TIME REVERSAL OF DIFFUSION PROCESSES UNDER A FINITE ENTROPY CONDITION

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ABSTRACT. Motivated by entropic optimal transport, time reversal of diffusion processes is revisited. An integration by parts formula is derived for the carré du champ of a Markov process in an abstract space. It leads to a time reversal formula for a wide class of diffusion processes in $\mathbb{R}^n$ possibly with singular drifts, extending the already known results in this domain.

The proof of the integration by parts formula relies on stochastic derivatives. This formula is applied to compute the semimartingale characteristics of the time-reversed P^* of a diffusion measure P provided that the relative entropy of P with respect to another diffusion measure R is finite, and the semimartingale characteristics of the time-reversed R^* are known (for instance when the reference path measure R is reversible).

As an illustration of the robustness of this method, the integration by parts formula is also employed to derive a time-reversal formula for a random walk on a graph.

CONTENTS

1. INTRODUCTION

The time-reversed $(Y_t^* := Y_{T-t}, 0 \leq t \leq T)$ of a Markov process $(Y_t, 0 \leq t \leq T)$ remains a Markov process. Consequently, the problem of finding its Markov generator arises naturally. The answer to this problem is given by the so-called time reversal formula. More precisely, we establish an integration by parts formula connecting the carré du champ of a Markov process (its Dirichlet form) with its backward and forward generators, see (1.9). This result extends the well known case of reversible processes where forward and backward generators are equal, and is valid under mild regularity assumptions. Then,

we apply this abstract integration by parts formula (IbP) to identify the semimartingale characteristics of a time-reversed diffusion process, see Theorems 1.12 and 1.16.

It is worth mentioning that this IbP formula allows a unified study of time reversal for diffusion processes and processes with jumps. See Section 7 below where time reversal of random walks on graphs is investigated to demonstrate the robustness of this strategy, and the companion paper [6] where the time-reversal of a Markov process with jumps is investigated.

During the eighties, Föllmer gave a smart proof of the time reversal formula for a diffusion process driven by a Brownian motion under a finite entropy condition, using Nelson stochastic derivatives [13, 14]. In the present article, we revisit Föllmer's proof working out in more detail some technical steps and extending it to more general diffusion processes. In doing so, we keep its powerful guideline based on stochastic derivatives and entropic arguments.

We stress that our version of the time reversal formula, as well as Föllmer's one, holds under a finite entropy hypothesis implying a low regularity of the drift field, L^2 being typically enough. Such a situation is not covered by the other main results in the field.

Besides being an interesting topic in its own right, last years have seen a renewed interest in time reversal because of its applications to the Schrödinger problem (a.k.a. entropic optimal transport), see (1.1) below, and functional inequalities. To give some examples, in [4] and [17] a fluid-dynamic (Benamou-Brenier) formulation of entropic optimal transport is derived leveraging time reversal arguments, and similar ideas are also used in [2] in a mean field setting. In all these applications, it is of fundamental importance of having a result applicable to diffusions whose drift is only square integrable, as no more than this can be expected assuming only finite entropy with respect to the Brownian motion. We refer to subsection *"Entropic and deterministic optimal transports"* below for a slightly more accurate discussion of the links between time reversal and the Schrödinger problem.

Regarding functional inequalities, it is worth mentioning that Fontbona and Jourdain [16] recover and extend the Bakry-Émery criterion using an approach based on time reversal. Also using time reversal in a crucial manner, a simple proof of the logarithmic Sobolev inequality is proposed by Léonard in [32], Gentil, Léonard, Ripani and Tamanini [18] derive the HWI inequality, and Karatzas, Schachermayer and Tschiderer [24] obtain pathwise results about the exponential rate of convergence to equilibrium of some Wasserstein gradient flows and another proof of the HWI inequality.

Outline of the article. Next Section 2 gathers basic notions about Nelson stochastic derivatives that will be used throughout the paper. Related technical results which are necessary during our proofs are postponed to the appendix Section A. The main general result of the article is the integration by parts formula stated at Theorem 3.17. Section 3 is devoted to its proof. The time reversal formulas that we obtain for diffusion processes in Sections 4 and 5, and random walks in Section 7 are corollaries of this theorem. These time reversal formulas are stated at Theorems 4.9, 1.16, 5.7 and 7.11. Finally, in Section 6, the current-osmosis decomposition of an entropic interpolation in a diffusion setting is discussed in detail to illustrate our main motivation for revisiting time reversal under a finite entropy condition.

Theorem 1.16 which is stated in this introductory section is an extension of Theorem 4.9. Its proof is almost verbatim the same as Theorem 4.9's proof.

Entropic and deterministic optimal transports. Let us start presenting some aspect of our main motivation for revisiting time reversal of Markov processes.
Entropic optimal transport dates back to the seminal articles [39, 40] by Schrödinger and

was rigorously rephrased in terms of large deviations of empirical measures of particle systems by Föllmer in his Saint-Flour lecture notes [15]. One wants to minimize the relative entropy

$$H(P|R) := E_P \log(dP/dR)$$

with respect to the law R of some reference Markov process on a time interval $[0, T]$ among all Markov measures P with prescribed initial and final marginals:

$$\inf\{H(P|R); \ P : P_0 = \mu_0, P_T = \mu_T\}. \tag{1.1}$$

To fix the ideas in this introductory discussion, following Schrödinger and Föllmer, our reference measure R is the law of a Brownian motion. We denote by P_t the t-marginal of P, i.e. the law under P of the position at time t, and P^* the time-reversal of P. The quantity $H(P|R) - H(P_0|R_0)$ appears as an average *forward* kinetic action (again a result by Föllmer). Since time-reversal is a one-one mapping, we have

$$H(P^*|R^*) = H(P|R) \tag{1.2}$$

which allows us to interpret $H(P|R) - H(P_T|R_T)$ as a *backward* kinetic action. Taking the half sum, we arrive at

$$H(P|R) = \text{function}(P_0, P_T) + A_{\text{cu}}(P) + A_{\text{os}}(P), \tag{1.3}$$

where the current action term $A_{\text{cu}}(P)$ is purely kinetic with a direct interpretation in terms of *deterministic optimal transport*:

$$\inf\{A_{\text{cu}}(P); \ P : P_0 = \mu_0, P_T = \mu_T)\} = T^{-1}W_2^2(\mu_0, \mu_T),$$

with W_2 the standard quadratic Wasserstein distance. This is the Benamou-Brenier formula. It turns out that the osmotic action term

$$A_{\text{os}}(P) = \mathsf{A}_{\text{os}}([P]) \tag{1.4}$$

only depends on the marginal flow $[P] := (P_t)_{0 \le t \le T}$ of P (it is directly linked to some Fisher information). This important identity follows from the *time reversal formula*, which is the main goal of this paper.

The decomposition (1.3) with (1.4) plays a major rôle in the comparison between deterministic and entropic optimal transports. In particular, we see that for a given flow of marginals $\mu := (\mu_t)_{0 \le t \le T}$,

$$\begin{aligned} \mathsf{A}(\mu) &:= \inf\{H(P|R); \ P : P_t = \mu_t, 0 \le t \le T\} \\ &= \text{function}(\mu_0, \mu_T) + \mathsf{A}_{\text{BB}}(\dot\mu) + \mathsf{A}_{\text{os}}(\mu) \end{aligned} \tag{1.5}$$

where $\mathsf{A}_{\text{BB}}(\dot\mu) := \inf\{A_{\text{cu}}(P); \ P : P_t = \mu_t, 0 \le t \le T\}$ is the Benamou-Brenier action of μ : the fundamental notion of Otto calculus on the Wasserstein space of probability measures leading to the definition of the tangent vector $\dot\mu$, see [1, 42]. *In view of (1.5), the osmotic action, whose appearance is tightly connected to time reversal, quantifies the difference between McCann displacement interpolations (attached to the standard deterministic quadratic transport) and their entropic approximations, i.e. marginal flows of solutions of the Schrödinger problem (1.1) (attached to Brownian path measures R with a diffusion coefficient tending to zero).*

In addition, at least formally, we see with (1.5) that the Schrödinger problem (1.1) is recast as a standard least action principle on the Wasserstein space of probability measures, where $\mathsf{A}_{\text{BB}}(\dot\mu)$ is the kinetic action and $-\mathsf{A}_{\text{os}}(\mu)$ is the action of some scalar potential which turns out to be minus some relative Fisher information, see Proposition

6.2. As the entropic interpolation, i.e. the marginal flow of the solution of the Schrödinger problem, solves the least action problem

$$\inf\{\mathsf{A}(\mu); \mu : \mu(0) = \mu_0, \mu(T) = \mu_T\},$$

this suggests that it also solves some Newton equation in the Wasserstein space. See [43, 5] for some progresses in this direction.

It is worth mentioning that similar considerations apply to large deviation functionals of mean-field interacting particles (as opposed to non-interacting particle systems leading to the relative entropy $H(P|R)$), as for example in [2]. It brings us with a new interpretation in terms of Wasserstein geometry of the celebrated contributions of Dawson and Gärtner on the large deviations of mean-field particle systems [8, 7].

Although this article focuses on time reversal, in order to clarify our motivation for studying time reversal thirty-five years after it was well understood, we give some details about these considerations at Section 6, where Proposition 6.2 is the rigorous statement of (1.3) and (1.4).

Time reversal formula for a diffusion process. General time reversal formulas for diffusion processes are well known since the 80's. Consider a diffusion process Y in $\mathbb{R}^n$ satisfying

$$dY_t = b_t(Y_t)\,dt + \sigma_t(Y_t)\,dB_t, \quad 0 \le t \le T,$$

with B a Brownian motion, b a drift vector field and σ a matrix field associated to the diffusion field $\mathsf{a} := \sigma\sigma^{\mathsf{t}}$, ($\sigma^{\mathsf{t}}$ is the transposed of σ.) Assuming that the law of Y_t is absolutely continuous at each time t, under various hypotheses on b and a, one can prove that the time-reversed process Y^* is again a diffusion process with diffusion matrix field $\mathsf{a}_t^* = \mathsf{a}_{T-t}$ and drift field

$$b_t^*(y) = -b_{T-t}(y) + \nabla\!\cdot\!(\mu_{T-t}\mathsf{a}_{T-t})(y)/\mu_{T-t}(y), \tag{1.6}$$

where μ_t is the density of the law of Y_t with respect to Lebesgue measure. This is not a straightforward result because a reversed semimartingale might not be a semimartingale anymore, see [44].

For this identity to hold, it is assumed in [19, 33] that b is locally Lipschitz (for a Sobolev-type relaxation of this regularity property, see [38]), and that either a is bounded away from zero or that the derivative $\nabla\mathsf{a}$ in the sense of distribution is controlled locally. Haussmann and Pardoux [19] take a PDE approach, while Millet, Nualart and Sanz [33] rely on stochastic calculus of variations. The existence of an absolutely continuous density follows from a Hörmander type condition (PDE formulation in [19] and consequence of Malliavin calculus in [33]).

Föllmer's approach significantly departs from these strategies. Under the simplifying hypothesis that a is the identity matrix, it is assumed in [14] that the law P of Y has a finite entropy

$$H(P|R) < \infty, \tag{1.7}$$

with respect to the law R of a Brownian motion with some given initial probability distribution. In particular, the drift field b of P satisfies $\int_{[0,T]\times\mathbb{R}^n} |b_t(y)|^2\,\mu_t(y)dtdy < \infty$ and might be singular, rather than locally Lipschitz as required in [19, 33]. As a consequence of this finite entropy assumption, Föllmer proves the time reversal formula

$$b_t^*(y) = -b_{T-t}(y) + \nabla\log\mu_{T-t}(y) \tag{1.8}$$

(recall $\mathsf{a} = \mathrm{Id}$) where the derivative is in the sense of distributions, without invoking any already known result about the regularity of μ.

