{[s,t]}|R_{[s,t]}) \leqslant H(P|R) < \infty$. On the other hand, we obtain with (5.7) and (5.16) that

$$H(P_s|R_s) = \int_{\mathbb{R}^n} \log(f_s g_s) \, dP_s = \int_{\mathbb{R}^n} (\varphi_s + \psi_s) \, dP_s,$$
$$H(P_{[s,t]}|R_{[s,t]}) = H(P_s|R_s) + E_P\Big(\psi(\overline{X}_t) - \psi(\overline{X}_s) + \int_{[s,t]} V(\overline{X}_r) \, dr\Big).$$

Hence (see Remark 4.1 and Proposition 4.2), $\log(f_s g_s) \in L^1(P_s)$, and $\psi(\overline{X}_t) - \psi(\overline{X}_s) + \int_{[s,t]} V(\overline{X}_r) \, dr \in L^1(P)$. We conclude with Fubini-Lebesgue theorem applied with the product measure $(\delta_s + \mathrm{Leb}_{[s,t]} + \delta_t) \otimes P_{[s,t]}$ on $[s,t] \times \Omega_{[s,t]}$. □

*Proof of Theorem 5.9.* The boundary condition $\psi_T = \log g_T$ is an obvious outcome of the expression of $\psi$.

By (4.11), (4.12) and the Markov property of $P$ proved at Proposition 5.6 and Lemma 4.14, there exists a vector field $\gamma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\int_{[0,T]\times\mathbb{R}^n} |\gamma|^2_{\mathsf{a}} \, d\bar{\mathsf{p}} < \infty \tag{5.18}$$

and for any $0 \leqslant s < t \leqslant T$,

$$\frac{dP_{[s,t]}}{dR_{[s,t]}} = \frac{dP_s}{dR_s}(X_s) \exp\left( \int_{[s,t]} \gamma(\overline{X}_r) \cdot dM^P_r + \int_{[s,t]} |\gamma|^2_{\mathsf{a}}(\overline{X}_r)/2 \; dr \right), \quad P\text{-a.e.}, \tag{5.19}$$

where $dM^P_r = dX_r - (\mathsf{b} + \mathsf{a}\gamma)(\overline{X}_r) \, dr$.

By Lemma 5.17, we have: $V(\overline{X}) \in L^1(\overline{P})$, and because of (5.18) we have also: $|\gamma|^2_{\mathsf{a}}(\overline{X}) \in L^1(\overline{P})$. Hence, applying Lemma 2.11 with $k^h = \frac{1}{h}\mathbf{1}_{[-h,0]}$, we obtain that

$$\lim_{h \to 0^+} \left[ t \mapsto h^{-1} \int_{[t,t+h]} (|\gamma|^2_{\mathsf{a}}/2 - V)(\overline{X}_r) \, dr \right] = (|\gamma|^2_{\mathsf{a}}/2 - V)(\overline{X}) \quad \text{in } L^1(\overline{P}).$$

Identifying (5.16) with (5.19) gives us for all $0 \leqslant s < t \leqslant T$,

$$\psi(\overline{X}_t) - \psi(\overline{X}_s) = \int_{[s,t]} (|\gamma|^2_{\mathsf{a}}/2 - V)(\overline{X}_r) \, dr + \int_{[s,t]} \gamma(\overline{X}_r) \cdot dM^P_r, \quad P\text{-a.e.}, \tag{5.20}$$

which implies that for any $0 \leqslant t < t+h \leqslant T$,

$$E_P\left[h^{-1}\big(\psi(\overline{X}_{t+h}) - \psi(\overline{X}_t)\big) \mid X_t\right] = E_P\left[ h^{-1} \int_{[t,t+h]} (|\gamma|^2_{\mathsf{a}}/2 - V)(\overline{X}_r) \, dr \,\middle|\, X_t \right].$$

By (5.20) again, $\psi(\overline{X})$ admits a continuous version because $t \mapsto \int_{[0,t]} (|\gamma|^2_{\mathsf{a}}/2 - V)(\overline{X}_r) \, dr$ is continuous $P$-a.e., since $(|\gamma|^2_{\mathsf{a}}/2 - V)(\overline{X})$ is $\overline{P}$-integrable, and $t \mapsto \int_{[0,t]} \gamma(\overline{X}_r) \cdot dM^P_r$ admits a continuous version as a Brownian stochastic integral. Therefore, we are in position to apply Proposition 2.21 which ensures that $\psi$ belongs to $\mathrm{dom}\,\mathcal{L}^P$, and

$$\mathcal{L}^P \psi(\overline{X}) = [|\gamma|^2_{\mathsf{a}}/2 - V](\overline{X}), \quad \overline{P}\text{-a.e.} \tag{5.21}$$

As $\psi$ is in $\mathrm{dom}\,\mathcal{L}^P$, the extended Itô formula (4.26) is valid and writes as

$$d\psi(\overline{X}_t) = \mathcal{L}^P \psi(\overline{X}_t) \, dt + \widetilde{\nabla}^P \psi(\overline{X}_t) \cdot dM^P_t, \qquad \overline{P}\text{-a.e.}, \tag{5.22}$$

for some vector field $\widetilde{\nabla}^P\psi$. Note that identifying the bounded variation parts of (5.20) and (5.22) also leaves us with (5.21), while the identification of the martingale parts yields $|\gamma - \widetilde{\nabla}^P\psi|^2_{\mathsf{a}}(\overline{X}) = 0$, $\overline{P}$-*a.e.*, or equivalently since $\mathsf{a}$ is invertible,

$$\gamma(\overline{X}) = \widetilde{\nabla}^P\psi(\overline{X}), \quad \overline{P}\text{-}a.e. \tag{5.23}$$

With (5.21), this gives us (5.12).
Because of

$$dM^P_t = dM^R_t - \mathsf{a}\gamma(\overline{X}_t)\,dt, \qquad \overline{P}\text{-}a.e.,$$

and (5.23), equation (5.20) becomes

$$d\psi(\overline{X}_t) = [\mathcal{L}^P\psi - |\widetilde{\nabla}^P\psi|^2_{\mathsf{a}}](\overline{X}_t)\,dt + \widetilde{\nabla}^P\psi(\overline{X}_t)\cdot dM^R_t, \qquad \overline{P}\text{-}a.e.$$

As $\int_{[0,T]} |\widetilde{\nabla}^P\psi|^2_{\mathsf{a}}(\overline{X}_t)\,dt = \int_{[0,T]} |\gamma|^2_{\mathsf{a}}(\overline{X}_t)\,dt < \infty$, $P$-*a.e.*, this proves that $\psi$ is in $\operatorname{dom}\mathcal{L}^{R,P}$ and that (5.13) holds. Remark that $M^R$ is a $P$-local $R$-martingale.
Finally, (5.12) and (5.13) directly imply (5.10), while (5.11) follows from (5.23) and (5.13). $\square$

**Feynman-Kac formula.** Let us go back to the function

$$g_t(x) := E_R\left(\exp\Big(\int_{[t,T]} V(\overline{X}_s)\,ds\Big) g_T(X_T) \,\Big|\, X_t = x\right), \quad (t,x) \in [0,T]\times\mathbb{R}^n.$$

introduced at (5.3).

**Theorem 5.24.** *Under the hypotheses of Theorem 5.9, $g$ belongs to $\operatorname{dom}\mathcal{L}^{R,P}$ and*

$$[\mathcal{L}^{R,P} + V]g(\overline{X}) = 0, \quad \overline{P}\text{-}a.e.$$

*Proof.* We know by Theorem 5.9 that $\psi(\overline{X})$ is a $P$-local $R$-semimartingale. Applying Itô formula to $g = e^\psi$, we immediately see that $g(\overline{X})$ is also a $P$-local $R$-semimartingale. More precisely, we have

$$\begin{aligned} dg(\overline{X}_t) &= g(\overline{X}_t)\Big[d\psi(\overline{X}_t) + \frac{1}{2} d\big\langle \psi(\overline{X}), \psi(\overline{X})\big\rangle_t\Big] \\ &= g(\overline{X}_t)\Big[\mathcal{L}^{R,P}\psi(\overline{X}_t)\,dt + dM^\psi_t + \frac{1}{2}|\widetilde{\nabla}^{R,P}\psi|^2_{\mathsf{a}}(\overline{X}_t)\,dt\Big], \quad \overline{P}\text{-}a.e., \end{aligned}$$

where we used $d\psi(\overline{X}_t) = \mathcal{L}^{R,P}\psi(\overline{X}_t)\,dt + dM^\psi_t$ with $M^\psi$ a $P$-local $R$-martingale, which is an alternate statement for $\psi \in \operatorname{dom}\mathcal{L}^{R,P}$, and we also wrote $d\big\langle \psi(\overline{X}), \psi(\overline{X})\big\rangle_t = |\widetilde{\nabla}^{R,P}\psi|^2_{\mathsf{a}}(\overline{X}_t)\,dt$ which is a direct consequence of (5.20) and (5.23). Finally, with (5.10): $\big(\mathcal{L}^{R,P}\psi + |\widetilde{\nabla}^{R,P}\psi|^2_{\mathsf{a}}/2\big)(\overline{X}) = -V(\overline{X})$, we arrive at

$$dg(\overline{X}_t) = -V\,g(\overline{X}_t)\,dt + dM^g_t, \quad \overline{P}\text{-}a.e.,$$

where $dM^g_t = g(\overline{X}_t)\,dM^\psi_t$ is the infinitesimal increment of a $P$-local $R$-martingale. This completes the proof of the theorem. $\square$

*Remark* 5.25. This proof relies on Theorem 5.9 about the HJB equation. We didn't find a direct proof keeping the same minimal hypotheses. The main advantage of starting from HJB is the identification based on Girsanov theory which led us to (5.21) and (5.23).