With entropic optimal transport in mind, the hypothesis (1.7) is mandatory. This rules out the Lipschitz regularity of b which is required in "non-Föllmerian" approaches. Therefore, developing the entropic approach to time reversal is a necessary step of the research program attached to *entropic optimal transport*. Following a previous unpublished work by Cattiaux and Petit [3], the present article fills this gap, keeping the powerful guideline of Föllmer's proof based on stochastic derivatives and entropic arguments.

Main results of the article. Our main results are the IbP formula for the carré du champ of a general Markov process and the time reversal formula for a diffusion process.

IbP formula for the carré du champ of a Markov process. Its expression is

$$E_P\Big((\overrightarrow{\mathcal{L}}_t^P u + \overleftarrow{\mathcal{L}}_t^P u)[X_t] v(X_t) + \overrightarrow{\Gamma}_t^P(u,v)[X_t]\Big) = 0, \qquad (1.9)$$

where $\overrightarrow{\mathcal{L}}^P, \overleftarrow{\mathcal{L}}^P$ are the forward and backward extended generators of the Markov measure P, and $\overrightarrow{\Gamma}^P$ is its forward extended carré du champ. See Section 2 for more detail about these notions. This IbP formula is valid for a sufficiently large class of regular functions u and v.

No entropic argument is used to prove this result whose precise statement is given at Theorem 3.17. On the contrary, the main technical problem we face is to show that this IbP is valid under minimal regularity assumptions on P to be able to apply it to general Markov measures typically satisfying a finite entropy condition.

The reason for calculating with extended generators is twofold:

(1) Unlike semigroup generators (which are associated to topological function spaces), extended generators are low-sophisticated objects which are tailor-made for *martingale problems*: the relevant notion we work with in this article. This allows us to consider lowly regular path measures P.

(2) As already noticed by Nelson in [36], one can view Markov generators as stochastic derivatives, see Appendix A. This natural idea permits to perform computations along trajectories, using *stochastic calculus* to obtain expressions for the generators and carré du champ operators. Our main technical result proved in this spirit is Lemma 3.9. It is the keystone of the proof of the IbP formula.

A discrete-time version of (1.9) was proposed by Feynman [12, Eq. (7-45)] to derive Heisenberg's uncertainty principle with path integrals[1].

Time reversal formula for a diffusion process. The law P of the above process Y solves the martingale problem

$$P \in \mathrm{MP}(b, \mathsf{a})$$

meaning that for any $u \in C_c^2(\mathbb{R}^n)$, the process $u(X_t) - \int_0^t \overrightarrow{\mathcal{L}}_s u(X_s)\, ds$ is a local P-martingale, where the forward generator $\overrightarrow{\mathcal{L}}$ is defined by

$$\overrightarrow{\mathcal{L}}_t u(x) = b(t,x) \cdot \nabla u(x) + \Delta_{\mathsf{a}_t} u(x)/2, \qquad (t,x) \in [0,T] \times \mathbb{R}^n,$$

with $\Delta_{\mathsf{a}} := \sum_{1 \leq i,j \leq n} \mathsf{a}_{ij} \partial_{ij}^2$. One also writes

$$P \in \mathrm{MP}(\mu, b, \mathsf{a})$$

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[1]We thank Jean-Claude Zambrini for having brought this to our attention.

to specify the initial marginal measure $P_0 = \mu$ if necessary.

The Markov generator of a *Kolmogorov diffusion* with potential U is

$$Au = \left(-\mathsf{a}\nabla U \cdot \nabla u + \nabla \cdot (\mathsf{a}\nabla u)\right)/2,$$

where a is a field on $\mathbb{R}^n$ (not depending on time) with values in the set S_+ of all symmetric positive matrices and U is a differentiable numerical function. The equilibrium measures of this dynamics are proportional to

$$\mathsf{m}(dx) = e^{-U(x)}\, dx.$$

Expanding the divergence term, we see that the drift field of the generator is

$$\mathsf{v}^{\mathsf{a},\mathsf{m}} := (\nabla \cdot \mathsf{a} - \mathsf{a}\nabla U)/2.$$

Hypotheses 1.10.

  (i) $U \in C^1(\mathbb{R}^n)$, a is invertible and in $C^1(\mathbb{R}^n, S_+)$,
  (ii) for some $K \geq 0$, $x \cdot \mathsf{v}^{\mathsf{a},\mathsf{m}}(x) + \operatorname{tr}\mathsf{a}(x) \leq K(1 + |x|^2)$ for all $x \in \mathbb{R}^n$.

It is a standard result that under these hypotheses, the martingale problem $\mathrm{MP}(\mathsf{m}, \mathsf{v}^{\mathsf{a},\mathsf{m}}, \mathsf{a})$ admits a unique solution denoted by

$$R \in \mathrm{MP}(\mathsf{m}, \mathsf{v}^{\mathsf{a},\mathsf{m}}, \mathsf{a}), \tag{1.11}$$

which is m-reversible. This implies in particular that $R^* = R$.

Theorem 1.12 (Time reversal formula)**.** *Under the Hypotheses 1.10 on R given at (1.11), let $P \in \mathrm{P}(\Omega)$ be Markov and such that*

$$H(P|R) < \infty.$$

Then, for all t the density $\mu_t := dP_t/d\mathrm{Leb}$ exists and the time reversal P^* of P is a solution of the martingale problem

$$P^* \in \mathrm{MP}(b^*, \mathsf{a})$$

with

$$b_t^*(x) = -b_{T-t}(x) + \nabla \cdot (\mu_{T-t}\mathsf{a})(x)/\mu_{T-t}(x), \quad dtP_t(dx)\text{-a.e.} \tag{1.13}$$

where the divergence is in the sense of distributions.

This is an extension of (1.6) to a low regularity setting which is made precise as follows. For almost every t the density $\rho_t := dP_t/d\mathsf{m}$ admits a distributional spatial derivative $\nabla \rho_t$ satisfying

$$\int_{[0,T]\times\mathbb{R}^n} |\nabla \log \rho_t|_{\mathsf{a}}^2 \, dP_t dt < \infty. \tag{1.14}$$

and (1.13) is equivalent to

$$(b_t + b_{T-t}^*)/2 - \mathsf{v}^{\mathsf{a},\mathsf{m}} = \mathsf{a}\nabla \log \sqrt{\rho_t}, \quad dtdP_t\text{-a.e.} \tag{1.15}$$

Furthermore, P^* is the unique solution of $\mathrm{MP}(P_T, b^*, \mathsf{a})$ among the set of all $Q \in \mathrm{P}(\Omega)$ such that $H(Q|R) < \infty$.

This theorem is a restatement of Theorem 4.9 which is stated in terms of stochastic velocities, especially the fundamental identity (1.15) which is synthetically expressed in terms of the *osmotic momentum* $\beta^{\mathrm{os},P|R}$ at (4.11).

An extension of Theorem 1.12. Note that unlike [19, 33], it is assumed in Theorem 1.12 that the diffusion matrix field a does not depend on t. However, our method allows to extend the results of [19, 33] to a finite entropy setting. Indeed, the method of proof of the present article is perturbative: *if one knows the time-reversal formula for some possibly unbounded reference path measure R, then a time-reversal also holds for any probability measure P such that $H(P|R) < \infty$.*

A careful inspection of the proof of Theorem 4.9 (a.k.a. Theorem 1.12) shows that it extends to the case where the reference measure R might not be reversible.

Theorem 1.16 (Time-reversal formula, again). *Let us assume that the possibly unbounded reference measure R and its time reversal R^* both solve uniquely their respective martingale problems $\mathrm{MP}(R_0, b^R, \mathsf{a})$ and $\mathrm{MP}(R_0^* = R_T, b^{R^*}, \mathsf{a}^*)$ in the sense of (4.2), where b^R, b^{R^*} are locally bounded fields and a is continuous on $[0, T] \times \mathbb{R}^n$. The following assertions are verified.*

(a) For all $0 \le t \le T$, we have $\mathsf{a}_t^* = \mathsf{a}_{T-t}$.

(b) Assume also that for all $0 < t < T$ the time marginal R_t is absolutely continuous with respect to Lebesgue measure, and that the path probability measure satisfies $H(P|R) < \infty$ again.

Then, P and P^* uniquely solve $\mathrm{MP}(b^P, \mathsf{a})$ and $\mathrm{MP}(b^{P^*}, \mathsf{a}^*)$ respectively, in the sense of (4.2). The identity (1.15) becomes

$$(b_t^P + b_{T-t}^{P^*})/2 - (b_t^R + b_{T-t}^{R^*})/2 = \mathsf{a}_t \nabla \log \sqrt{\rho_t}, \quad dt dP_t\text{-a.e.,} \qquad (1.17)$$

where $\rho_t = dP_t/dR_t$ and the gradient is in the sense of distribution, and (1.14) still holds:

$$\int_{[0,T] \times \mathbb{R}^n} |\nabla \log \rho_t|_{\mathsf{a}_t}^2 \, dP_t dt < \infty.$$

Proof. Statement (a) is a consequence of (3.14) at Lemma 3.13-(b) applied to R, whose assumptions are satisfied by Theorem 3.17-(b).

The proof of item (b) is similar to the proof of Theorem 4.9, almost verbatim. The uniqueness of the solution to the martingale problems for R and R^* is necessary for invoking Girsanov's theory at Proposition 4.6. Finally, the local boundedness of the semimartingale characteristics of R and R^* implies the boundedness of $\overrightarrow{\mathcal{L}}^R u$ and $\overleftarrow{\mathcal{L}}^R u$ for any $u \in C_c^2(\mathbb{R}^n)$. This enters the proof of Lemma 4.7 in an essential manner. $\square$

Remark 1.18. A typical hypothesis for a path measure Q to be the unique solution of its martingale problem $\mathrm{MP}(\mathsf{a}, b^Q)$ in the sense of (4.2) is that $\mathsf{a} = \sigma \sigma^*$ with σ and b^Q locally Lipschitz in space and time.

In particular, with R satisfying the regularity hypotheses of the main results of [19, 33], we see that Theorem 1.16 extends the time reversal formula (1.6) to the wider class of all path measures P such that $H(P|R) < \infty$.

Literature about time reversal of Markov processes. The first investigations in the theory of time reversal of Markov processes date back to 1936 with a pair of articles [25, 26] by Kolmogorov providing sufficient conditions for a Markov chain or a diffusion process to be reversible. Then, in 1958 time reversal of Markov processes was used by Hunt [20] in his study of potential theory. During the same year, Nelson published an article [35] entitled "The adjoint Markoff process". Several papers went on in the direction initiated by Hunt: [34, 28, 10] (to cite a few of them). All these articles deal with *stationary* Markov processes and their results are expressed in terms of transition probabilities

rather than semimartingale characterics, which is quite natural in the framework of potential theory. The above mentioned articles [14, 19, 33] and [37] are the first ones where the expression of semimartingale characteristics of a time reversed process are obtained rigorously, see (1.6). They are restricted to a diffusion setting.

We also mention the article [24] by Karatzas, Schachermayer and Tschiderer both for its well written appendix section on time reversal of diffusion processes and its results connecting deterministic optimal transport and diffusion processes, where time reversal plays a crucial role. The recent article [23] by Karatzas, Maas and Schachermayer also makes use of time reversal in the context of Markov chains. Recently, Izydorczyk, Oudjane, Russo and Tessitore [21, 22] used time reversal of diffusion processes to prove the well-posedness of some backward Fokker-Planck equations, and to design efficient algorithms solving some Hamilton-Jacobi-Bellman equations with terminal conditions.