## 6. Growth conditions

We present at Theorem 6.10 and Theorem 6.26 some growth conditions on the coefficients $\mathsf{a}, \mathsf{b}, V, g_T$ and $f_0$ which are sufficient for $H(P|R) < \infty$.
Consider the Markov diffusion generator

$$A^U := \partial_t + \mathsf{v}^U \cdot \nabla + \Delta_{\mathsf{a}}/2,$$

where the velocity field is of the gradient form

$$\mathsf{v}^U(t,x) := -\mathsf{a}(t,x)\nabla U(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^n,$$

with $U$ some differentiable numerical function. The reference measure in next Theorem 6.10 solves

$$R = \mathrm{MP}(\mathsf{a}, \mathsf{v}^U + \mathsf{v}_*)$$

with $\mathsf{v}_*$ a range($\mathsf{a}$)-valued bounded vector field without any regularity.
In this section, it is not necessary to assume that $\mathsf{a}$ is invertible.

**Hypotheses 6.1.**

*(i)* $U \in C^{1,2}([0,T] \times \mathbb{R}^n)$,
*(ii)* $\mathsf{a} = \sigma\sigma^{\mathsf{t}}$ *for some* $\sigma$ *which is locally Lipschitz:*

$$\sup_{x,y \in B, x \neq y, 0 \leqslant t \leqslant T} \frac{|\sigma(t,y) - \sigma(t,x)|}{|y-x|} < \infty, \text{ for any bounded subset } B \subset \mathbb{R}^n,$$

*(iii) for some* $K \geqslant 0$, $\sup_{0 \leqslant t \leqslant T}\{x \cdot \mathsf{v}^U(t,x) + \mathrm{tr}\, \mathsf{a}(t,x)\} \leqslant K(1+|x|^2)$ *for all* $x \in \mathbb{R}^n$.
*(iv) The measurable vector field* $\mathsf{v}_* : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *is bounded in the sense that*

$$\sup_{[0,T] \times \mathbb{R}^n} |\mathsf{v}_*|_{\mathsf{g}} < \infty. \tag{6.2}$$

We denote $\mathsf{g} := \mathsf{a}^{-1}$ the generalized inverse of $\mathsf{a}$ and $|\mathsf{v}|^2_{\mathsf{g}} := \mathsf{v} \cdot \mathsf{g}\mathsf{v}$ with $|\mathsf{v}|_{\mathsf{g}} = \infty$ when $\mathsf{v}$ is outside the range of $\mathsf{a}$.

It is a standard result that under the hypotheses (i), (ii) and (iii), any martingale problem associated with the characteristics $\mathsf{v}^U$ and $\mathsf{a}$ admits a unique solution. The uniqueness is a consequence of the Lipschitz hypothesis 6.1-(ii). It implies in particular (4.7). Let us denote by

$$R^U = \mathrm{MP}(\mathsf{a}, \mathsf{v}^U; \mathsf{m}^U),$$

the solution of this martingale problem with some initial marginal $\mathsf{m}^U$. It is also true that adding the hypothesis (iv), for any initial marginal $\mathsf{m}_*$ the following martingale problem admits a unique (in the sense of (4.7)) solution

$$R = \mathrm{MP}(\mathsf{a}, \mathsf{v}^U + \mathsf{v}_*; \mathsf{m}_*). \tag{6.3}$$

Let us give some details about this last assertion, when $\mathsf{m}_* \ll \mathsf{m}^U$ to keep minimal notation. We denote

$$\beta := \mathsf{g}\mathsf{v}_*,$$

recall that $\mathsf{v}_*$ is assumed to live in range($\mathsf{a}$). By Novikov's criterion, Girsanov's formula is valid under the finite energy estimate

$$\sup \int_{[0,T]} |\beta_t|^2_{\mathsf{a}}\, dt \leqslant T \sup |\mathsf{v}_*|^2_{\mathsf{g}} < \infty, \tag{6.4}$$

implied by the hypothesis (6.2). This formula is:

$$\frac{dR}{dR^U} = \frac{d\mathsf{m}_*}{d\mathsf{m}^U}(X_0) Z_T^{(\beta)}, \tag{6.5}$$

where we set

$$Z_t^{(\xi)} := \exp\Big(\int_{[0,t]} \xi_s \cdot dM_s^{R^U} - \int_{[0,t]} |\xi_s|_{\mathsf{a}}^2/2\, ds\Big), \quad 0 \leqslant t \leqslant T, \tag{6.6}$$

with $dM_t^{R^U} = dX_t - \mathsf{v}^U(X_t)\, dt$ the increment of a local $R^U$-martingale.
The notation $Z^{(\xi)}$ will be used again later in the proof of Theorem 6.10. For any adapted process $\xi$, $Z^{(\xi)}$ an $R^U$-supermartingale. When $\xi = \beta$ and under (6.4), it is a genuine martingale.

Now, we establish some preliminaries for the proof of Theorem 6.10 at Lemma 6.7 and Lemma 6.9.

**Lemma 6.7.** *For any function $H$ in $C^{1,2}([0,T] \times \mathbb{R}^n)$, we set*

$$\mathcal{H} := e^H A^U e^{-H} = -\partial_t H + \mathsf{a}\nabla U \cdot \nabla H - \Delta_{\mathsf{a}} H/2 + |\nabla H|_{\mathsf{a}}^2/2.$$

*The process*

$$Z_t = \exp\left(H(\overline{X}_0) - H(\overline{X}_t) - \int_{[0,t]} \mathcal{H}(\overline{X}_s)\, ds\right), \quad 0 \leqslant t \leqslant T, \tag{6.8}$$

*is a $R^U$-supermartingale. In particular, if $\int_{\mathbb{R}^n} e^{-2H_0}\, d\mathsf{m}^U < \infty$, then*

$$E_{R^U} \exp\left(-H(\overline{X}_0) - H(\overline{X}_T) - \int_{[0,T]} \mathcal{H}(\overline{X}_t)\, dt\right) < \infty.$$

*Proof.* Because it is a nonnegative local $R^U$-martingale, the process

$$Z_t := \exp\left(-\int_{[0,T]} \nabla H(\overline{X}_t) \cdot (dX_t - \mathsf{v}^U(\overline{X}_t)\, dt) - \int_{[0,T]} |\nabla H|_{\mathsf{a}}^2(\overline{X}_t)/2\, dt\right), \qquad 0 \leqslant t \leqslant T,$$

is a $R^U$-supermartingale. With Itô formula

$$-\int_{[0,T]} \nabla H(\overline{X}_t) \cdot dX_t = H(\overline{X}_0) - H(\overline{X}_T) + \int_{[0,T]} \{\partial_t + \Delta_{\mathsf{a}}/2\} H(\overline{X}_t)\, dt, \quad R^0\text{-a.e.},$$

we see that $Z$ is expressed by (6.8). This implies

$$\begin{aligned} E_{R^U} \exp\left(-H(\overline{X}_0) - H(\overline{X}_T) - \int_{[0,T]} \mathcal{H}(\overline{X}_t)\, dt\right) &= E_{R^U}[e^{-2H(\overline{X}_0)} Z_T] \\ &\leqslant E_{R^U}[e^{-2H(\overline{X}_0)} Z_0] = E_{R^U} e^{-2H(\overline{X}_0)} = \int_{\mathbb{R}^n} e^{-2H_0}\, d\mathsf{m}^U < \infty, \end{aligned}$$

where the inequality is due to the supermartingale property of $Z$. □

**Lemma 6.9.** *For all measures $\mathsf{r}, \mathsf{q}$ and any probability measure $\mathsf{p}$ on the same measurable space, such that $\mathsf{p} \ll \mathsf{q} \ll \mathsf{r}$, $E_{\mathsf{p}} \log_+(d\mathsf{p}/d\mathsf{q}) < \infty$ and $E_{\mathsf{q}} \max(1, d\mathsf{q}/d\mathsf{r}) < \infty$, we have:*

$$H(\mathsf{p}|\mathsf{r}) \leqslant 2H(\mathsf{p}|\mathsf{q}) + E_{\mathsf{q}}\,(d\mathsf{q}/d\mathsf{r}) \in [-\infty, \infty),$$

*where we set $H(\mathsf{p}|\mathsf{r}) = -\infty$ when $E_{\mathsf{p}} \log_-(d\mathsf{p}/d\mathsf{r}) = \infty$ and $E_{\mathsf{p}} \log_+(d\mathsf{p}/d\mathsf{r}) < \infty$.*

*Proof.* We start considering positive parts of integrands to manipulate well-defined integrals:

$$E_{\mathsf{p}} \log_+(d\mathsf{p}/d\mathsf{r}) = E_{\mathsf{p}} \log_+(d\mathsf{p}/d\mathsf{q} \times d\mathsf{q}/d\mathsf{r}) \leqslant E_{\mathsf{p}} \log_+(d\mathsf{p}/d\mathsf{q}) + E_{\mathsf{p}} \log_+(d\mathsf{q}/d\mathsf{r}).$$

With the convex inequality:

$$ab \leqslant a\log a + e^{b-1} \leqslant a \log a + e^b \leqslant a \log_+ a + e^b, \quad \text{for all } a \geqslant 0, b \in \mathbb{R},$$

we see that

$$E_{\mathsf{p}} \log_+(d\mathsf{q}/d\mathsf{r}) = E_{\mathsf{q}}\big(d\mathsf{p}/d\mathsf{q} \times \log_+(d\mathsf{q}/d\mathsf{r})\big) \leqslant E_{\mathsf{p}} \log_+(d\mathsf{p}/d\mathsf{q}) + E_{\mathsf{q}} \max(1, d\mathsf{q}/d\mathsf{r}) < \infty$$

which is finite by hypothesis. It follows that the integrals $H(\mathsf{p}|\mathsf{r}) := E_{\mathsf{p}} \log(d\mathsf{p}/d\mathsf{r})$, $H(\mathsf{p}|\mathsf{q}) := E_{\mathsf{p}} \log(d\mathsf{p}/d\mathsf{q})$ and $E_{\mathsf{p}} \log(d\mathsf{q}/d\mathsf{r})$ are well-defined in $[-\infty, \infty)$ and we are allowed to write

$$H(\mathsf{p}|\mathsf{r}) = E_{\mathsf{p}} \log(d\mathsf{p}/d\mathsf{r}) = E_{\mathsf{p}} \log(d\mathsf{p}/d\mathsf{q}) + E_{\mathsf{p}} \log(d\mathsf{q}/d\mathsf{r}) = H(\mathsf{p}|\mathsf{q}) + E_{\mathsf{p}} \log(d\mathsf{q}/d\mathsf{r}).$$