Nelson's contribution. While investigating large deviations of the empirical measure of weakly interacting Brownian particles as in [39, 40] or [8], Föllmer established the time reversal formula (1.8) using entropic arguments, among which the identity (1.2) is decisive. At the same period, Zambrini obtained in [45] a time-symmetric description of the backward and forward drifts of the solution to the Schrödinger problem (1.1). These two authors used in a crucial manner the notion of stochastic derivatives introduced by Nelson in 1967 in [36]. Time reversal is at the core of Nelson's theory of Brownian motion. Indeed, his expression of the osmotic velocity (a notion introduced by him, guided by the seminal article [11] by Einstein on the Brownian motion) in terms of the density of the process is nothing but the time reversal formula. He proves it in an informal manner, i.e. assuming that all the derivatives exist in a classical sense, using PDEs, namely Fokker-Planck equations in both directions of time, also called forward and backward Kolmogorov's equations after [26]. In the present article, stochastic derivatives also play a major role.

Back to the roots. As a concluding remark about the history of time reversal of Markov processes, it appears that the very starting point of this adventure is, again, the paper [39] by Schrödinger. Indeed, in the first paragraph of [25], Kolmogorov refers to [39] as his main motivation[2].

Notation. The set of all probability measures on a measurable set A is denoted by $P(A)$ and the set of all nonnegative σ-finite measures on A is $M(A)$. The push-forward of a measure $q \in M(A)$ by the measurable map $f : A \to B$ is $f_\# q(\bullet) := q(f \in \bullet) \in M(B)$.

Relative entropy. The relative entropy of $p \in P(A)$ with respect to the reference measure $r \in M(A)$ is

$$H(p|r) := \int_A \log(dp/dr)\, dp \in (-\infty, \infty]$$

if p is absolutely continuous with respect to r ($p \ll r$) and $\int_A \log_-(dp/dr)\, dp < \infty$, and $H(p|r) = +\infty$ otherwise. If $r \in P(A)$ is a probability measure, then $H(p|r) \in [0, \infty]$. See Section B for details.

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[2]We thank Jean-Claude Zambrini for having brought this to our attention.

Path measures. The configuration space is a Polish space $\mathcal{X}$ equipped with its Borel σ-field. The path space is the set $\Omega := D([0, T], \mathcal{X})$ of all $\mathcal{X}$-valued càdlàg trajectories on the time index set $[0, T]$, and the canonical process $(X_t)_{0 \leq t \leq T}$ is defined by $X_t(\omega) = \omega_t$ for any $0 \leq t \leq T$ and any path $\omega = (\omega_s)_{0 \leq s \leq T} \in \Omega$. It is equipped with the canonical σ-field $\sigma(X_{[0,T]})$ and the canonical filtration $\left(\sigma(X_{[0,t]}); 0 \leq t \leq T \right)$ where for any subset $\mathcal{T} \subset [0, T]$, $X_{\mathcal{T}} := (X_t, t \in \mathcal{T})$ and $\sigma(X_{\mathcal{T}})$ is the σ-field generated by the collection of maps $(X_t, t \in \mathcal{T})$.

The càdlàg setting is necessary at Section 3 for the abstract IbP formula and Section 7 where random walks are investigated. At Sections 4, 5 and 6, diffusion processes are time-reversed and the path space is the set $\Omega = C([0, T], \mathbb{R}^n)$ of all continuous trajectories.

We call any positive measure $Q \in M(\Omega)$ on Ω a path measure. For any $\mathcal{T} \subset [0, T]$, we denote $Q_{\mathcal{T}} = (X_{\mathcal{T}})_{\#} Q$. In particular, for any $0 \leq r \leq s \leq T$, $X_{[r,s]} = (X_t)_{r \leq t \leq s}$, $Q_{[r,s]} = (X_{[r,s]})_{\#} Q$, and $Q_t = (X_t)_{\#} Q \in M(\mathcal{X})$ denotes the law of the position X_t at time t. If $Q \in P(\Omega)$ is a probability measure, then $Q_t \in P(\mathcal{X})$.

The time-space canonical process is

$$\overline{X}_t := (t, X_t) \in [0, T] \times \mathcal{X},$$

and for any function $u : [0, T] \times \mathcal{X} \to \mathbb{R}$, we denote $u(\overline{X}) : (t, \omega) \mapsto u(t, \omega_t)$. We also denote

$$\overline{Q}(dtd\omega) := dtQ(d\omega), \qquad dtd\omega \subset [0, T] \times \Omega,$$
$$\overline{\mathsf{q}}(dtdx) := dtQ_t(dx), \qquad dtdx \subset [0, T] \times \mathcal{X}.$$

2. Stochastic derivatives

Let us recall the definitions of Markov measures, extended generators and stochastic derivatives. The precise definitions of these notions together with some useful related technical results are recalled at the appendix Section A. Stochastic derivatives were introduced by Nelson in 1967 [36].

Conditionable path measure. A path measure Q such that Q_t is σ-finite for all t is called a conditionable path measure. This notion is necessary to define properly the conditional expectations $E_Q(\bullet \,|\, X_t)$, $E_Q(\bullet \,|\, X_{[0,t]})$ and $E_Q(\bullet \,|\, X_{[t,T]})$, for any t, see [31]. If Q has a finite mass, then it is automatically conditionable.

Extended forward generator. Let Q be a conditionable measure. A measurable function u on $[0, T] \times \mathcal{X}$ is said to be in the domain of the extended forward generator of Q if there exists a real-valued process $\overrightarrow{\mathcal{L}}^Q u(t, X_{[0,t]})$ which is adapted with respect to the forward filtration such that $\int_{[0,T]} |\overrightarrow{\mathcal{L}}^Q u(t, X_{[0,t]})| \, dt < \infty$, Q-a.e. and the process

$$M_t^u := u(\overline{X}_t) - u(\overline{X}_0) - \int_{[0,t]} \overrightarrow{\mathcal{L}}^Q u(s, X_{[0,s]}) \, ds, \quad 0 \leq t \leq T,$$

is a local Q-martingale. We say that $\overrightarrow{\mathcal{L}}^Q$ is the extended forward generator of Q. Its domain is denoted by dom $\overrightarrow{\mathcal{L}}^Q$. Otherwise stated, we say that Q solves the *martingale problem* with generator $\overrightarrow{\mathcal{L}}$ and domain $\mathcal{U}$, if $\mathcal{U} \subset$ dom $\overrightarrow{\mathcal{L}}^Q$ and for any $u \in \mathcal{U}$, $\overrightarrow{\mathcal{L}}^Q u = \overrightarrow{\mathcal{L}} u$.

Stochastic forward derivative. Nelson's definition [36] of the stochastic forward derivative is the following. For any conditionable measure Q and any measurable real function u on $[0,T] \times \mathcal{X}$ such that $E_Q|u(\overline{X}_s)| < \infty$ for all $0 \le s \le T$, we say that u admits a stochastic forward derivative under Q at time $t \in [0,T)$ if the following limit

$$\overrightarrow{L}^Q u(t, X_{[0,t]}) := \lim_{h \to 0^+} E_Q\left(\frac{1}{h}[u(\overline{X}_{t+h}) - u(\overline{X}_t)] \mid X_{[0,t]}\right) \tag{2.1}$$

exists in $L^1(\overline{Q})$. In this case, $\overrightarrow{L}^Q u(t, \bullet)$ is called the stochastic forward derivative of u at time t.

Extended generators and stochastic derivatives are essentially the same. It is the content of Proposition A.10. If u is in dom $\overrightarrow{\mathcal{L}}^Q$ and satisfies $E_Q \int_{[0,T]} |\overrightarrow{\mathcal{L}}^Q u(t, X_{[0,t]})| \, dt < \infty$, one can compute $\overrightarrow{\mathcal{L}}^Q u$ using the stochastic derivative:

$$\overrightarrow{\mathcal{L}}^Q u = \overrightarrow{L}^Q u, \quad \overline{Q}\text{-a.e.}$$

Beware of the notation: calligraphic $\mathcal{L}$ refers to the martingale problem MP($\mathcal{L}$), while the roman font L refers to the stochastic derivative (2.1) which provides us with a mean of calculating $\mathcal{L}$ via (2.1) using stochastic calculus.

Reversing time. Let $Q \in M(\Omega)$ be any path measure. Its time reversal is

$$Q^* := (X^*)_{\#} Q \in M(\Omega),$$

where

$$\begin{cases} X_t^* := \lim_{h \to 0^+} X_{T-t+h}, & 0 \le t < T, \\ X_T^* := X_0, & t = T, \end{cases}$$

is the reversed canonical process. We assume that Q is such that

$$Q(X_{T^-} \ne X_T) = 0,$$

i.e. its sample paths are left-continuous at $t = T$. This implies that the time reversal mapping X^* is (almost surely) one-one on Ω.

As a notation, the σ-field generated by $X_{[t^-,T]}$ is $\sigma(X_{[t^-,T]}) := \cap_{h>0}\sigma(X_{[t-h,T]}) = \sigma(X_{t^-}) \vee \sigma(X_{[t,T]})$, and the predictable backward filtration is defined by: $(\sigma(X_{[t^-,T]}); 0 \le t \le T)$.

We introduce the backward extended generator and the backward stochastic derivative

$$\begin{aligned} \overleftarrow{\mathcal{L}}^Q u(t, X_{[t^-,T]}) &:= \overrightarrow{\mathcal{L}}^{Q^*} u^*(t^*, X_{[0,t^*]}^*), \\ \overleftarrow{L}^Q u(t, X_{[t^-,T]}) &:= \overrightarrow{L}^{Q^*} u^*(t^*, X_{[0,t^*]}^*), \end{aligned} \tag{2.2}$$

where $u^*(t^*, x) := u(t, x)$, with $t^* := T - t$, and $\overrightarrow{\mathcal{L}}^{Q^*}$ and $\overrightarrow{L}^{Q^*}$ stand respectively for the standard (forward) generator and derivative of Q^*. These definitions match with Definitions A.6 and A.7. In particular, for any $t \in (0,T]$,

$$\overleftarrow{L}^Q u(t, X_{[t^-,T]}) := \lim_{h \to 0^+} E_Q\left(\frac{1}{h}[u(\overline{X}_{t-h}) - u(\overline{X}_t)] \mid X_{[t^-,T]}\right)$$

if this limit exists in $L^1(Q)$. Remark that the definition of $\overleftarrow{L}^Q$ is consistent with (2.2).

The linear operators $\overrightarrow{\mathcal{L}}, \overrightarrow{L}, \overleftarrow{\mathcal{L}}$ and $\overleftarrow{L}$ are defined for any measurable function $u : [0,T] \times \mathcal{X} \to \mathbb{R}$ such that the above expressions are meaningful where this *meaningful* addresses the problem of their domains, see the appendix section A.

As for the forward generator, if u is in dom $\overleftarrow{\mathcal{L}}^Q$ and satisfies $E_Q \int_{[0,T]} \left| \overleftarrow{\mathcal{L}}^Q u(t, X_{[t,T]}) \right| dt < \infty$, then

$$\overleftarrow{\mathcal{L}}^Q u = \overleftarrow{L}^Q u, \quad \overline{Q}\text{-a.e.}$$

Markov measure. A path measure $Q \in M(\Omega)$ is said to be Markov if it is conditionable and for any $0 \le t \le T$, $Q(X_{[t,T]} \in \bullet \mid X_{[0,t]}) = Q(X_{[t,T]} \in \bullet \mid X_t)$. It is known that Q^* is also Markov and the stochastic derivatives and extended generators at time t only depend of the present position X_t. Therefore it is possible to consider the sum and difference of the forward and backward generators: they remain functions of the present position.

Current and osmotic generators. In restriction to dom $\overrightarrow{\mathcal{L}}^Q \cap \mathrm{dom}\, \overleftarrow{\mathcal{L}}^Q$, we define the *current* extended generator of Q by

$$\mathcal{L}^{\mathrm{cu},Q} := (\overrightarrow{\mathcal{L}}^Q - \overleftarrow{\mathcal{L}}^Q)/2.$$

Similarly, the *osmotic* extended generator of Q is

$$\mathcal{L}^{\mathrm{os},Q} := (\overrightarrow{\mathcal{L}}^Q + \overleftarrow{\mathcal{L}}^Q)/2.$$

The osmotic generator plays an important role in this article. This is the reason why our results about time reversal are restricted to Markov measures.