Using the convex inequality again, we obtain

$$E_{\mathsf{p}} \log(d\mathsf{q}/d\mathsf{r}) = E_{\mathsf{q}}\big(d\mathsf{p}/d\mathsf{q} \times \log(d\mathsf{q}/d\mathsf{r})\big) \leqslant H(\mathsf{p}|\mathsf{q}) + E_{\mathsf{q}}(d\mathsf{q}/d\mathsf{r})$$

and conclude plugging this estimate into the above identity. □

Besides $U$, let us introduce another function $U^\diamond \in C^{1,2}([0,T] \times \mathbb{R}^n)$ and denote

$$\begin{aligned}\mathcal{U} &:= -e^U A^U e^{-U} = |\nabla U|^2_{\mathsf{a}}/2 - \partial_t U - \Delta_{\mathsf{a}} U/2,\\ \mathcal{U}^\diamond &:= -e^{U^\diamond} A^{U^\diamond} e^{-U^\diamond} = |\nabla U^\diamond|^2_{\mathsf{a}}/2 - \partial_t U^\diamond - \Delta_{\mathsf{a}} U^\diamond/2.\end{aligned}$$

**Theorem 6.10.** *Let* $\mathsf{a}$, $U$ *and* $\mathsf{v}_*$ *entering the definition of* $R$ *at* (6.3) *satisfy the Hypotheses 6.1. Take a function* $U^\diamond$ *in* $C^{1,2}([0,T] \times \mathbb{R}^n)$ *and a nonnegative measurable function* $U_* : \mathbb{R}^n \to [0, \infty)$ *verifying*

$$\begin{gathered}(a) \quad \int_{\mathbb{R}^n} e^{2U_0 - 2U_0^\diamond}\, d\mathsf{m}^U < \infty, \qquad (b) \int_{\mathbb{R}^n} U_* e^{2U_0 - 2U_0^\diamond}\, d\mathsf{m}^U < \infty,\\ (c) \quad \int_{\mathbb{R}^n} e^{-U_*}\, d\mathsf{m}^U < \infty,\end{gathered} \tag{6.11}$$

*recall the notation* $\mathsf{m}^U := R_0^U$. *We assume that* $R_0 := \mathsf{m}_*$ *is absolutely continuous with respect to* $\mathsf{m}^U$ *and satisfies*

$$\sup \frac{|\log(d\mathsf{m}_*/d\mathsf{m}^U)|}{1 + U_*} < \infty. \tag{6.12}$$

*Let* $f_0, g_T$ *and* $V$ *verify* (5.2)*-(c) and suppose that there exist* $c, \kappa \geqslant 0$ *such that*

$$\begin{aligned} f_0 \log_+ f_0 &\leqslant \kappa \exp(U_0 - U_0^\diamond),\\ g_T \log_+ g_T &\leqslant \kappa \exp(U_T - U_T^\diamond),\\ V + \log(1 + V_+) &\leqslant \mathcal{U} - \mathcal{U}^\diamond + c.\end{aligned} \tag{6.13}$$

*Then* $P$ *defined by* (5.1) *can be normalized as a probability measure and the relative entropy* $H(P|R)$ *is finite.*

*Proof.* We shall prove in a moment that $E_P \log_+(dP/dR^U) < \infty$. Since its proof does not require a priori that $P$ is a finite measure, this estimate implies that $dP/dR^U$ is finite. It follows with (6.19) that $dP/dR$ is also finite, implying that (5.2)-(a,b) is satisfied.

We divide the rest of the proof into five steps. First, we consider the reference measure $R = R^U$ and show that under the assumption (6.13) where $U^\diamond$ and $U_*$ satisfy (6.11), we have $H(P|R^U) < \infty$, requiring two steps: (i): $E_P \log_+(dP/dR^U) < \infty$, and (ii): $E_P \log_-(dP/dR^U) < \infty$. Then, we introduce $\mathsf{v}_*$ and consider a specific $\mathsf{m}_*$, given at (6.18). We first show that (iii): $H(P|R) < \infty$ , then (iv): $H(P|R) > -\infty$ with this $\mathsf{m}_*$. Finally, (v): we extend the result to the case where $\mathsf{m}_*$ satisfies (6.12).

• Proof of $E_P \log_+(dP/dR^U) < \infty$. By definition (5.1) of $P$, with notation (5.8)

$$\begin{aligned}\log_+\left(\frac{dP}{dR^U}\right) &= \left(\varphi_0(X_0) + \psi_T(X_T) + \int_{[0,T]} V(\overline{X}_t)\,dt\right)_+ \\ &\leqslant A := [\varphi_0]_+(X_0) + [\psi_T]_+(X_T) + \Big[\int_{[0,T]} V(\overline{X}_t)\,dt\Big]_+. \end{aligned}\tag{6.14}$$

As

$$E_P \log_+\left(\frac{dP}{dR^U}\right) \leqslant E_P A = E_{R^U} \exp\left(\log A + \varphi_0(X_0) + \psi_T(X_T) + \int_{[0,T]} V(\overline{X}_t)\,dt\right),$$

by Lemma 6.7 for the estimate (+) to hold it suffices that

$$\log A + \varphi_0(X_0) + \psi_T(X_T) + \int_{[0,T]} V(\overline{X}_t)\,dt \leqslant -H_0(X_0) - H_T(X_T) - \int_{[0,T]} \mathcal{H}(\overline{X}_t)\,dt + c,$$

for some real $c \geqslant 0$ and some function $H$ in $C^{1,2}([0,T]\times\mathbb{R}^n)$ verifying

$$\int_{\mathbb{R}^n} e^{-2H_0}\,d\mathsf{m}^U < \infty. \tag{6.15}$$

Since

$$\log A \leqslant \log(1+[\varphi_0]_+(X_0)) + \log(1+[\psi_T]_+(X_T)) + \log\left(1 + \Big[\int_{[0,T]} V(\overline{X}_t)\,dt\Big]_+\right),$$

this is implied by

$$\begin{aligned}\varphi_0(X_0) + \log(1+[\varphi_0]_+(X_0)) &\leqslant -H_0(X_0) + c,\\ \psi_T(X_T) + \log(1+[\psi_T]_+(X_T)) &\leqslant -H_T(X_T) + c,\\ \int_{[0,T]} V(\overline{X}_t)\,dt + \log\left(1 + \Big[\int_{[0,T]} V(\overline{X}_t)\,dt\Big]_+\right) &\leqslant -\int_{[0,T]} \mathcal{H}(\overline{X}_t)\,dt + c.\end{aligned}$$

Because $[\int_{[0,T]} V(\overline{X}_t)\,dt]_+ \leqslant \int_{[0,T]} V_+(\overline{X}_t)\,dt$, for the above inequalities to be fulfilled, it is enough that there exists $c \geqslant 0$ such that

$$\begin{aligned}\varphi_0 + \log(1+[\varphi_0]_+) &\leqslant -H_0 + c,\\ \psi_T + \log(1+[\psi_T]_+) &\leqslant -H_T + c,\\ V + \log(1+V_+) &\leqslant -\mathcal{H} + c.\end{aligned}\tag{6.16}$$

Writing

$$H = -U + U^\diamond,$$

these inequalities are (6.13) because

$$\begin{aligned}-\mathcal{H} &= \partial_t(-U+U^\diamond) - \mathsf{a}\nabla U\cdot\nabla(-U+U^\diamond) + \Delta_\mathsf{a}(-U+U^\diamond)/2 - |\nabla(-U+U^\diamond)|_\mathsf{a}^2/2\\ &= -\partial_t U - \Delta_\mathsf{a} U/2 + |\nabla U|_\mathsf{a}^2/2 + \partial_t U^\diamond + \Delta_\mathsf{a} U^\diamond/2 - |\nabla U^\diamond|^2/2 = \mathcal{U} - \mathcal{U}^\diamond.\end{aligned}$$

Finally (6.15) becomes $\int_{\mathbb{R}^n} e^{-2U_0^\diamond + 2U_0}\,d\mathsf{m}^U < \infty$, which is (6.11)-(a). These last considerations prove that (6.11)-(a) and (6.13) imply $E_P \log_+(dP/dR^U) < \infty$.

• Proof of $E_P \log_-(dP/dR^U) < \infty$. By Proposition 4.2 it is enough to obtain

$$E_P U_*(X_0) < \infty \tag{6.17}$$

because it is assumed at (6.11)-(c) that $E_{R^U} e^{-U_*(X_0)} < \infty$. We have

$$\begin{aligned}
E_P U_*(X_0) &= E_{R^U}\left[U_*(X_0)\exp\Big(\varphi_0(X_0)+\psi_T(X_T)+\int_{[0,T]} V(\overline{X}_t)\,dt\Big)\right]\\
&\overset{(a)}{\leqslant} E_{R^U}\left[U_*(X_0)\exp\Big(-H_0(X_0)-H_T(X_T)-\int_{[0,T]}\mathcal{H}(\overline{X}_t)\,dt+3c\Big)\right]\\
&\overset{(b)}{=} e^{3c}E_{R^U}\left[U_*(X_0)\exp(-2H_0(X_0))Z_T\right]\\
&\overset{(c)}{\leqslant} e^{3c}E_{R^U}\left[U_*(X_0)\exp(-2H_0(X_0))\right]\\
&\overset{(d)}{=} e^{3c}\int_{\mathbb{R}^n} U_* e^{-2H_0}\,d\mathsf{m}^U = e^{3c}\int_{\mathbb{R}^n} U_* e^{2U_0-2U_0^\diamond}\,d\mathsf{m}^U.
\end{aligned}$$

We used (6.16) at (a), the definition (6.8) of the process $Z$ at (b), the fact that $Z = Z^{(-\nabla H)}$ is an $R^U$-supermartingale (see the proof of Lemma 6.7) at (c), and $R_0^U = \mathsf{m}^U$ at (d). Finally, our assumption (6.11)-(b) implies that $E_P U_*(X_0)$ is finite, completing the proof of $E_P \log_-(dP/dR^U) < \infty$ and $-\infty < H(P|R^U) < \infty$.

Now, we consider the "bounded perturbation" $R$ of $R^U$ under the assumption (6.2). We choose for a while

$$\mathsf{m}_* = e^{U_*}\,\mathsf{m}^U. \tag{6.18}$$

This will be relaxed later.

• Proof of $H(P|R) < \infty$ under (6.18). We already know that $H(P|R^U) < \infty$. Hence, Lemma 6.9 gives us

$$-\infty \leqslant H(P|R) \leqslant 2H(P|R^U) + E_{R^U}(dR^U/dR) < \infty,$$

provided that

$$E_{R^U}(dR^U/dR) < \infty. \tag{6.19}$$

Let us show that our assumptions (6.2) and (6.18) imply (6.19).
The identities $dX_t = \mathsf{v}(\overline{X}_t)\,dt + dM_t^R$, $R$-a.e., $dX_t = \mathsf{v}^U(X_t)\,dt + dM_t^{R^U}$, $R^U$-a.e., with $\mathsf{v} = \mathsf{v}^U + \mathsf{v}_* = \mathsf{v}^U + \mathsf{a}\beta$, imply that $dM^R = dM^{R^U} - \mathsf{a}\beta\,dt$. With Girsanov's formula (6.5), we have

$$\begin{aligned}
&E_{R^U}(dR^U/dR)\\
&\quad= E_{R^U}\left[\frac{d\mathsf{m}^U}{d\mathsf{m}_*}(X_0)\exp\Big(-\int_{[0,T]}\beta_t\cdot dM_t^R-\int_{[0,T]}|\beta|_{\mathsf{a}}^2/2\,dt\Big)\right]\\
&\quad= E_{R^U}\left[\frac{d\mathsf{m}^U}{d\mathsf{m}_*}(X_0)\exp\Big(-\int_{[0,T]}\beta_t\cdot dM_t^{R^U}+\int_{[0,T]}|\beta|_{\mathsf{a}}^2/2\,dt\Big)\right]\\
&\quad= E_{R^U}\left[\frac{d\mathsf{m}^U}{d\mathsf{m}_*}(X_0)Z_T^{(-\beta)}\exp\Big(\int_{[0,T]}|\beta|_{\mathsf{a}}^2\,dt\Big)\right] \leqslant \exp(T\sup|\beta|_{\mathsf{a}}^2)E_{R^U}\left[\frac{d\mathsf{m}^U}{d\mathsf{m}_*}(X_0)Z_T^{(-\beta)}\right]\\
&\quad\leqslant \exp(T\sup|\beta|_{\mathsf{a}}^2)E_{R^U}\Big(\frac{d\mathsf{m}^U}{d\mathsf{m}_*}(X_0)\Big) = \exp(T\sup|\mathsf{v}_*|_{\mathsf{g}}^2)\int_{\mathbb{R}^n}\frac{d\mathsf{m}^U}{d\mathsf{m}_*}\,d\mathsf{m}^U,
\end{aligned}$$

where last inequality holds because $Z^{(-\beta)}$ is a $R^U$-supermartingale, recall notation (6.6). It is assumed that $\int_{\mathbb{R}^n} e^{-U_*}\,d\mathsf{m}^U < \infty$ and we know that $\int_{\mathbb{R}^n} U_*\,dP_0 < \infty$, see (6.17). Hence $\int_{\mathbb{R}^n} d\mathsf{m}^U/d\mathsf{m}_*\,d\mathsf{m}^U = \int_{\mathbb{R}^n} e^{-U_*}\,d\mathsf{m}^U < \infty$, proving (6.19) when (6.18) holds.

• Proof of $H(P|R) > -\infty$ under (6.18). By Proposition 4.2, this will be done by showing that

$$E_R e^{-2U_*(X_0)} < \infty,$$

because we already know that $E_P U_*(X_0) < \infty$, see (6.17). Let us control

$$E_R e^{-2U_*(X_0)} = E_{R^U}\left(\frac{dR}{dR^U} e^{-2U_*(X_0)}\right) \leqslant E_{R^U}\left[\left(\frac{dR}{dR^U}\right)^2 e^{-3U_*(X_0)}\right]^{1/2} \left(E_{R^U} e^{-U_*(X_0)}\right)^{1/2}$$

by Cauchy-Schwarz inequality. As

$$\begin{aligned}\left(\frac{dR}{dR^U}\right)^2 = \left(\frac{d\mathsf{m}_*}{d\mathsf{m}^U}(X_0)\ Z^{(\beta)}\right)^2 &= \left(\frac{d\mathsf{m}_*}{d\mathsf{m}^U}(X_0)\right)^2 Z_T^{(2\beta)} \exp\left(\int_{[0,T]} |\beta_t|^2_{\mathsf{a}}\, dt\right)\\ &\leqslant \exp\left(T \sup |\beta|^2_{\mathsf{a}}\right) \left(\frac{d\mathsf{m}_*}{d\mathsf{m}^U}(X_0)\right)^2 Z_T^{(2\beta)}\end{aligned}$$

where $Z^{(2\beta)}$ is a $R^U$-supermartingale, we obtain

$$\begin{aligned}E_{R^U}\left[\left(\frac{dR}{dR^U}\right)^2 e^{-3U_*(X_0)}\right] &\leqslant \exp\left(T \sup |\beta|^2_{\mathsf{a}}\right) \int_{\mathbb{R}^n} \left(\frac{d\mathsf{m}_*}{d\mathsf{m}^U}\right)^2 e^{-3U_*}\, d\mathsf{m}^U\\ &= \exp\left(T \sup |\beta|^2_{\mathsf{a}}\right) \int_{\mathbb{R}^n} e^{-U_*}\, d\mathsf{m}^U,\end{aligned}$$

showing that

$$E_R e^{-2U_*(X_0)} \leqslant \exp\left(T \sup |\mathsf{v}_*|^2_{\mathsf{g}}/2\right) \int_{\mathbb{R}^n} e^{-U_*}\, d\mathsf{m}^U < \infty$$

and completing the proof of

$$-\infty < H(P|R) < \infty$$

when $\mathsf{m}_*$ is given by (6.18).

• Relaxation of (6.18). Let us extend this property to any $\mathsf{m}_*$ satisfying (6.12). As for any positive function $r$ on $\mathbb{R}^n$ we have

$$H(P|r(X_0)\, R) = H(P|R) - E_P \log r(X_0),$$

one can extend our previous result from $R$ to $r(X_0)\, R$ provided that $E_P|\log r(X_0)| < \infty$. Simply requiring that $|\log r| \leqslant \kappa(1 + U_*)$ for some $\kappa \geqslant 0$ so that $E_P|\log r(X_0)| \leqslant \kappa(1+E_P U_*(X_0)) < \infty$, permits us to extend our result from $\mathsf{m}_* = e^{U_*}\, \mathsf{m}^U$ to $\mathsf{m}_* = r e^{U_*}\, \mathsf{m}^U$ with $\sup\{|\log r|/(1 + U_*)\} < \infty$, which is the assumption (6.12). $\square$

*Remarks* 6.20. (a) In the case where $R^U$, hence $\mathsf{m}^U$, is a bounded measure, one can choose $U_* = 0$ because it verifies $\int_{\mathbb{R}^n} e^{-U_*}\, d\mathsf{m}^U < \infty$. Moreover, the assumption (6.11) simplifies as $\int_{\mathbb{R}^n} e^{2U_0 - 2U_0^\diamond}\, d\mathsf{m}^U < \infty$ and (6.12) becomes $\sup|\log(d\mathsf{m}_*/d\mathsf{m}^U)| < \infty$.

(b) One can always choose $\mathsf{m}_* = \mathsf{m}^U$ in (6.12).

(c) There is some freedom in the choice of $\mathsf{m}^U$ because the initial measure $R_0$ does not appear explicitly in the Feynman-Kac equation (FK) and it is erased by the conditional expectation in the Feynman-Kac formula (1.1). A possible choice is

$$\mathsf{m}^U = e^{-2U_0}\, \mathrm{Leb}.$$

In this case, the hypotheses (6.11) become

$$\int_{\mathbb{R}^n} e^{-2U_0^\diamond}\, d\mathrm{Leb} < \infty, \qquad \int_{\mathbb{R}^n} U_* e^{-2U_0^\diamond}\, d\mathrm{Leb} < \infty, \qquad \int_{\mathbb{R}^n} e^{-2U_0 - U_*}\, d\mathrm{Leb} < \infty,$$

and when $\int_{\mathbb{R}^n} e^{-2U_0}\,d\mathrm{Leb} < \infty$, taking $U_* = 0$, a possible choice for $U^\diamond$ is

$$U^\diamond(t,x) = U_0^\diamond(x) = \gamma \log \sqrt{1+|x|^2}, \qquad \text{with } \gamma > n/2$$

to ensure $\int_{\mathbb{R}^n} e^{-2U_0^\diamond}\,d\mathrm{Leb} < \infty$. It gives

$$\mathcal{U}^\diamond(t,x) = \frac{\gamma(2+\gamma)|x|^2_{\mathsf{a}(t,x)} - \gamma \operatorname{tr}\mathsf{a}(t,x)(1+|x|^2)}{2(1+|x|^2)^2}.$$

(d) Because for any $\varepsilon > 0$ and $0 < q \leqslant 1$, there is some $c(\varepsilon) \geqslant 0$ such that

$$a + \log(1+a_+) \leqslant a + \varepsilon a_+^q + c(\varepsilon), \quad \forall a \in \mathbb{R},$$

for the upper bounds (6.13) to hold, it suffices that there exist $\varepsilon > 0$, $0 < q \leqslant 1$ and $c \geqslant 0$ such that

$$\begin{aligned} \varphi_0 + \varepsilon[\varphi_0]_+^q &\leqslant U_0 - U_0^\diamond + c, \\ \psi_T + \varepsilon[\psi_T]_+^q &\leqslant U_T - U_T^\diamond + c, \\ V + \varepsilon V_+^q &\leqslant \mathcal{U} - \mathcal{U}^\diamond + c. \end{aligned}$$

Roughly speaking, the hypothesis (6.13) imposes that $V$ should not grow faster than the opposite of the confinement potential $\mathcal{U}$ associated to $R^U$. It implies in particular that $V$ is locally upper bounded because $\mathcal{U}$ is continuous. Next result presents a set of hypotheses where this is relaxed.