3. Integration by parts formula

The main technical result of this paper is the integration by parts formula stated at Theorem 3.17. This section is dedicated to its statement and its proof.

Carré du champ. Let Q be a path measure on Ω. Its forward carré du champ is the forward-adapted process defined by

$$\overrightarrow{\Gamma}_t^Q(u,v) := \overrightarrow{\mathcal{L}}_t^Q(uv) - u\overrightarrow{\mathcal{L}}_t^Q v - v\overrightarrow{\mathcal{L}}_t^Q u, \quad (u,v) \in \mathrm{dom}\, \overrightarrow{\Gamma}_t^Q, \ 0 \le t \le T,$$

where dom $\overrightarrow{\Gamma}_t^Q := \left\{ (u,v); \ u,v,uv \in \mathrm{dom}\, \overrightarrow{\mathcal{L}}_t^Q \right\}$.
We introduce a class $\mathcal{U}$ of functions on $\mathcal{X}$ such that

$$\mathcal{U} \subset \mathrm{dom}\, \overrightarrow{\mathcal{L}}_t^Q \cap C_b(\mathcal{X}) \tag{3.1}$$

for all $0 \le t \le T$ and any path measure Q of interest, where $C_b(\mathcal{X})$ is the space of all bounded continuous functions on $\mathcal{X}$. *We assume that $\mathcal{U}$ is an algebra,* i.e.

$$u, v \in \mathcal{U} \implies uv \in \mathcal{U}. \tag{3.2}$$

In particular,

$$u, v \in \mathcal{U} \implies (u,v) \in \mathrm{dom}\, \overrightarrow{\Gamma}_t^Q. \tag{3.3}$$

We shall mainly consider functions in $\mathcal{U}$ and make an intensive use of their carré du champ. In each setting, this algebra will be chosen rich enough to determine a Markov dynamics, i.e. to solve in a unique way some relevant martingale problem. For instance, in the diffusion setting, $\mathcal{U} = C_c^2(\mathbb{R}^n)$ is a good choice.

Remark 3.4. The requirement that $\mathcal{U}$ is an algebra (it is necessary that uv belongs to dom $\overrightarrow{\mathcal{L}}^Q$ to consider $\overrightarrow{\mathcal{L}}^Q(uv)$), is strong.
Indeed, let us say that a semimartingale is *nice* if its bounded variation part is *absolutely continuous*. The product of two semimartingales is a semimartingale, but the product of two nice semimartingales might not be nice anymore.
However, this is true for instance when the semimartingales are adapted to a Brownian

filtration because in this case any local martingale is represented as a stochastic integral with respect to a Brownian motion. In general, a martingale representation theorem is needed to verify the stability of the product of nice semimartingales.

Similarly the backward carré du champ is the backward-adapted process defined by

$$\overleftarrow{\Gamma}_t^Q(u, v) := \overleftarrow{\mathcal{L}}_t^Q(uv) - u\overleftarrow{\mathcal{L}}_t^Q v - v\overleftarrow{\mathcal{L}}_t^Q u,$$

for any $0 \le t \le T$ and $(u, v) \in \text{dom}\, \overleftarrow{\Gamma}_t^Q$. To emphasize the fact that $\overrightarrow{\Gamma}^Q(u, v)$ and $\overleftarrow{\Gamma}^Q(u, v)$ are processes rather than functions, we often write

$$\overrightarrow{\Gamma}_t^Q(u, v) = \overrightarrow{\Gamma}_t^Q(u, v)(X_{[0,t]}) = \overrightarrow{\Gamma}_t^Q(u, v)(X),$$
$$\overleftarrow{\Gamma}_t^Q(u, v) = \overleftarrow{\Gamma}_t^Q(u, v)(X_{[t,T]}) = \overleftarrow{\Gamma}_t^Q(u, v)(X).$$

The quadratic covariation $[u(X), v(X)]$ is a Q-semimartingale. We denote by $\langle u(X), v(X) \rangle^Q$ its bounded variation part, i.e.

$$d[u(X), v(X)]_t = d\langle u(X), v(X) \rangle_t^Q + dM_t^{Q,[u,v]}, \qquad \overline{Q}\text{-a.e.} \qquad (3.5)$$

where, here and below, $M^{Q,\bullet}$ stands for any forward local Q-martingale. As next lemma indicates, we are interested in situations where the bounded variation process $\langle u(X), v(X) \rangle^Q$ is predictable (as a continuous process). Therefore, in the remainder of the article $\langle u(X), v(X) \rangle^Q$ is the usual sharp bracket (sometimes called conditional quadratic variation) of stochastic process theory.

Lemma 3.6. *Let $\mathcal{U}$ satisfy the hypotheses* (3.1) *and* (3.2).
(a) For any $u, v \in \mathcal{U}$, the process $\langle u(X), v(X) \rangle^Q$ is absolutely continuous Q-a.e. and

$$d\langle u(X), v(X) \rangle_t^Q = \overrightarrow{\Gamma}_t^Q(u, v)(X_{[0,t]})\, dt, \qquad \overline{Q}\text{-a.e.}$$

(b) For any $u, v \in \mathcal{U}$, the process $\langle u(X), v(X) \rangle^{Q^*}$ is absolutely continuous Q^*-a.e. and

$$d\langle u(X), v(X) \rangle_{|T-t}^{Q^*}(X^*) = \overleftarrow{\Gamma}_t^Q(u, v)(X_{[t,T]})\, dt, \qquad \overline{Q}\text{-a.e.}$$

Proof. • Proof of (a). As a definition of the forward generator

$$du(X)_t = \overrightarrow{\mathcal{L}}_t^Q u(X)\, dt + dM_t^u, \qquad dv(X)_t = \overrightarrow{\mathcal{L}}_t^Q v(X)\, dt + dM_t^v,$$
$$d(uv)(X)_t = \overrightarrow{\mathcal{L}}_t^Q(uv)(X)\, dt + dM_t^{uv},$$

and applying Itô's formula in the forward sense of time

$$
\begin{aligned}
d(uv)(X)_t &= u(X_t)dv(X)_t + v(X_t)du(X)_t + d[u(X), v(X)]_t \\
&= u(X_t)dv(X)_t + v(X_t)du(X)_t + d\langle u(X), v(X) \rangle_t + dM_t^{Q,[u,v]} \\
&= [u(X_t)\overrightarrow{\mathcal{L}}_t^Q v(X) + v(X_t)\overrightarrow{\mathcal{L}}_t^Q u(X)]\, dt + d\langle u(X), v(X) \rangle_t \\
&\qquad\qquad + u(X_t)dM_t^v + v(X_t)dM_t^u + dM_t^{Q,[u,v]}.
\end{aligned}
$$

The Doob-Meyer decomposition theorem allows us to identify the bounded variation and martingale parts of $uv(X)$, leading us to

$$\overrightarrow{\mathcal{L}}_t^Q(uv)(X)\, dt = [u(X_t)\overrightarrow{\mathcal{L}}_t^Q v(X) + v(X_t)\overrightarrow{\mathcal{L}}_t^Q u(X)]\, dt + d\langle u(X), v(X) \rangle_t^Q, \qquad \overline{Q}\text{-a.e.}$$

which gives the announced result.

• Proof of (b). Analogous, with Q^* instead of Q. ☐

Remark that the main hypothesis of this lemma is (3.2): $u, v \in \mathcal{U}$, and its consequence (3.3).

Let us prepare some notation for next Lemma 3.9 which is the main technical result of this section. We introduce the class of functions

$$\mathcal{U}_2^Q := \left\{ u \in \mathcal{U}; \overrightarrow{\mathcal{L}}^Q u(X) \in L^2(\overline{Q}), \ \overrightarrow{\Gamma}^Q(u)(X) \in L^1(\overline{Q}) \right\}. \tag{3.7}$$

If Q is Markov, $\overrightarrow{\Gamma}_t^Q(u, v)(X) = \overrightarrow{\Gamma}_t^Q(u, v)[X_t]$ only depends on the current position X_t, and we denote

$$(t, x) \mapsto \overrightarrow{\Gamma}_t^Q(u, v)[x] := E_Q(\overrightarrow{\Gamma}_t^Q(u, v)(X) \mid X_t = x).$$

Consider the following convolution kernels

$$k^h := h^{-1} \mathbf{1}_{[-h,0]}, \qquad k^{-h} := h^{-1} \mathbf{1}_{[0,h]},$$

with $h > 0$. Let $f : [0, T] \to \mathbb{R}$ be any absolutely continuous function with derivative $\dot{f}$. The following expressions will be used during the proof of next lemma:

$$\begin{aligned}
h^{-1}[f(t + h) - f(t)] &= h^{-1} \int_{[t,t+h]} \dot{f}(r) \, dr = k^h * \dot{f}(t), \quad 0 \le t \le T - h, \\
h^{-1}[f(t) - f(t - h)] &= h^{-1} \int_{[t-h,t]} \dot{f}(r) \, dr = k^{-h} * \dot{f}(t), \quad h \le t \le T.
\end{aligned} \tag{3.8}$$

Lemma 3.9. *Let Q be any path measure and take any u, v in the class $\mathcal{U}_2^Q$.*

(a) The following limit holds

$$\lim_{h \to 0^+} E_Q \int_0^{T-h} \left| E_Q \left[h^{-1} \{ u(X_{t+h}) - u(X_t) \} \{ v(X_{t+h}) - v(X_t) \} \mid X_{[0,t]} \right] \right.$$
$$\left. - \overrightarrow{\Gamma}_t^Q(u, v)(X) \right| dt = 0.$$

(b) If in addition Q is Markov and $(t, x) \mapsto \overrightarrow{\Gamma}_t^Q(u, v)[x]$ is continuous, then

$$\lim_{h \to 0^+} E_Q \int_h^T \left| E_Q \left[h^{-1} \{ u(X_t) - u(X_{t-h}) \} \{ v(X_t) - v(X_{t-h}) \} \mid X_{t-h} \right] \right.$$
$$\left. - \overrightarrow{\Gamma}_t^Q(u, v)[X_t] \right| dt = 0. \tag{3.10}$$

Proof. ● Proof of (a). Let us start with a remark about our assumptions. The (a priori local) martingale $M_t^u = u(X_t) - u(X_0) - \int_0^t \overrightarrow{\mathcal{L}}_s^Q u(X) \, ds$, is a square integrable martingale because

$$E_Q \sup_{0 \le t \le T} |M_t^u|^2 \le C_2 E_Q[u(X)]_T^Q = C_2 E_Q \langle u(X) \rangle_T^Q = C_2 T E_Q \overrightarrow{\Gamma}_T^Q(u)(X) < \infty, \tag{3.11}$$

where the first inequality is Doob's maximal inequality with $C_2 = 4$, and the rest follows from the assumptions $\overrightarrow{\Gamma}^Q(u)(X) \in L^1(\overline{Q})$ and Lemma 3.6. For each $0 \le t \le T - h$ with $0 < h \le T$,

$$\begin{aligned}
[u(X_{t+h}) &- u(X_t)][v(X_{t+h}) - v(X_t)] \\
&= \left[\int_t^{t+h} dM_s^u + \int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q u(X) \, ds \right] \left[\int_t^{t+h} dM_s^v + \int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q v(X) \, ds \right] \\
&= A_t^h + B_t^h + C_t^h + D_t^h, \qquad Q\text{-a.e.,}
\end{aligned}$$

where

$$A_t^h = \int_t^{t+h} dM_s^u \int_t^{t+h} dM_s^v, \qquad\qquad B_t^h = \int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q u(X)\, ds \int_t^{t+h} dM_s^v,$$

$$C_t^h = \int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q v(X)\, ds \int_t^{t+h} dM_s^u, \qquad D_t^h = \int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q u(X)\, ds \int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q v(X)\, ds.$$

Let us control A_t^h. Denoting $N_{t,s}^u := M_s^u - M_t^u$ and $N_{t,s}^v := M_s^v - M_t^v$,

$$A_t^h = \int_t^{t+h} d(N_{t,s}^u N_{t,s}^v)$$

$$= \int_t^{t+h} N_{t,s}^u dM_s^v + \int_t^{t+h} N_{t,s}^v dM_s^u + \int_t^{t+h} dM_s^{Q,[u,v]} + \int_t^{t+h} d\langle M^u, M^v \rangle_s^Q,$$

where $M^{Q,[u,v]}$ is the martingale part of the semimartingale $[u(X), v(X)]$, see (3.5). With Lemma 3.6, we obtain

$$h^{-1} E_Q(A_t^h \mid X_{[0,t]}) = h^{-1} \int_t^{t+h} E_Q[\overrightarrow{\Gamma}_s^Q(u,v)(X_{[0,s]}) \mid X_{[0,t]}]\, ds. \qquad (3.12)$$

Remark that under our integrability assumptions, the stochastic integrals $\int_t^{t+h} dM_s^u$, $\int_t^{t+h} dM_s^v$, $\int_t^{t+h} N_{t,s}^u dM_s^v$ and $\int_t^{t+h} N_{t,s}^v dM_s^u$ are integrable Q-martingales. The first ones because of (3.11), and the last ones by Burkholder-Davis-Gundy inequality:

$$E_Q \sup_{0 \le t \le T} \left| \int_0^t M_s^u\, dM_s^v \right| \le C_1 E_Q \left[\left(\int_0^T |M_t^u|^2 d[M^v]_t \right)^{1/2} \right] \le C_1 E_Q \left(\sup_{0 \le t \le T} |M_t^u| [M^v]_T^{1/2} \right)$$

$$\le C_1 \sqrt{E_Q \sup_{0 \le t \le T} |M_t^u|^2} \sqrt{E_Q [M^v]_T} \le C_1 C_2^{1/2} \sqrt{E_Q \langle u(X) \rangle_T} \sqrt{E_Q \langle v(X) \rangle_T} < \infty,$$

with C_1 a universal constant and where we used (3.11) when C_2 appears. By Burkholder-Davis-Gundy inequality again, we also have $M^{Q,[u,v]} \in L^1(Q)$. These considerations justify the cancellation of the expectations of the martingale terms.

The remaining terms B^h, C^h and D^h are controlled using our integrability assumptions and Cauchy-Schwarz inequality. Let us start with B^h:

$$\left(E_Q \int_0^{T-h} |B_t^h|\, dt \right)^2$$

$$\le E_Q \int_0^{T-h} \left(\int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q u(X)\, ds \right)^2 dt\ E_Q \int_0^{T-h} \left(\int_t^{t+h} dM_s^v \right)^2 dt$$

$$\le E_Q \int_0^{T-h} \left(\int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q u(X)\, ds \right)^2 dt\ E_Q \int_0^{T-h} \int_t^{t+h} \overrightarrow{\Gamma}_s^Q(v)(X)\, ds dt$$

$$\le o(h^2)\, E_Q \int_0^{T-h} k^h * (\overrightarrow{\mathcal{L}}^Q u)^2(t, X_{[0,t]})\, dt$$

$$= o(h^2) \left(E_Q \int_0^T (\overrightarrow{\mathcal{L}}_t^Q u)^2(X)\, dt + o_{h \to 0^+}(1) \right),$$

where the third inequality follows from Lebesgue's dominated convergence theorem under the assumption that $\overrightarrow{\Gamma}^Q(v)(X) \in L^1(\overline{Q})$, and use we took $k^h := h^{-1} \mathbf{1}_{[-h,0]}$ as our

convolution kernel, see (3.8). The last identity is a consequence of Lemma A.8 under the assumption $\overrightarrow{\mathcal{L}}^Q u(X) \in L^2(\overline{Q})$. This gives

$$E_Q \int_0^{T-h} h^{-1}|B_t^h|\, dt \leq o_{h \to 0^+}(1)\|\overrightarrow{\mathcal{L}}^Q u(X)\|_{L^2(\overline{Q})} + o_{h \to 0^+}(1)$$

and similarly

$$E_Q \int_0^{T-h} h^{-1}|C_t^h|\, dt \leq o_{h \to 0^+}(1)\|\overrightarrow{\mathcal{L}}^Q v(X)\|_{L^2(\overline{Q})} + o_{h \to 0^+}(1).$$

The control of D^h is analogous:

$$\left(E_Q \int_0^{T-h} |D_t^h|\, dt\right)^2$$
$$\leq E_Q \int_0^{T-h} \left(\int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q u(X)\, ds\right)^2 dt \; E_Q \int_0^{T-h} \left(\int_t^{t+h} \overrightarrow{\mathcal{L}}_s^Q v(X)\, ds\right)^2 dt$$
$$\leq h^4 E_Q \int_0^{T-h} k^h * (\overrightarrow{\mathcal{L}}^Q u)^2(t, X_{[0,t]})\, dt \; E_Q \int_0^{T-h} k^h * (\overrightarrow{\mathcal{L}}_v^Q)^2(t, X_{[0,t]})\, dt$$
$$= h^4 \left(E_Q \int_0^T (\overrightarrow{\mathcal{L}}_t^Q u)^2(X)\, dt + o_{h \to 0^+}(h)\right) \left(E_Q \int_0^T (\overrightarrow{\mathcal{L}}_t^Q v)^2(X)\, dt + o_{h \to 0^+}(h)\right),$$

leading to

$$E_Q \int_0^{T-h} h^{-1}|D_t^h|\, dt \leq h\|\overrightarrow{\mathcal{L}}^Q u(X)\|_{L^2(\overline{Q})}\|\overrightarrow{\mathcal{L}}^Q v(X)\|_{L^2(\overline{Q})} + o_{h \to 0^+}(h).$$

Putting everything together, we obtain

$$\lim_{h \to 0^+} E_Q \int_0^{T-h} \bigg| E_Q\big[h^{-1}\{u(X_{t+h}) - u(X_t)\}\{v(X_{t+h}) - v(X_t)\} \mid X_{[0,t]}\big]$$
$$- h^{-1} \int_t^{t+h} E_Q[\overrightarrow{\Gamma}_s^Q(u, v)(X_{[0,s]}) \mid X_{[0,t]}]\, ds \bigg|\, dt = 0.$$

On the other hand, by Corollary A.9 applied with the convolution kernel $k^h = \frac{1}{h}\mathbf{1}_{[-h,0]}$ and $\mathcal{A}_t = \sigma(X_{[0,t]})$, under the assumptions $\overrightarrow{\Gamma}^Q(u)(X), \overrightarrow{\Gamma}^Q(v)(X) \in L^1(\overline{Q})$, we obtain

$$\lim_{h \to 0^+} E_Q \int_0^{T-h} \bigg| h^{-1} \int_t^{t+h} E_Q[\overrightarrow{\Gamma}_s^Q(u, v)(X_{[0,s]}) \mid X_{[0,t]}]\, ds - \overrightarrow{\Gamma}_t^Q(u, v)(X_{[0,t]}) \bigg|\, dt = 0.$$

The conclusion of the proof of (a) follows from these last two limits.

• Proof of (b). Changing a little bit the previous arguments, in particular using the assumed Markov property of Q, the convolution kernel $k^{-h} := \frac{1}{h}\mathbf{1}_{[0,h]}$ instead of k^h, and applying Corollary A.9 with $\mathcal{A}_t = \sigma(X_t)$, we obtain similarly

$$\lim_{h \to 0^+} E_Q \int_h^T \bigg| E_Q\big[h^{-1}\{u(X_{t-h}) - u(X_t)\}\{v(X_{t-h}) - v(X_t)\} \mid X_{t-h}\big]$$
$$- h^{-1} \int_{t-h}^t E_Q[\overrightarrow{\Gamma}_s^Q(u, v)[X_s] \mid X_{t-h}]\, ds \bigg|\, dt = 0.$$

On the other hand, as in the proof Corollary A.9 we obtain

$$E_Q \int_h^T \left| h^{-1} \int_{t-h}^t E_Q[\overrightarrow{\Gamma}_s^Q(u,v)[X_s] \mid X_{t-h}] \, ds - \overrightarrow{\Gamma}_{t-h}^Q(u,v)[X_{t-h}] \right| dt$$

$$\leq E_Q \int_h^T \left| h^{-1} \int_{t-h}^t \overrightarrow{\Gamma}_s^Q(u,v)[X_s] \, ds - \overrightarrow{\Gamma}_{t-h}^Q(u,v)[X_{t-h}] \right| dt$$

$$= E_Q \int_h^T \left| k^{-h} * \overrightarrow{\Gamma}_t^Q(u,v)[X_t] - \overrightarrow{\Gamma}_{t-h}^Q(u,v)[X_{t-h}] \right| dt$$

$$\leq E_Q \int_h^T \left| k^{-h} * \overrightarrow{\Gamma}_t^Q(u,v)[X_t] - \overrightarrow{\Gamma}_t^Q(u,v)[X_t] \right| dt$$

$$+ E_Q \int_h^T \left| \overrightarrow{\Gamma}_t^Q(u,v)[X_t] - \overrightarrow{\Gamma}_{t-h}^Q(u,v)[X_{t-h}] \right| dt.$$

We know by Lemma A.8 that $\lim_{h \to 0^+} E_Q \int_h^T \left| k^{-h} * \overrightarrow{\Gamma}_t^Q(u,v)[X_t] - \overrightarrow{\Gamma}_t^Q(u,v)[X_t] \right| dt = 0$. With the additional hypothesis that $(t,x) \mapsto \overrightarrow{\Gamma}_t^Q(u,v)[x]$ is continuous, and because $\overrightarrow{\Gamma}^Q(u,v)$ is integrable, we see that $\lim_{h \to 0^+} E_Q \int_h^T \left| \overrightarrow{\Gamma}_{t-h}^Q(u,v)[X_{t-h}] - \overrightarrow{\Gamma}_t^Q(u,v)[X_t] \right| dt = 0$. Putting everything together we arrive at (3.10). □

Corollary 3.13. *Let Q be any path measure and take any u,v in the class $\mathcal{U}_2^Q$. Then, for almost all t,*

$$\lim_{h \to 0^+} E_Q \big[h^{-1} \{u(X_{t+h}) - u(X_t)\} \{v(X_{t+h}) - v(X_t)\} \big] = E_Q \overrightarrow{\Gamma}_t^Q(u,v)(X).$$

If in addition, $\mathcal{U} \subset \text{dom} \overleftarrow{\mathcal{L}}^Q$, $\overleftarrow{\mathcal{L}}^Q u(X), \overleftarrow{\mathcal{L}}^Q v(X) \in L^2(\overline{Q})$, $(u,u),(v,v) \in \text{dom} \overleftarrow{\Gamma}^Q$ and $\overleftarrow{\Gamma}^Q(u)(X), \overleftarrow{\Gamma}^Q(v)(X) \in L^1(\overline{Q})$, and the hypotheses of Lemma 3.9-(b) are satisfied, then for almost all t,

$$\lim_{h \to 0^+} E_Q \big[h^{-1} \{u(X_{t-h}) - u(X_t)\} \{v(X_{t-h}) - v(X_t)\} \big]$$

$$= E_Q \overrightarrow{\Gamma}_t^Q(u,v)(X) = E_Q \overleftarrow{\Gamma}_t^Q(u,v)(X). \tag{3.14}$$

Proof. The first statement follows directly from statement (a) of Lemma 3.9 with Fubini and Jensen. Our additional hypotheses on $\mathcal{U}$, u and v mean that u and v belong to $\mathcal{U}_2^{Q^*}$. Applying (a) to Q^* instead of Q, we have

$$\lim_{h \to 0^+} E_Q \int_h^T \bigg| E_Q \big[h^{-1} \{u(X_{t-h}) - u(X_t)\} \{v(X_{t-h}) - v(X_t)\} \mid X_{[t,T]} \big]$$

$$- \overleftarrow{\Gamma}_t^Q(u,v)(X) \bigg| dt = 0.$$

With Fubini and Jensen again, we see that (b) of Lemma 3.9, and this identity imply (3.14). □

Integration by parts formula. The following easy result is pointed out because it is a technical argument of the proof of next Theorem 3.17.