**Kato class.** It is known since the article [28] by Khas'minskii that if for some $\tau > 0$,

$$\sup_{x\in\mathbb{R}^n} \sup_{0\leqslant s<s+\tau\leqslant T} E_Q\left( \int_{[s,s+\tau]} |W|(\overline{X}_t)\,dt \,\middle|\, X_s = x \right) =: \alpha < 1,$$

then

$$\exp(\Lambda(\tau)) := \sup_{x\in\mathbb{R}^n} \sup_{0\leqslant s<s+\tau\leqslant T} E_Q\left[ \exp\left( \int_{[s,s+\tau]} e^{|W|(\overline{X}_t)}\,dt \right) \,\middle|\, X_s = x \right] \leqslant \frac{1}{1-\alpha}.$$

The Markov property of $Q$ is essential to prove this result. On the other hand, the Markov property also implies that $\tau \mapsto \Lambda(\tau)$ is subadditive. Consequently, the finiteness of $\Lambda(\tau)$ for a small enough positive $\tau$ implies that there exists $c > 0$ such that $\Lambda(t) \leqslant c + ct$ for all $0 \leqslant t \leqslant T$, proving that

$$\sup_{x\in\mathbb{R}^n} E_Q\left[ \exp\left( \int_{[0,T]} e^{|W|(\overline{X}_t)}\,dt \right) \,\middle|\, X_0 = x \right] \leqslant e^{c+cT}. \tag{6.21}$$

**Definition 6.22** (Kato class)**.** *A measurable function $W : [0,T]\times\mathbb{R}^n \to (-\infty,\infty)$ belongs to the Kato class $J(Q)$ of the Markov measure $Q \in \mathrm{M}(\Omega)$ if*

$$\sup_{x\in\mathbb{R}^n} E_Q\left( \int_{[0,T]} |W|(\overline{X}_t)\,dt \,\middle|\, X_0 = x \right) < \infty$$

*and*

$$\lim_{h\downarrow 0} \sup_{x\in\mathbb{R}^n} \sup_{0\leqslant s<s+h\leqslant T} E_Q\left( \int_{[s,s+h]} |W|(\overline{X}_t)\,dt \,\middle|\, X_s = x \right) = 0.$$

*We say that $W$ is in $J^*(Q)$ if the function $W^*$ defined by $W^*(t,x) := W(T-t,x)$, $(t,x) \in [0,T]\times\mathbb{R}^n$, is in $J(Q^*)$ where $Q^*$ is the time reversal of $Q$.*

It follows from the above considerations that any $W$ in $J(Q)$ verifies (6.21) and we see immediately that $J(Q)$ contains all the bounded functions.
Let us introduce the Feynman-Kac operators

$$Su(x) := E_Q\left[\exp\left(\int_{[0,T]} e^{|W|(\overline{X}_t)}\,dt\right)u(X_T)\,\middle|\, X_0 = x\right], \quad x \in \mathbb{R}^n, \tag{6.23}$$

and

$$S^*u(y) := E_Q\left[u(X_0)\,\exp\left(\int_{[0,T]} e^{|W|(\overline{X}_t)}\,dt\right)\,\middle|\, X_T = y\right], \quad y \in \mathbb{R}^n,$$

defined for any measurable function $u : \mathbb{R}^n \to \mathbb{R}$ such that the above conditional expectations are meaningful. Note that $Su$ is defined $Q_0$-*a.e.* and $S^*u$ is defined $Q_T$-*a.e.* Of course, if $Q$ is reversible and $W$ does not depend explicitly on the time variable, i.e. $W(t,x) = W(x)$, then $S^* = S$.

**Lemma 6.24.** *Let $Q \in \mathrm{M}(\Omega)$ be a Markov measure and $W : [0,T] \times \mathbb{R}^n \to (-\infty, \infty)$ a measurable function in the Kato class $J(Q) \cap J^*(Q)$. Then, for any $1 \leqslant p \leqslant \infty$, the linear operators $S : L^p(Q_T) \to L^p(Q_0)$ and $S^* : L^p(Q_0) \to L^p(Q_T)$ are bounded.*

*Proof.* By (6.21), $S : L^\infty(Q_T) \to L^\infty(Q_0)$ is a bounded operator with $\|S\|_{\infty,\infty} \leqslant e^{c+cT}$. Similarly, as $W \in J^*(Q)$, we see by time-reversal that $S^* : L^\infty(Q_0) \to L^\infty(Q_T)$ is a bounded operator with $\|S^*\|_{\infty,\infty} \leqslant e^{c^*+c^*T}$. For any $g \in L^1(Q_T)$,

$$\int_{\mathbb{R}^n} |Sg|\,dQ_0 \leqslant \int_{\mathbb{R}^n} S|g|\,dQ_0 = \int_{\mathbb{R}^n} |g| S^*\mathbf{1}\,dQ_T \leqslant \|S^*\|_{\infty,\infty}\,\|g\|_{L^1(Q_T)},$$

proving that $S : L^1(Q_T) \to L^1(Q_0)$ is a bounded operator with $\|S\|_{1,1} \leqslant \|S^*\|_{\infty,\infty}$. The term $Sg$ in the first integral is justified a posteriori by the finiteness of the second integral. The equality follows from the properties of the conditional expectations of nonnegative functions.
By the Riesz-Thorin interpolation theorem it is also true that for any $1 \leqslant p \leqslant \infty$, $S : L^p(Q_T) \to L^p(Q_0)$ is a bounded operator, and a similar proof works with $S^*$. □

It is uneasy to obtain practical sufficient conditions for a function to belong to $J(Q)$ except when some upper bound for the transition kernel is known. Of course, the Gaussian case is well understood, corresponding to $Q$ being the law of a Brownian or an Ornstein-Uhlenbeck process. For a clear exposition of the properties of the Kato class and the semigroup of Feynman-Kac operators of the Wiener measure, see [8, Ch. 3]. In particular, when $Q$ is the Wiener measure $\mathcal{W}^\epsilon$ of the reversible Brownian motion with diffusion coefficient $\epsilon > 0$, that is

$$\mathcal{W}^\epsilon = \mathrm{MP}(\epsilon \mathrm{Id}, 0; \mathrm{Leb}),$$

with Markov generator $\epsilon\Delta/2$ and the Lebesgue measure as its initial marginal, we know that $\mathcal{W}^\epsilon = (\mathcal{W}^\epsilon)^*$ and $W : \mathbb{R}^n \to \mathbb{R}$ is in $J(\mathcal{W}^\epsilon)$ if and only if

$$\lim_{\alpha \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|y-x| \leqslant \alpha} |\mathsf{g}(y-x)W(y)|\,dy = 0 \tag{6.25}$$

where $\mathsf{g}$ is related to the Green potential and defined by

$$\mathsf{g}(z) := \begin{cases} |z|^{2-n}, & n \neq 2; \\ \log(1/|z|), & n = 2. \end{cases}$$

**A variant of Theorem 6.10.** We are now ready to prove another criterion for $H(P|R^U) < \infty$ with a possibly locally unbounded potential $V$.

Taking $U = 0$ in the definition of $R^U$, we obtain a law $R^{0,\mathsf{a}} = \mathrm{MP}(\Delta_{\mathsf{a}}/2) = \mathrm{MP}(\mathsf{a}, 0)$ of a Markov Brownian martingale with diffusion matrix $\mathsf{a}$, whose generator is $A^0 = \partial_t + \Delta_{\mathsf{a}}/2$. In this section we prefer looking at the law

$$R^{\mathsf{a}} = \mathrm{MP}(\nabla\cdot(\mathsf{a}\nabla\bullet)/2; \mathrm{Leb})$$

of the Markov process with initial measure $R^{\mathsf{a}}_0 = \mathrm{Leb}$ and generator $\nabla\cdot(\mathsf{a}\nabla\bullet)/2$ in divergence form. When $\mathsf{a}$ does not depend on time explicitly, it is reversible with Lebesgue measure as reversing measure; in particular $J^*(R^{\mathsf{a}}) = J(R^{\mathsf{a}})$. In the more general case where $\mathsf{a}$ depends on $t$, $R^{\mathsf{a}}$ is not reversible anymore but its marginal flow remains constantly equal to Leb.

**Theorem 6.26.** *Let*

$$R = \mathrm{MP}\left((\mathsf{v}^U + \mathsf{v}_*)\cdot\nabla + \nabla\cdot(\mathsf{a}\nabla\bullet)/2\right)$$

*with* $\mathsf{a}$*,* $U$ *and* $\mathsf{v}_*$ *satisfying the Hypotheses 6.1. Let* $h_0, h_T$ *and* $U_*$ *be nonnegative measurable functions on* $\mathbb{R}^n$ *such that*

$$(1+U_*)h_0 \in L^p(\mathrm{Leb}), \qquad h_T \in L^{p'}(\mathrm{Leb}), \tag{6.27}$$

*with* $1 \leqslant p \leqslant \infty$*,* $1/p + 1/p' = 1$*, and*

$$\int_{\mathbb{R}^n} e^{-U_* - 2U_0}\, d\mathrm{Leb} < \infty. \tag{6.28}$$

*We assume that* $R_0$ *is absolutely continuous and satisfies*

$$\sup \frac{|\log(dR_0/d\mathrm{Leb}) + 2U_0|}{1+U_*} < \infty. \tag{6.29}$$

*Let* $f_0, g_T$ *and* $V$ *verify* (5.2)*-(c) and suppose that there exists a measurable function*

$$W \in J(R^{\mathsf{a}}) \cap J^*(R^{\mathsf{a}})$$

*such that*

$$f_0 \leqslant e^{U_0}\, h_0, \qquad g_T \leqslant e^{U_T}\, h_T, \qquad V + \log(1+V_+) \leqslant \mathcal{U} + W. \tag{6.30}$$

*Then* $P$ *defined by* (5.1) *can be normalized as a probability measure and the relative entropy* $H(P|R)$ *is finite.*

*Proof.* As in the proof of Theorem 6.10, the announced estimate $E_P \log_+(dP/dR) < \infty$ implies that $dP/dR$ is finite, hence (5.2)-(a,b) is satisfied.