Lemma 3.15. *For any measurable bounded function $u \in \text{dom} \overrightarrow{\mathcal{L}}^Q$ such that $\overrightarrow{\mathcal{L}}^Q u[\overline{X}] \in L^1(\overline{Q})$, and all $0 \leq s \leq t \leq T$,*

$$E_Q[u(\overline{X}_t) - u(\overline{X}_s) \mid X_s] = E_Q \left[\int_s^t \overrightarrow{\mathcal{L}}^Q u(\overline{X}_r) \, dr \mid X_s \right].$$

For any measurable bounded function $u \in \mathrm{dom}\,\overleftarrow{\mathcal{L}}^Q$ such that $\overleftarrow{\mathcal{L}}^Q u[\overline{X}] \in L^1(\overline{Q})$, and all $0 \le s \le t \le T$,

$$E_Q[u(\overline{X}_t) - u(\overline{X}_s) \mid X_t] = -E_Q\left[\int_s^t \overleftarrow{\mathcal{L}}^Q u(\overline{X}_r)\,dr \mid X_t\right].$$

Proof. The first equality is obvious. Let us look at the second one:

$$E_Q[u(\overline{X}_t) - u(\overline{X}_s) \mid X_t] = E_{Q^*}[u^*(T-t, X_{T-t}) - u^*(T-s, X_{T-s}) \mid X_{T-t}]$$

$$= -E_{Q^*}\left[\int_{T-t}^{T-s} \overrightarrow{\mathcal{L}}^{Q^*} u^*(r, X_r)\,dr \mid X_{T-t}\right] = -E_{Q^*}\left[\int_s^t \overrightarrow{\mathcal{L}}^{Q^*} u^*(T-r, X_{T-r})\,dr \mid X_{T-t}\right]$$

$$= -E_{Q^*}\left[\int_s^t \overleftarrow{\mathcal{L}}^Q u(r, X_{T-r})\,dr \mid X_{T-t}\right] = -E_Q\left[\int_s^t \overleftarrow{\mathcal{L}}^Q u(r, X_r)\,dr \mid X_t\right],$$

as announced. □

Next Theorem 3.17 is the cornerstone of the proofs of time reversal formulas. Before stating it, let us introduce some notation. For any path measure Q, we define

$$\overrightarrow{\mathcal{L}}_t^Q u[X_t] := E_Q\left(\overrightarrow{\mathcal{L}}_t^Q u(X_{[0,t]}) \mid X_t\right), \qquad \overrightarrow{\Gamma}_t^Q(u,v)[X_t] := E_Q\left(\overrightarrow{\Gamma}_t^Q(u,v)(X_{[0,t]}) \mid X_t\right),$$

$$\overleftarrow{\mathcal{L}}_t^Q u[X_t] := E_Q\left(\overleftarrow{\mathcal{L}}_t^Q u(X_{[t,T]}) \mid X_t\right), \qquad \overleftarrow{\Gamma}_t^Q(u,v)[X_t] := E_Q\left(\overleftarrow{\Gamma}_t^Q(u,v)(X_{[t,T]}) \mid X_t\right),$$

where we use square brackets $[X_t]$ to specify the conditional expectation knowing X_t, provided it is well defined. Of course, if Q is Markov, then $\overrightarrow{\mathcal{L}}_t^Q u[X_t] = \overrightarrow{\mathcal{L}}_t^Q u(X_t)$, and so on. We introduce the class of functions

$$\mathcal{U}^Q := \left\{u \in \mathcal{U};\ \overrightarrow{\mathcal{L}}^Q u[\bullet] \in L^1(\bar{\mathsf{q}}),\ \overrightarrow{\Gamma}^Q(u)[\bullet] \in L^1(\bar{\mathsf{q}})\right\}. \tag{3.16}$$

Comparing with (3.7), we see that the differences with $\mathcal{U}_2^Q$ are the conditional expectations with respect to X_t and that $\overrightarrow{\mathcal{L}}^Q u[\bullet]$ stands in $L^1(\bar{\mathsf{q}})$ instead of $L^2(\bar{\mathsf{q}})$. The integrability improvement is useful at Section 7 and in the companion paper [6] when establishing time reversal formulas for jump processes under a finite entropy hypothesis.

Theorem 3.17 (IbP of the carré du champ). *Let $P \in \mathrm{M}(\Omega)$ be any path measure. Take two functions u, v in $\mathcal{U}^P$.*

(a) If

$$u \in \mathrm{dom}\,\overleftarrow{\mathcal{L}}^P \quad and \quad \overleftarrow{\mathcal{L}}^P u(X) \in L^1(\overline{P}), \tag{3.18}$$

then for almost every t

$$E_P\left((\overrightarrow{\mathcal{L}}_t^P u + \overleftarrow{\mathcal{L}}_t^P u)[X_t]v(X_t) + \overrightarrow{\Gamma}_t^P(u,v)[X_t]\right) = 0. \tag{3.19}$$

(b) Suppose that P is Markov,

$$(t, x) \mapsto \overrightarrow{\Gamma}_t^P(u,v)(x) \text{ is continuous}, \tag{3.20}$$

the class of functions $\mathcal{U}^P$ determines the weak convergence of Borel measures on $\mathcal{X}$, and the linear form

$$w \in \mathcal{U}^{\overline{P}} \mapsto E_P \int_{[0,T]} \overrightarrow{\Gamma}_t^P(u, w_t)(X_t)\,dt \tag{3.21}$$

on $\mathcal{U}^{\overline{P}} := \left\{w \in C_b([0,T] \times \mathcal{X});\ w(t, \bullet) \in \mathcal{U}^P,\ \forall 0 \le t \le T\right\}$ defines a finite measure on $[0, T] \times \mathcal{X}$.

Then, (3.18) holds and therefore (3.19) is satisfied.

Remarks 3.22.

(a) The assumption (3.21) is an integration by parts formula.

(b) Statement (a) is really significant when P is a Markov measure because in this case $\overrightarrow{\mathcal{L}}^P[X] = \overrightarrow{\mathcal{L}}^P(X)$, $\overleftarrow{\mathcal{L}}^P[X] = \overleftarrow{\mathcal{L}}^P(X)$ and $\overrightarrow{\Gamma}^P[X] = \overrightarrow{\Gamma}^P(X)$: we do not loose any information and this carries all the necessary material to derive a time reversal formula. We state it in the general form to stress that the Markov property does not play any role in the proof of statement (a).

(c) Using the notion of osmotic extended generator

$$\mathcal{L}_t^{\mathrm{os},P} u(x) := (\overrightarrow{\mathcal{L}}_t^P + \overleftarrow{\mathcal{L}}_t^P)u[x]/2,$$

the IbP formula writes as

$$\int_{\mathcal{X}} v \mathcal{L}_t^{\mathrm{os},P} u \, dP_t = -\frac{1}{2} \int_{\mathcal{X}} \overrightarrow{\Gamma}_t^P(u,v)[x] \, P_t(dx) = -\frac{1}{2} \int_{\mathcal{X}} \overleftarrow{\Gamma}_t^P(u,v)[x] \, P_t(dx),$$

where last equality is Corollary 3.13, provided that the extra hypotheses of this corollary are satisfied. We see that it extends the usual integration by parts formula stated at Proposition 3.36 below, which is only valid for *stationary* Markov measures.

(d) The symmetry of the carré du champ implies

$$\int_{\mathcal{X}} v \mathcal{L}_t^{\mathrm{os},P} u \, dP_t = \int_{\mathcal{X}} u \mathcal{L}_t^{\mathrm{os},P} v \, dP_t.$$

(e) By Proposition A.10 we know that for any $u \in \mathrm{dom}\, \mathcal{L}^{\mathrm{os},P} := \mathrm{dom}\, \overrightarrow{\mathcal{L}}^P \cap \mathrm{dom}\, \overleftarrow{\mathcal{L}}^P$ such that $E_P \int_{[0,T]} (|\overrightarrow{\mathcal{L}}_t^P u| + |\overleftarrow{\mathcal{L}}_t^P u|)(X) \, dt < \infty$, the limit

$$\mathcal{L}_t^{\mathrm{os},P} u(X_t) = \lim_{h \to 0^+} \frac{1}{h} E_P\Big(\frac{u(X_{t+h}) + u(X_{t-h})}{2} - u(X_t) \mid X_t\Big)$$

takes place in $L^1(\overline{P})$.

Proof of Theorem 3.17. We start proving the IbP formula (3.19) assuming that u and v belong to $\mathcal{U}_2^P$, and using both hypotheses (3.18) and (3.20). Once this is done, we extend the result to the case where u and v are in $\mathcal{U}^P$. Finally, we shall see at the end of the proof that it is a simple matter to remove one assumption among (3.18) and (3.20).

● Proof of (3.19) under the hypotheses: $u, v \in \mathcal{U}_2^P$, (3.18) and (3.20). It is based on the elementary identity

$$
\begin{aligned}
[(u_{t+h} - u_t) & + (u_{t-h} - u_t)]v_t \\
&= -(u_{t-h} - u_t)(v_{t-h} - v_t) + v_t(u_{t+h} - u_t) - v_{t-h}(u_t - u_{t-h}),
\end{aligned}
\tag{3.23}
$$

which implies

$$E_P\Big(\big\{E_P[u(X_{t+h}) - u(X_t) \mid X_t] + E_P[u(X_{t-h}) - u(X_t) \mid X_t]\big\}\, v(X_t)\Big)$$

$$= -E_P\Big(\{u(X_{t-h}) - u(X_t)\}\{v(X_{t-h}) - v(X_t)\}\Big)$$

$$+ E_P\Big(v(X_t)E_P[u(X_{t+h}) - u(X_t) \mid X_t]\Big) - E_P\Big(v(X_{t-h})E_P[u(X_t) - u(X_{t-h}) \mid X_{t-h}]\Big).$$

Dividing both sides by $h > 0$, letting $h \to 0^+$,

$$E_P[(\overrightarrow{\mathcal{L}}_t u + \overleftarrow{\mathcal{L}}_t u)[X_t] \, v(X_t)]$$
$$= -E_P \overrightarrow{\Gamma}_t^P(u,v)[X_t] + \lim_{h \to 0^+} E_P\Big(v(X_t)E_P[u(X_{t+h}) - u(X_t) \mid X_t]\Big) \tag{3.24}$$
$$- \lim_{h \to 0^+} E_P\Big(v(X_{t-h})E_P[u(X_t) - u(X_{t-h}) \mid X_{t-h}]\Big),$$

and the proof will be complete once we show that the last two terms cancel each other. Let us present some justifications for (3.24). We denote for any $0 < h \leq T$,

$$\overline{X}_t^h := \overline{X}_{t+h} = (t+h, X_{t+h}), \quad 0 \leq t \leq T - h,$$
$$\overline{X}_t^{-h} := \overline{X}_{t-h} = (t-h, X_{t-h}), \quad h \leq t \leq T.$$

Because u is taken in $\mathcal{U}_2^P$, by the first part of Proposition A.10 (and Jensen's inequality), we have $\lim_{h \to 0^+} h^{-1} E_P[u(\overline{X}^h) - u(\overline{X}) \mid \overline{X}] = \overrightarrow{\mathcal{L}}^P u[\overline{X}]$ in $L^2(\overline{P})$, and a fortiori in $L^1(\overline{P})$. But $v(\overline{X})$ is a bounded function. Hence,

$$\lim_{h \to 0^+} E_P\Big(v(\overline{X}) \, h^{-1}[u(\overline{X}^h) - u(\overline{X})] \mid \overline{X}\Big) = v(\overline{X})\overrightarrow{\mathcal{L}}^P u[\overline{X}] \quad \text{in } L^1(\overline{P}).$$

Similarly, with the second part of Proposition A.10, under the assumption (3.18)

$$\lim_{h \to 0^+} E_P\Big(v(\overline{X}) \, h^{-1}[u(\overline{X}^{-h}) - u(\overline{X})] \mid \overline{X}\Big) = v(\overline{X})\overleftarrow{\mathcal{L}}^P u[\overline{X}] \quad \text{in } L^1(\overline{P}). \tag{3.25}$$

With Fubini and Jensen, this proves

$$\lim_{h \to 0^+} \int_h^{T-h} \Big| h^{-1} E_P\Big([\{u(X_{t+h}) - u(X_t)\} + \{u(X_{t-h}) - u(X_t)\}] \, v(X_t)\Big)$$
$$- E_P\Big((\overrightarrow{\mathcal{L}}_t^P u + \overleftarrow{\mathcal{L}}_t^P u)[X_t] \, v(X_t)\Big)\Big| \, dt = 0. \tag{3.26}$$

Under the assumption (3.20) and because u and v are assumed to belong to $\mathcal{U}_2^P$, by Lemma 3.9-(b) we have also

$$\lim_{h \to 0^+} \int_h^T \Big| E_P\Big(h^{-1}\{u(X_t) - u(X_{t-h})\}\{v(X_t) - v(X_{t-h})\}\Big)$$
$$- E_P \overrightarrow{\Gamma}_t^P(u,v)[X_t]\Big| \, dt = 0. \tag{3.27}$$

It remains to prove that the last terms of (3.24) cancel each other by showing that

$$\lim_{h \to 0^+} \int_0^{T-h} \Big| E_P\Big(v(X_t)h^{-1}[u(X_{t+h}) - u(X_t)]\Big) - E_P(v(\overline{X}_t)\overrightarrow{\mathcal{L}} u[\overline{X}_t])\Big| \, dt = 0, \tag{3.28}$$

$$\lim_{h \to 0^+} \int_h^T \Big| E_P\Big(v(X_{t-h})h^{-1}[u(X_t) - u(X_{t-h})]\Big) - E_P(v(\overline{X}_t)\overrightarrow{\mathcal{L}} u[\overline{X}_t])\Big| \, dt = 0. \tag{3.29}$$

The leftmost integrand of (3.28) is $E_P(v(X_t)\{k^h * \overrightarrow{\mathcal{L}} u[\overline{X}]\}_t)$, so that the identity follows because $v(\overline{X}_t)$ is in $L^\infty(P)$ and $\lim_{h \to 0^+}\{k^h * \overrightarrow{\mathcal{L}} u[\overline{X}]\} = \overrightarrow{\mathcal{L}} u[\overline{X}]$ in $L^1(\overline{P})$ by Lemma A.8. On the other hand, (3.29) is true because

(i) $E_P\Big(v(X_{t-h})h^{-1}[u(X_t) - u(X_{t-h})]\Big) = E_P\Big(v(X_{t-h})\{k^{-h} * \overrightarrow{\mathcal{L}} u[\overline{X}]\}_t\Big)$;

(ii) $\lim_{h \to 0^+} k^{-h} * \overrightarrow{\mathcal{L}} u[\overline{X}] = \overrightarrow{\mathcal{L}} u[\overline{X}]$ in $L^1(\overline{P})$;

(iii) $\lim_{h \to 0^+} v(X_{t-h}) = v(X_{t-}) = v(X_t)$, $\overline{P}$-a.e.

Item (i) follows from Lemma 3.15 and (3.8), (ii) is a direct consequence of Lemma A.8, and (iii) follows because the sample paths are left-limited, it is assumed that v is continuous and bounded, and $X_t = X_{t-}$ for almost every t, P-a.e. because the sample paths are càdlàg.

We have proved

$$E_P \int_{[0,T]} \left| (\overrightarrow{\mathcal{L}}_t^P u + \overleftarrow{\mathcal{L}}_t^P u)[X_t] v(X_t) + \overrightarrow{\Gamma}_t^P(u,v)[X_t] \right| dt = 0, \qquad (3.30)$$

and therefore (3.19), under the hypotheses: $u, v \in \mathcal{U}_2^P$, (3.18) and (3.20). Let us relax this hypothesis by considering functions u and v in $\mathcal{U}^P$ instead of $\mathcal{U}_2^P$.

• Proof of (3.19) under the hypotheses: $u, v \in \mathcal{U}^P$, (3.18) and (3.20). The proof of this extension relies on a localization argument. For any $u, v \in \mathcal{U}^P$ and any $k \geq 1$, we define the stopping time

$$\tau^k := \inf \left\{ t \in [0, T]; \int_0^t |\overrightarrow{\mathcal{L}}_s^P u(X_{[0,s]})| \, ds + \int_0^t |\overrightarrow{\mathcal{L}}_s^P v(X_{[0,s]})| \, ds \geq k \right\}$$

and consider the sequence of stopped path measures $P^k := (X^{\tau^k})_\# P$, $k \geq 1$. Clearly

$$\lim_{k \to \infty} \tau^k = \infty, \ P\text{-a.e.}$$

because u and v belong to dom $\overrightarrow{\mathcal{L}}^P$. For any $k \geq 1$, the functions u and v are in $\mathcal{U}_2^{P^k}$, therefore we have just proved that P^k verifies (3.30):

$$E_{P^k} \int_{[0,T]} \left| (\overrightarrow{\mathcal{L}}_t^{P^k} u + \overleftarrow{\mathcal{L}}_t^{P^k} u)[X_t] v(X_t) + \overrightarrow{\Gamma}_t^{P^k}(u,v)[X_t] \right| dt = 0.$$

On the other hand, $\overrightarrow{\mathcal{L}}_t^{P^k} u[x] = E_P[\mathbf{1}_{\{t < \tau^k\}} \overrightarrow{\mathcal{L}}_t^P u(X) \mid X_t = x]$, $\overleftarrow{\mathcal{L}}_t^{P^k} u[x] = E_P \mathbf{1}_{\{t \leq \tau^k\}} \overleftarrow{\mathcal{L}}_t^P u$, and $\overrightarrow{\Gamma}_t^{P^k}(u,v) = \mathbf{1}_{\{t < \tau^k\}} \overrightarrow{\Gamma}_t^P(u,v)$. Hence

$$0 = E_{P^k} \int_{[0,T]} \left| (\mathbf{1}_{\{t < \tau^k\}} \overrightarrow{\mathcal{L}}_t^P u + \mathbf{1}_{\{t \leq \tau^k\}} \overleftarrow{\mathcal{L}}_t^P u)[X_t] v(X_t) + \mathbf{1}_{\{t < \tau^k\}} \overrightarrow{\Gamma}_t^P(u,v)[X_t] \right| dt$$

$$= E_P \int_{[0,T]} \left| (\mathbf{1}_{\{t < \tau^k\}} \overrightarrow{\mathcal{L}}_t^P u + \mathbf{1}_{\{t \leq \tau^k\}} \overleftarrow{\mathcal{L}}_t^P u)[X_t] v(X_t) + \mathbf{1}_{\{t < \tau^k\}} \overrightarrow{\Gamma}_t^P(u,v)[X_t] \right| dt$$

$$= E_P \int_{[0,T]} \left| (\overrightarrow{\mathcal{L}}_t^P u + \overleftarrow{\mathcal{L}}_t^P u)[X_t] v(X_t) + \overrightarrow{\Gamma}_t^P(u,v)[X_t] \right| dt.$$

The second equality holds because P and P^k match on $\{t \leq \tau^k\}$, and last equality follows letting k tend to infinity by dominated convergence under our integrability assumptions. We have proved (3.19) under the hypotheses: $u, v \in \mathcal{U}^P$, (3.18) and (3.20).

This proof was based on the convergence of the identity (3.23) as h tends to zero. But for this convergence to hold, it is sufficient that only three of its four terms converge. We take advantage of this remark to complete the proof.

• Proof of (a). Let us remove (3.20). This assumption was used to obtain (3.27) and was not used anywhere else. Hence, the limits of the other three terms of (3.23) are valid even in absence of (3.20), showing in return that in addition to (3.19), (3.27) holds true.

• Proof of (b). Let us remove (3.18). This assumption was used to obtain (3.25) and was not used anywhere else. Hence, the limits of the other three terms of (3.23) are valid

even in absence of (3.18), showing in return that in addition to (3.19), the expectation of (3.25) holds true: the limit

$$\lim_{h \to 0^+} E_P \int_{[h,T]} w_t(X_t) h^{-1} \{u(X_{t-h}) - u(X_t)\} \, dt =: \overleftarrow{\ell}^P_u(w) \tag{3.31}$$

exists for all $w \in \mathcal{U}^{\overline{P}}$ (passing from $\mathcal{U}^P$ to $\mathcal{U}^{\overline{P}}$ is obvious), and we have

$$\overleftarrow{\ell}^P_u(w) = E_P \int_{[0,T]} \left(-\overrightarrow{\mathcal{L}}^P_t u[X_t] w_t(X_t) - \overrightarrow{\Gamma}^P_t(u,w)[X_t] \right) dt, \qquad w \in \mathcal{U}^{\overline{P}}. \tag{3.32}$$

As we assume that $\mathcal{U}^P$ is separating and $w \mapsto E_P \int_{[0,T]} \overrightarrow{\Gamma}^P_t(u,w)[X_t] \, dt$ defines a finite measure, $\overleftarrow{\ell}^P_u$ is also a finite measure on $[0,T] \times \mathcal{X}$ because $\overrightarrow{\mathcal{L}}^P u \in L^1(\overline{P})$. It is absolutely continuous with respect to $\bar{\mathsf{p}}$ because $\int_{[0,T] \times \mathcal{X}} |w| \, d\bar{\mathsf{p}} = 0$ implies $\int_{[0,T] \times \mathcal{X}} \overrightarrow{\Gamma}^P_t(u,w_t)[x] \, \bar{\mathsf{p}}(dtdx) = 0$. Moreover, since $\mathcal{U}^P$ is convergence-determining, so is $\mathcal{U}^{\overline{P}}$, and with (3.31) and Proposition A.11 we see that $\overleftarrow{\mathcal{L}}^P u = d\overleftarrow{\ell}^P_u / d\bar{\mathsf{p}}$. We conclude with (3.32) that the IbP formula (3.19) is satisfied. $\square$

Remark 3.33. Another very similar proof is based on the elementary identity

$$[(u_{t+h} - u_t) + (u_{t-h} - u_t)]v_t$$
$$= -(u_{t+h} - u_t)(v_{t+h} - v_t) + v_t(u_{t-h} - u_t) - v_{t+h}(u_t - u_{t+h}).$$

Doing this, one sees that (a) is still valid. But (b) does not follow so easily because one cannot drop (3.18).

Corollary 3.34. *Let u be in $\mathcal{U}^P$ and suppose that (3.18) is fulfilled. Then, $u \in \mathrm{dom}\, \mathcal{L}^{\mathrm{os},P}$ and for almost every t*

$$\int_{\mathcal{X}} \mathcal{L}^{\mathrm{os},P}_t u \, dP_t = 0.$$

Proof. Apply Theorem 3.17 with $v = 1$. $\square$

Stationary Markov measure. To make the point of Remark 3.22-(c) precise, let us recall what time reversal does with stationary Markov measures.