• Proof of $E_P \log_+ \frac{dP}{dR^U} < \infty$. In this proof we change a little bit the path measure

$$R^U = \mathrm{MP}(\mathsf{v}^U\cdot\nabla + \nabla\cdot(\mathsf{a}\nabla\bullet)/2; \mathsf{m}^U), \qquad \mathsf{m}^U = e^{-2U_0}\,\mathrm{Leb}. \tag{6.31}$$

We replace $\Delta_{\mathsf{a}}$ by $\nabla\cdot(\mathsf{a}\nabla\bullet)$ but keep the same notation $R^U$.
Clearly (6.30) implies

$$\begin{aligned} &f_0 \log_+ f_0 + f_0 \leqslant 3e^{U_0}\, h_0, \quad g_T \log_+ g_T + g_T \leqslant 3e^{U_T}\, h_T, \\ &V + \log(1+V_+) \leqslant \mathcal{U} + W. \end{aligned} \tag{6.32}$$

As in the proof of Theorem 6.10 we obtain

$$\frac{dR^U}{dR^{\mathsf{a}}} = \frac{d\mathsf{m}^U}{d\mathrm{Leb}}(X_0) \exp\left(U(\overline{X}_0) - U(\overline{X}_T) - \int_{[0,T]} \mathcal{U}(\overline{X}_t)\, dt\right)$$

and with (6.32) we see that

$$\begin{aligned}
E_P \log_+ \frac{dP}{dR^U} &\leqslant E_{R^U} \exp\left(\log \frac{dP}{dR^U} + \log\left(1+\log_+ \frac{dP}{dR^U}\right)\right)\\
&\leqslant E_{R^U} \exp\left(\log A + \varphi_0(X_0) + \psi_T(X_T) + \int_{[0,T]} V(\overline{X}_t)\, dt\right)\\
&\leqslant 9\, E_{R^U} \left\{ \left(\frac{dR^U}{dR^{\mathsf{a}}}\right)^{-1} h_0(X_0) h_T(X_T) \exp\left(\int_{[0,T]} W(\overline{X}_t)\, dt\right)\right\}\\
&= 9\, E_{R^{\mathsf{a}}} \left\{ h_0(X_0) \exp\left(\int_{[0,T]} W(\overline{X}_t)\, dt\right) h_T(X_T)\right\}\\
&= 9\, E_{R^{\mathsf{a}}} \left\{ h_0(X_0)[S h_T](X_0)\right\},
\end{aligned}$$

where the notation $A$ is used at (6.14) and $S$ is the Feynman-Kac operator associated with $R^{\mathsf{a}}$ and $W \in J(R^{\mathsf{a}}) \cap J^*(R^{\mathsf{a}})$. We conclude with Lemma 6.24 under the assumption (6.27) that $E_P \log_+ \frac{dP}{dR^U} < \infty$.

• Proof of $E_P \log_- \frac{dP}{dR^U} < \infty$. By Proposition 4.2, under the assumption (6.27) it suffices to prove that $E_P U_*(X_0) < \infty$. Proceeding as above

$$\begin{aligned}
E_P U_*(X_0) &= E_{R^U} \left[ U_*(X_0) \exp\left(\varphi_0(X_0) + \psi_T(X_T) + \int_{[0,T]} V(\overline{X}_t)\, dt\right)\right]\\
&\leqslant 9\, E_{R^{\mathsf{a}}} \left\{ U_* h_0(X_0) [S h_T](X_0)\right\}
\end{aligned}$$

and we conclude with Lemma 6.24 under the assumption (6.27) that $E_P U_*(X_0) < \infty$.

• Proof of the rest. Already done at Theorem 6.10. □

*Remarks* 6.33.

(a) Since any Kato class contains all the bounded functions, in (6.30) it is valid to choose

$$W = c.$$

(b) Again, if $\int_{\mathbb{R}^n} e^{-2U_0}\, d\mathrm{Leb} < \infty$, taking $U_* = 0$ in (6.27) and (6.28) is all right.

**Corollary 6.34** (Classical cases)**.**

*(a)* Brownian motion. *When* $R = \mathrm{MP}(\epsilon\Delta/2; \mathrm{Leb})$ *with* $\epsilon > 0$, *for the relative entropy* $H(P|R)$ *to be finite, it suffices that for some* $1 \leqslant p \leqslant \infty$,

$$\begin{aligned}
&(1 + \log_+ |\bullet|) f_0 \in L^p(\mathrm{Leb}), && g_T \in L^{p'}(\mathrm{Leb}),\\
&\sup_{t\in[0,T]} V_-(t, \bullet) \in L^1_{\mathrm{loc}}(\mathrm{Leb}), && \sup_{t\in[0,T]} V_+(t, \bullet) \textit{ verifies } (6.25).
\end{aligned}$$

*(b)* Ornstein-Uhlenbeck process. *When* $R = \mathrm{MP}\left(-kx\cdot\nabla + \epsilon\Delta/2; \mathcal{N}_{\epsilon/(2k)}\right)$ *with* $\epsilon, k > 0$, *for the relative entropy* $H(P|R)$ *to be finite, it suffices that for some* $1 < p < \infty$ *and some* $W$ *verifying* (6.25)

$$f_0 \in L^p(\mathcal{N}_{\epsilon/(pk)}), \qquad g_T \in L^{p'}(\mathcal{N}_{\epsilon/(p'k)}), \qquad \sup_{t\in[0,T]} V_-(t, \bullet) \in L^1_{\mathrm{loc}}(\mathrm{Leb})$$

$$\sup_{t\in[0,T]} V_+(t, x) \leqslant \frac{k^2}{2\epsilon} |x|^2 - 2\log_+ |x| + W(x), \quad \forall x \in \mathbb{R}^n,$$

*where* $\mathcal{N}_a$ *stands for the normal distribution with zero mean and variance* $a\mathrm{Id}$.

*Proof.* • Proof of (a). Apply Theorem 6.26 with $\mathsf{v}_* = 0$, $U = 0$, $U_* = (n+1)\log_+ |\bullet|$, and remark that a Kato class is a vector space which is stable by the lattice operations. The assumption about $V_-$ implies (5.2)-(c).

• Proof of (b). Apply Theorem 6.26 with $\mathsf{v}_* = 0$, $U(x) = k|x|^2/(2\epsilon)$, $\mathcal{U}(x) = k^2|x|^2/(2\epsilon) - nk/2$ and $U_* = 0$. Remember that a Kato class is a lattice vector space which contains the constants, and take advantage of: $\forall a, b \geqslant 0, a \leqslant b - \log(1+b) \implies a + \log(1+a) \leqslant b$, applied to $a = \sup_t V_+$. The assumption about $V_-$ implies (5.2)-(c). □

*Remarks* 6.35.

(a) In both cases $R$ is chosen to be reversible.
(b) By time symmetry, (a) also holds if the hypothesis on $f_0, g_T$ is replaced by

$$f_0 \in L^p(\mathrm{Leb}), \qquad (1 + \log_+ |\bullet|) g_T \in L^{p'}(\mathrm{Leb}).$$

## Appendix A. Carré du champ

Lemma A.2 below is a simplified version of [4, Lemma 3.9], which was used during the proof of Lemma 4.30. For the confort of the reader, we give its detailed proof which is slightly simpler, but essentially the same as [4]'s one.

Let $Q \in \mathrm{M}(\Omega)$ be a conditionable path measure. Its forward carré du champ is defined by

$$\Gamma^Q(u,v) := \mathcal{L}^Q(uv) - u\mathcal{L}^Q v - v\mathcal{L}^Q u, \quad 0 \leqslant t \leqslant T,$$

for any functions $u, v$ in $\operatorname{dom} \mathcal{L}^Q$ such that their product $uv$ also belongs to $\operatorname{dom} \mathcal{L}^Q$.

The quadratic covariation $[u(X), v(X)]$ is a $Q$-semimartingale. We denote by $\langle u(X), v(X)\rangle^Q$ its bounded variation part, i.e.

$$d[u(X), v(X)]_t = d\langle u(X), v(X)\rangle^Q_t + dM^{Q,[u,v]}_t, \qquad \overline{Q}\text{-a.e.}$$

where, here and below, $M^{Q,\bullet}$ or $M^\bullet$ stands for any local $Q$-martingale. As next lemma indicates, we are interested in situations where the bounded variation process $\langle u(X), v(X)\rangle^Q$ is predictable (as a continuous process). Therefore, in the whole article $\langle u(X), v(X)\rangle^Q$ is the usual sharp bracket (sometimes called conditional quadratic variation) of stochastic process theory.

**Lemma A.1.** *For any $u, v \in \operatorname{dom} \mathcal{L}^Q$ such that $uv \in \operatorname{dom} \mathcal{L}^Q$, the process $\langle u(X), v(X)\rangle^Q$ is absolutely continuous $Q$-a.e. and*

$$d\langle u(\overline{X}), v(\overline{X})\rangle^Q_t = \Gamma^Q(u,v)(t, X_{[0,t]})\, dt, \qquad Q\text{-a.e.}$$

*Proof.* As a definition of the forward generator $d(uv)(\overline{X}_t) = \mathcal{L}^Q_t(uv)(t, X_{[0,t]})\, dt + dM^{uv}_t$. Comparing this expression with (4.23), the Doob-Meyer decomposition theorem gives the announced result. □

We say that a process $Y$ *can be localized as* a bounded (resp. integrable) process if there exists a sequence of stopping times $(\sigma_k)$ tending almost surely to infinity and such that for each $k$, the stopped process $Y^{\sigma_k}$ is bounded almost surely (resp. integrable).

**Lemma A.2.** *For any conditionable path measure $Q \in \mathrm{M}(\Omega)$, almost every $t \in [0,T]$, and any locally bounded functions $u, v \in \operatorname{dom} \mathcal{L}^Q$ such that $uv \in \operatorname{dom} \mathcal{L}^Q$, and $M^{Q,[u,v]}$ as defined at Lemma A.1 can be localized as an integrable $Q$-martingale, there exist an* *increasing sequence $(\tau_k)$ of $Q$-integration times of $u$ and $v$, and a sequence $(h_n)$ of positive numbers such that $\lim_{k\to\infty} \tau_k = \infty$, $Q$-a.e., $\lim_{n\to\infty} h_n = 0$ and for each $k$ we have*

$$\begin{aligned}\Gamma^Q(u,v)(t,X_{[0,t]})\\ &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{h_n} E_Q\left[\{u(\overline{X}^{\tau_k}_{t+h_n}) - u(\overline{X}^{\tau_k}_t)\}\{v(\overline{X}^{\tau_k}_{t+h_n}) - v(\overline{X}^{\tau_k}_t)\} \,\Big|\, X_{[0,t]}\right], \quad Q\text{-a.e.}\end{aligned}$$

*Proof.* Since $u$ and $v$ are assumed to be locally bounded, $u(\overline{X})$ and $v(\overline{X})$ can be localized as bounded processes. Furthermore, the processes $\int_0^\bullet |\mathcal{L}^Q u(t,X_{[0,t]})|\,dt$ and $\int_0^\bullet |\mathcal{L}^Q v(t,X_{[0,t]})|\,dt$ can also be localized as bounded processes. It follows that the local $Q$-martingales $M^u, M^v$, (where $M^u_t := u(\overline{X}_t) - \int_0^t \mathcal{L}^Q u(s,X_{[0,s]})\,ds$) can also be localized as bounded processes. Localizing as in the proof of Proposition 3.14, it is enough to show that

$$\begin{aligned}\lim_{h\to 0^+} E_Q \int_0^{T-h} \Big| E_Q\big[h^{-1}\{u(X_{t+h}) - u(X_t)\}\{v(X_{t+h}) - v(X_t)\} \,\big|\, X_{[0,t]}\big]\\ - \Gamma^Q(u,v)(t,X_{[0,t]})\Big|\,dt = 0,\end{aligned} \tag{A.3}$$

and we can assume that all the above mentioned processes are bounded.
For each $0 \leqslant t \leqslant T-h$ with $0 < h \leqslant T$,

$$\begin{aligned}&[u(X_{t+h}) - u(X_t)][v(X_{t+h}) - v(X_t)]\\ &= \Big[\int_t^{t+h} dM^u_s + \int_t^{t+h} \mathcal{L}^Q u(\overline{X}_s)\,ds\Big]\Big[\int_t^{t+h} dM^v_s + \int_t^{t+h} \mathcal{L}^Q v(\overline{X}_s)\,ds\Big]\\ &= A^h_t + B^h_t + C^h_t + D^h_t, \qquad Q\text{-a.e.},\end{aligned}$$

where

$$\begin{aligned}A^h_t &= \int_t^{t+h} dM^u_s \int_t^{t+h} dM^v_s, & B^h_t &= \int_t^{t+h} \mathcal{L}^Q u(\overline{X}_s)\,ds \int_t^{t+h} dM^v_s,\\ C^h_t &= \int_t^{t+h} \mathcal{L}^Q v(\overline{X}_s)\,ds \int_t^{t+h} dM^u_s, & D^h_t &= \int_t^{t+h} \mathcal{L}^Q u(\overline{X}_s)\,ds \int_t^{t+h} \mathcal{L}^Q v(\overline{X}_s)\,ds.\end{aligned}$$

Let us control $A^h_t$. Denoting $U_{t,s} := M^u_s - M^u_t$ and $V_{t,s} := M^v_s - M^v_t$,

$$\begin{aligned}A^h_t &= \int_t^{t+h} d(U_{t,s}V_{t,s})\\ &= \int_t^{t+h} U_{t,s}\,dM^v_s + \int_t^{t+h} V_{t,s}\,dM^u_s + \int_t^{t+h} dM^{Q,[u,v]}_s + \int_t^{t+h} d\langle M^u, M^v\rangle^Q_s,\end{aligned}$$

and with Lemma A.1

$$h^{-1}E_Q\left[A^h_t \,\big|\, X_{[0,t]}\right] = h^{-1}\int_t^{t+h} E_Q[\Gamma^Q(u,v)(\overline{X}_s) \mid X_{[0,t]}]\,ds. \tag{A.4}$$

Remark that the boundedness properties obtained above by localization, together with the extra assumption that $M^{Q,[u,v]}$ is integrable, justify the cancelation of the expectations of the martingale terms.

Let us control $B^h$:

$$
\begin{aligned}
h^{-1}E_Q\int_0^{T-h}|B^h_t|\,dt &\leqslant E_Q\int_0^{T-h} h^{-1}\Big|\int_t^{t+h}\mathcal{L}^Q u(\overline{X}_s)\,ds\Big|\,|M^v_{t+h}-M^v_t|\,dt\\
&= E_Q\int_0^{T-h}\Big|k^h * (\mathcal{L}^Q u)(\overline{X}_t)\Big|\,|M^v_{t+h}-M^v_t|\,dt\\
&= o_{h\to 0^+}(1),
\end{aligned}
$$

where we took $k^h := h^{-1}\mathbf{1}_{[-h,0]}$ as our convolution kernel. The last identity is a consequence of Lemma 2.11 under the assumption $\mathcal{L}^Q u(\overline{X})\in L^1(\overline{Q})$ (because $\int_0^\bullet |\mathcal{L}^Q u(t, X_{[0,t]})|\,dt$ is bounded), the uniform boundedness and right-continuity of $M^v$ and the dominated convergence theorem.

Similarly, $\lim_{h\to 0^+} h^{-1}E_Q\int_0^{T-h}|C^h_t|\,dt=0$.

The control of $D^h$ is analogous:

$$
\begin{aligned}
h^{-1}E_Q\int_0^{T-h}|D^h_t|\,dt &\leqslant E_Q\int_0^{T-h}\Big|k^h * (\mathcal{L}^Q u)(\overline{X}_t)\Big|\,\Big|\int_t^{t+h}\mathcal{L}^Q v(\overline{X}_s)\,ds\Big|\,dt\\
&= o_{h\to 0^+}(1),
\end{aligned}
$$

thanks to the uniform boundedness of $\int_{[0,T]}|\mathcal{L}^Q v(\overline{X}_s)|\,ds$.

Putting everything together, we obtain

$$
\begin{aligned}
\lim_{h\to 0^+} E_Q\int_0^{T-h}\Big| & E_Q\big[h^{-1}\{u(X_{t+h})-u(X_t)\}\{v(X_{t+h})-v(X_t)\}\,\Big|\,X_{[0,t]}\big]\\
&- h^{-1}\int_t^{t+h} E_Q[\Gamma^Q(u,v)(\overline{X}_s)\mid X_{[0,t]}]\,ds\Big|\,dt=0.
\end{aligned}
$$

On the other hand, with Corollary 2.16 we obtain

$$
\lim_{h\to 0^+} E_Q\int_0^{T-h}\Big|h^{-1}\int_t^{t+h}E_Q[\Gamma^Q(u,v)(\overline{X}_s)\mid X_{[0,t]}]\,ds-\Gamma^Q(u,v)(t,X_{[0,t]})\Big|\,dt=0.
$$

The limit (A.3) follows from these last two limits. □

## Appendix B. About Nelson velocities

This section refers to the diffusion measure $Q$ of Section 4. Its content is not used directly in this article. We propose it to the reader to stress the importance for our purpose of considering the relative momentum field $\beta^{Q|R}$ rather than the absolute velocity $\mathsf{v}^Q$.

Denoting by Id the identity mapping on $\mathbb{R}^n$, we see that the vector field $\mathsf{v}^Q$ appearing in the martingale problems satisfy $\mathsf{v}^Q=\mathcal{L}^Q[\mathrm{Id}]$. Because of the identification $\mathcal{L}^Q=L^Q$ which was obtained at Section 2, one suspects that $\mathsf{v}^Q$ should satisfy

$$
\mathsf{v}^Q_t = L^Q_t[\mathrm{Id}] = \lim_{h\to 0^+} E_Q\Big(\frac{X_{t+h}-X_t}{h}\,\Big|\,X_{[0,t]}\Big),
$$

*whenever this expression is meaningful.* The r.h.s. of this identity is the forward Nelson velocity of $Q$. But in general it is not well defined, due to a possible lack of integrability. In order to give sense to limits of this type in a general setting, one must introduce integration times and work as in Proposition 3.14. Next result presents a situation where integration times can be avoided.

**Proposition B.1.** *Under the hypothesis* (4.9), *suppose that* $\mathsf{a}$ *is bounded from above. Then, the limit*

$$\mathsf{v}^{Q|R}(t,\omega) = \lim_{h\to 0^+} E_Q\left(\frac{X_{t+h}-X_t}{h} - \frac{1}{h}\int_{[t,t+h]} \mathsf{b}(\overline{X}_s)\,ds \,\middle|\, X_{[0,t]} = \omega_{[0,t]}\right), \quad (t,\omega)\in\overline{\Omega},$$

*takes place in* $L^2(\overline{Q})$.

*Proof.* Under (4.9), we know that $E_{\overline{Q}}|\beta^{Q|R}|^2_{\mathsf{a}} < \infty$. Because of the assumed upper boundedness of $\mathsf{a}$, this implies that $E_{\overline{Q}}|\mathsf{v}^{Q|R}|^2 < \infty$, where $\mathsf{v}^{Q|R} := \mathsf{a}\beta^{Q|R}$. Rewrite the assertion: $Q \in \mathrm{MP}(\mathsf{a}, \mathsf{b}+\mathsf{v}^{Q|R})$ as:

$$X_{t+h} - X_t - \int_{[t,t+h]} \mathsf{b}(\overline{X}_s)\,ds = \int_{[t,t+h]} \mathsf{v}^{Q|R}(\overline{X}_s)\,ds + M^Q_{t+h} - M^Q_t,$$

where $M^Q$ is a local $Q$-martingale. The assumption $E_{\overline{Q}}|\mathsf{v}^{Q|R}|^2 < \infty$, expressed with the Euclidean norm $|\bullet|$ rather than the Riemannian norm $|\bullet|_{\mathsf{g}}$, permits us to apply the convolution Lemma 2.11 to $v = a\beta$ componentwise with $p = 2$. The critical step where this is used is Jensen's inequality right below (2.12). With this at hand, proceeding as in the proof of Proposition 2.18 leads us to the announced result. □

*Remarks* B.2.

(a) In the setting of this proposition, if the Nelson velocity $L^R[\mathrm{Id}]$ is ill defined because $E_R \int_{[0,T]} |\mathsf{b}_t|\,dt = \infty$, it might happen that $L^Q[\mathrm{Id}]$ is also ill defined. Nevertheless, we have: $\int_{[0,T]} |\mathsf{b}_t|\,dt < \infty$, $R$-*a.e.*, and $Q \in \mathrm{MP}(\mathsf{a}, \mathsf{v}^Q)$ where $\mathsf{v}^Q = \mathsf{b} + \mathsf{v}^{Q|R}$ satisfies $\int_{[0,T]} |\mathsf{v}^Q_t|\,dt < \infty$, $Q$-*a.e.*

(b) Requiring that the diffusion matrix field $\mathsf{a}$ is upper bounded is not a strong restriction for the applications, because in general temperature is upper bounded.

(c) If $\mathsf{a}$ is only locally bounded, then there exists a sequence $(h_n)$ of positive numbers such that $\lim_{n\to\infty} h_n = 0$ and the limit

$$\mathsf{v}^{Q|R}(t,\omega) = \lim_{k\to\infty}\lim_{n\to\infty} E_Q\left(\frac{X^{\tau_k}_{t+h_n} - X^{\tau_k}_t}{h_n} - \frac{1}{h_n}\int_{[t,t+h_n]} \mathbf{1}_{\{s\leqslant\tau_k\}}\mathsf{b}(\overline{X}_s)\,ds \,\middle|\, X_{[0,t]} = \omega_{[0,t]}\right)$$

holds $\overline{Q}$-*a.e.*, where for each integer $k \geqslant 1$, $\tau_k := \inf\{t\in[0,T] : |X_t| \geqslant k\}$. The proof of this statement is similar to Proposition 3.14's proof.

## References


[1] S. Albeverio. *Theory of Dirichlet forms and applications, in Ecole d'Eté de Probabilités de Saint-Flour XXX-2000*, volume 1816 of *Lecture Notes in Mathematics*. Springer, Berlin, 2003.

[2] E. Bernton, P. Ghosal, and M. Nutz. Entropic optimal transport: Geometry and large deviations. Preprint arXiv:2102.04397.

[3] D. Bohm. A suggested interpretation of the quantum theory in tems of "hidden" variables I, II. *Physical Review*, 85:166–179, 180–193, 1952.

[4] P. Cattiaux, G. Conforti, I. Gentil, and C. Léonard. Time reversal of diffusion processes under a finite entropy condition. Preprint arXiv:2104.07708.

[5] P. Cattiaux and C. Léonard. Minimization of the Kullback information of diffusion processes. *Ann. Inst. H. Poincaré. Probab. Statist.*, 30:83–132, 1994.

[6] P. Cattiaux and C. Léonard. Minimization of the Kullback information for some Markov processes. In *Seminar on Probability, tome 30 (Univ. Strasbourg, Strasbourg, 1996)*, volume 1626 of *Lecture Notes in Math.*, pages 288–311. Springer, Berlin, 1996.

[7] K.L. Chung and J.C. Zambrini. *Introduction to random time and quantum randomness*. World Scientific Publishing Co. Inc., 2003.

[8] K.L. Chung and Z. Zhao. *From Brownian Motion to Schrödinger's Equation*, volume 312 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1995.

[9] G. Conforti. A second order equation for Schrödinger bridges with applications to the hot gas experiment and entropic transportation cost. *Probability Theory and Related Fields*, 174(1):1–47, 2019.

[10] R. Cont and D.-A. Fournié. Functional Itô calculus and stochastic integral representation of martingales. *Ann. Probab.*, 41(1):109–133, 2013.

[11] A.B. Cruzeiro, L. Wu, and J.-C. Zambrini. Bernstein processes associated with a Markov process. In R. Rebolledo, editor, *Stochastic analysis and mathematical physics, ANESTOC'98. Proceedings of the Third International Workshop*, Trends in Mathematics, pages 41–71, Boston, 2000. Birkhäuser.

[12] C. Dellacherie and P.-A. Meyer. *Probabilités et Potentiel. Ch. XII à XVI. Théorie du potentiel associée à une résolvante, théorie des processus de Markov*. Hermann. Paris, 1987.

[13] J.L. Doob. Conditional Brownian motion and the boundary limits of harmonic functions. *Bull. Soc. Math. France*, 85:431–458, 1957.

[14] J.L. Doob. *Classical Potential Theory and Its Probabilistic Counterpart*. Classics in Mathematics. Springer, 2nd edition, 2000. (reprint of the 1984 first edition).

[15] D. Dürr and S. Teufel. *Bohmian mechanics. The physics and mathematics of quantum theory*. Springer-Verlag, Berlin, 2009.

[16] L.C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Series in Mathematics*. American Mathematical Society, 1998.

[17] S. Fang. Inégalité du type de Poincaré sur l'espace des chemins riemanniens. *C. R. Acad. Sci. Paris*, 318:257–260, 1994.

[18] R. Feynman. Space-time approach to nonrelativistic quantum mechanics. *Rev. Mod. Phys.*, 20:367–387, 1948.

[19] R. Feynman and A. Hibbs. *Quantum Mechanics and Path Integrals*. International Series in Pure and Applied Physics. McGraw-Hill, 1965.

[20] W.H. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions*, volume 25 of *Applications of Mathematics*. Springer, second edition, 2006.

[21] H. Föllmer. An entropy approach to the time reversal of diffusion processes. In *Stochastic Differential Systems - Filtering and Control*, volume 69 of *Lecture Notes in Control and Information Sciences*, pages 156–163. Springer, 1985.

[22] H. Föllmer. Time reversal on Wiener space. In *Stochastic Processes - Mathematics and Physics*, volume 1158 of *Lecture Notes in Math.*, pages 119–129. Springer, Berlin, 1986.

[23] H. Föllmer. *Random fields and diffusion processes, in École d'été de Probabilités de Saint-Flour XV-XVII-1985-87*, volume 1362 of *Lecture Notes in Mathematics*. Springer, Berlin, 1988.

[24] E. Hsu. Analysis on path and loop spaces. volume 5 of *IAS/Park City Math. Series*. Amer. Math. Soc., 1997.

[25] J. Jacod. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, 1979.

[26] M. Kac. On the distribution of certain Wiener functionals. *Trans. Amer. Math. Soc.*, 65:1–13, 1949.

[27] M. Kac. On some connections between probability theory and differential and integral equations. In J. Neyman, editor, *Proc. Second Berkeley Symp. Math. Stat. Prob.*, pages 189–215. Univ. of California Press, 1951.

[28] R. Khas'minskii. On positive solutions of the equation $Au+Vu=0$. *Theory Probab. Appl.*, 4:309–318, 1959.

[29] N.V. Krylov. *Nonlinear Elliptic and Parabolic Equations of the Second Order*. Reidel, Dordrecht, 1987.

[30] H. Kunita. Absolute continuity of Markov processes and generators. *Nagoya Mathematical Journal,*, 36:1–26, 1969.

[31] H. Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1997.

[32] C. Léonard. Girsanov theory under a finite entropy condition. In *Séminaire de probabilités, vol. 44.*, pages 429–465. Lecture Notes in Mathematics 2046. Springer, 2012.

[33] C. Léonard. Some properties of path measures. In *Séminaire de probabilités, vol. 46.*, pages 207–230. Lecture Notes in Mathematics 2123. Springer, 2014.

[34] C. Léonard. A survey of the Schrödinger problem and some of its connections with optimal transport. *Discrete Contin. Dyn. Syst. A*, 34(4):1533–1574, 2014.

[35] G. Lowther. Cadlag modifications. *Almost Sure - A random mathematical blog.* `https://almostsuremath.com/2009/12/18/cadlag-modifications/`.

[36] Z.-M. Ma and M. Röckner. *Introduction to the theory of (non-symmetric) Dirichlet forms.* Universitext. Springer, 1992.

[37] P.-A. Meyer and W.A. Zheng. Tightness criteria for laws of semimartingales. *Ann. Inst. H. Poincaré. Probab. Statist.*, 20(4):353–372, 1984.

[38] P.-A. Meyer and W.A. Zheng. Construction de processus de Nelson réversibles. In *Séminaire de probabilités. Tome 19*, volume 1123 of *Lecture Notes in Mathematics*, pages 12–26. Springer, 1985.

[39] E. Nelson. *Dynamical theories of Brownian motion.* Princeton University Press, 1967.

[40] Y. Oshima. *Semi-Dirichlet forms and Markov processes*, volume 48 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., 2013.

[41] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften.* Springer, 3rd edition, 1999.

[42] E. Schrödinger. Über die Umkehrung der Naturgesetze. *Sitzungsberichte Preuss. Akad. Wiss. Berlin. Phys. Math.*, 144:144–153, 1931.

[43] E. Schrödinger. Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique. *Ann. Inst. H. Poincaré*, 2:269–310, 1932.

[44] W. Stannat. *The theory of generalized Dirichlet forms and its applications in analysis and stochastics*, volume 678 of *Mem. Amer. Math. Soc.* American Mathematical Society, 1999.

[45] D.W. Stroock and S.R.S. Varadhan. Diffusion processes with continuous coefficients, I. *Communications on Pure and Applied Mathematics*, 22(3):345–400, 1969.

[46] D.W. Stroock and S.R.S. Varadhan. Diffusion processes with continuous coefficients, II. *Communications on Pure and Applied Mathematics*, 22(4):479–530, 1969.

[47] D.W. Stroock and S.R.S. Varadhan. *Multidimensional diffusion processes.* Number 233 in Grundlehren der mathematischen Wissenschaften. Springer Verlag, 1979.

[48] M. von Renesse. An optimal transport view on Schrödinger's equation. *Canad. Math. Bull.*, 55(4):858–869, 2011.

[49] J.-C. Zambrini. Variational processes and stochastic versions of mechanics. *J. Math. Phys.*, 27:2307–2330, 1986.

[50] J.-C. Zambrini. The research program of stochastic deformation (with a view toward geometric mechanics). In *Stochastic Analysis: A series of lectures.* Springer, 2015. arXiv:1212.4186.

[51] X. Zhang. Clark-Ocone formula and variational representation for Poisson functionals. *Ann. Probab.*, 37(2):506–529, 2009.

[52] W.A. Zheng. Tightness results for laws of diffusion processes. Application to stochastic mechanics. *Ann. Inst. H. Poincaré. Probab. Statist.*, 21(2):103–124, 1985.



Modal'X, UMR CNRS 9023. UPL, Univ Paris Nanterre, F92000 Nanterre, France
*Email address*: `christian.leonard@math.cnrs.fr`