We consider a *stationary* Markov measure $P \in \mathrm{M}(\Omega)$ with stationary measure m, i.e. $P_t = \mathsf{m}$, for all $t \in [0,T]$. Stationary means that for any real numbers $t_1, \ldots, t_k$ and h, the laws of $(X_{t_1+h}, \ldots, X_{t_k+h})$ and $(X_{t_1}, \ldots, X_{t_k})$ under P are the same. As P is Markov, it is sufficient that this property holds for $k = 2$. We restrict the time interval to $[0,T]$.

Define the class of functions

$$\mathcal{V} := \left\{ v \in \mathrm{dom}\, \overrightarrow{\mathcal{L}}^P \cap L^2(\mathsf{m}); \; \overrightarrow{\mathcal{L}}^P v \in L^1(\mathsf{m}) \cap L^2(\mathsf{m}) \right\}.$$

The adjoint $(\overrightarrow{\mathcal{L}}^P_{|\mathcal{V}})^*$ in $L^2(\mathsf{m})$ of the restriction $\overrightarrow{\mathcal{L}}^P_{|\mathcal{V}}$ to $\mathcal{V}$ of the forward generator $\overrightarrow{\mathcal{L}}^P$ of P is defined by: $\int_{\mathcal{X}} v(\overrightarrow{\mathcal{L}}^P_{|\mathcal{V}})^* u \, d\mathsf{m} = \int_{\mathcal{X}} u \overrightarrow{\mathcal{L}}^P v \, d\mathsf{m}$, for any $u, v \in \mathcal{V}$.

Lemma 3.35. *Suppose that P is Markov and stationary, then: $\overleftarrow{\mathcal{L}}^P_{|\mathcal{V}} = (\overrightarrow{\mathcal{L}}^P_{|\mathcal{V}})^*$.*

Proof. Fix t, h such that $0 \le t \le t + h \le T$ and take $u, v \in \mathcal{V}$. By stationarity

$$E_P[u(X_t)\{v(X_{t+h}) - v(X_t)\}]$$
$$= E_P[\{u(X_{t-h}) - u(X_t)\}v(X_t)] + E_P[u(X_t)v(X_{t+h}) - u(X_{t-h})v(X_t)]$$
$$= E_P[\{u(X_{t-h}) - u(X_t)\}v(X_t)].$$

Dividing by $h > 0$ and letting it tend to zero, we conclude with Proposition A.10. $\square$

This is a well-known result. One of its versions in the framework of discrete time was published by Nelson in 1958 [35]. We find it pleasant to provide an elementary proof in the continuous-time setting, based on stochastic derivatives: a tool developed by Nelson himself a decade later.

Proposition 3.36 (Integration by parts). *Suppose that $P \in \mathrm{M}(\Omega)$ is Markov and stationary, then for any $u, v \in \mathcal{V}$, such that $uv \in \mathcal{V}$,*

$$\int_{\mathcal{X}} u\mathcal{L}^{\mathrm{sym},P} v \, d\mathsf{m} = -\frac{1}{2} \int_{\mathcal{X}} \overrightarrow{\Gamma}^{P}(u,v) \, d\mathsf{m},$$

where

$$\mathcal{L}^{\mathrm{sym},P} := (\overrightarrow{\mathcal{L}}^{P} + (\overrightarrow{\mathcal{L}}^{P})^{*})/2 = (\overrightarrow{\mathcal{L}}^{P} + \overleftarrow{\mathcal{L}}^{P})/2 = \mathcal{L}^{\mathrm{os},P} \qquad (3.37)$$

is the algebraic symmetrization of $\overrightarrow{\mathcal{L}}^{P}$.

Proof. Let us denote for simplicity $A := \overrightarrow{\mathcal{L}}^{P}_{|\mathcal{V}}$. Of course, $\int_{\mathcal{X}} Au \, d\mathsf{m} = 0$ because $\int_{\mathcal{X}} Au \, d\mathsf{m} = \int_{\mathcal{X}} \mathbf{1} Au \, d\mathsf{m} = \int_{\mathcal{X}} uA^{*}\mathbf{1} \, d\mathsf{m}$ and the stationarity implies that $A^{*}\mathbf{1} = 0$. Therefore,

$$\begin{aligned}
\int_{\mathcal{X}} \overrightarrow{\Gamma}^{P}(u,v) \, d\mathsf{m} &= \int_{\mathcal{X}} \{A(uv) - uAv - vAu\} \, d\mathsf{m} \\
&= -\int_{\mathcal{X}} \{uAv + vAu\} \, d\mathsf{m} = -\int_{\mathcal{X}} \{uA^{*}v + vA^{*}u\} \, d\mathsf{m} \\
&= -2 \int_{\mathcal{X}} u\bar{A}v \, d\mathsf{m}
\end{aligned}$$

with $\bar{A} := (A + A^{*})/2$. We conclude with Lemma 3.35. $\qquad \square$

Finite entropy. Up to now the entropy did not play any role. Let us write some words about it in preparation to forthcoming time reversal formulas.

Comparing statements (a) and (b) of Theorem 3.17, we see that (b) is easier to verify than (a), because (a) requires that u is in the domain of the backward generator: a property which is not known a priori. On the other hand, the assumption (3.20) in (b) is too much demanding for some applications we have in mind, where a finite entropy condition destroys this regularity in presence of jumps, see [6].

We are going to investigate time reversal of Markov measures P verifying the finite entropy condition (1.7): $H(P|R) < \infty$, where the time reversal R^{*} of a reference Markov measure R is accessible via Theorem 3.17-(b). Then, taking advantage of the elementary identity $H(P^{*}|R^{*}) = H(P|R) < \infty$, a deep insight of Föllmer already encountered at (1.2), we shall be in position to build a large enough class $\mathcal{U}^{P}$ and to verify the assumptions of Theorem 3.17-(a) for P.

4. Time reversal of a diffusion process in $\mathbb{R}^{n}$

In this section, the IbP formula of Theorem 3.17 is used to obtain at Theorem 4.9 a time reversal formula for diffusion measures.

Reference diffusion measure. The path space is the set $\Omega = C([0,T], \mathbb{R}^{n})$ of all continuous trajectories from $[0,T]$ to $\mathbb{R}^{n}$. The main reference measure we have in mind is the reversible Kolmogorov diffusion R defined at (1.11).

Finite entropy in a diffusion setting. Take $Q \in \mathrm{P}(\Omega)$ such that

$$H(Q|R) < \infty. \tag{4.1}$$

We know by the Girsanov theory under a finite entropy condition [30], that when R fulfils the uniqueness condition:

$$\forall R' \in \mathrm{M}(\Omega), \ [R' \in \mathrm{MP}(R_0, \mathsf{a}, b^R) \text{ and } R' \ll R] \implies R' = R, \tag{4.2}$$

there exists some $\mathbb{R}^n$-valued predictable process $\beta^{Q|R}$ which is defined $\overline{Q}$-a.e. such that Q solves the martingale problem

$$Q \in \mathrm{MP}(Q_0, \mathsf{v}^{\mathsf{a},\mathsf{m}} + \mathsf{a}\beta^{Q|R}, \mathsf{a}). \tag{4.3}$$

Recall Remark 1.18 for a setting where the uniqueness condition (4.2) is satisfied. Furthermore, because of the uniqueness of the solution to $\mathrm{MP}(\mathsf{m}, \mathsf{v}^{\mathsf{a},\mathsf{m}}, \mathsf{a})$, we know that

$$\begin{aligned}
\frac{dQ}{dR} &= \mathbf{1}_{\{dQ/dR>0\}} \, \frac{dQ_0}{dR_0}(X_0) \exp\left(\int_{[0,T]} \beta_t^{Q|R} \cdot dM_t^R - \int_{[0,T]} |\beta_t^{Q|R}|^2_{\mathsf{a}(X_t)}/2 \, dt \right) \\
&= \mathbf{1}_{\{dQ/dR>0\}} \, \frac{dQ_0}{dR_0}(X_0) \exp\left(\int_{[0,T]} \beta_t^{Q|R} \cdot dM_t^Q + \int_{[0,T]} |\beta_t^{Q|R}|^2_{\mathsf{a}(X_t)}/2 \, dt \right),
\end{aligned}$$

where

$$dM_t^R = dX_t - \mathsf{v}^{\mathsf{a},\mathsf{m}}(X_t) \, dt \quad \text{and} \quad dM^Q = dX_t - (\mathsf{v}^{\mathsf{a},\mathsf{m}}(X_t) + \mathsf{a}(X_t)\beta_t^{Q|R}) \, dt,$$

and we denote

$$|\beta|^2_{\mathsf{a}} := \beta \cdot \mathsf{a}\beta.$$

Moreover,

$$H(Q|R) = H(Q_0|R_0) + E_Q \int_{[0,T]} |\beta_t^{Q|R}|^2_{\mathsf{a}(X_t)}/2 \, dt. \tag{4.4}$$

Of course, in view of this identity, $H(Q|R) < \infty$ implies that $E_Q \int_{[0,T]} |\beta_t^{Q|R}|^2_{\mathsf{a}(X_t)} \, dt$ is finite.

Claim 4.5. *If in addition Q is Markov, then the process $\beta^{Q|R}$ turns out to be a vector field:*

$$\beta_t^{Q|R} = \beta^{Q|R}(\overline{X}_t), \ \overline{Q}\text{-a.e.}$$

Proof. Indeed, we see with (4.3) that

$$\begin{aligned}
[\mathsf{v}^{\mathsf{a},\mathsf{m}}(X_t) + \mathsf{a}(X_t)\beta_t^{Q|R}] \, dt &= E_Q(dX_t \mid X_{[0,t]}) = E_Q(dX_t \mid X_t) \\
&= \mathsf{v}^{\mathsf{a},\mathsf{m}}(X_t) + \mathsf{a}(X_t)E_Q(\beta_t^{Q|R} \mid X_t)] \, dt, \quad Q\text{-a.e.}
\end{aligned}$$

Remark that all the above conditional expectations are well-defined; in particular $E_Q(\beta_t^{Q|R} \mid X_t)$ is meaningful because of (4.4) and the finite entropy assumption (4.1). It follows that for all $0 \le t \le T$, $\mathsf{a}(X_t)\beta_t^{Q|R} = \mathsf{a}(X_t)E_Q(\beta_t^{Q|R} \mid X_t)$, $\overline{Q}$-a.e. $\square$

Moreover, we observe that

$$H(Q|R) - H(Q_0|R_0) = H(Q|R^{Q_0}) = E_Q \int_{[0,T]} \frac{1}{2}|\overrightarrow{\mathsf{v}}^{Q|R}|^2_{\mathsf{g}}(\overline{X}_t) \, dt$$

is an average kinetic action, where $\overrightarrow{\mathsf{v}}^{Q|R} := \mathsf{a}\beta^{Q|R}$ and

$$\mathsf{g} = \mathsf{a}^{-1}.$$

Nelson's velocities. The *forward stochastic velocity* $\overrightarrow{\mathsf{v}}^Q$ is

$$\overrightarrow{\mathsf{v}}^Q(t,x) := \overrightarrow{L}_t^Q[\mathrm{Id}](x) = \lim_{h\to 0^+} E_Q\Big(\frac{X_{t+h}-X_t}{h} \mid X_t = x\Big),$$

and similarly, we define the *backward velocity*

$$\overleftarrow{\mathsf{v}}^Q(t,x) := \overleftarrow{L}_t^Q[\mathrm{Id}](x) = \lim_{h\to 0^+} E_Q\Big(\frac{X_{t-h}-X_t}{h} \mid X_t = x\Big),$$

whenever these expressions are meaningful. These velocities might not be well defined because of a lack of integrability. However, under a finite entropy condition, Proposition 4.6 below tells us that they are well defined in the setting we are interested in.
The *current velocity* is

$$\mathsf{v}^{\mathrm{cu},Q} := (\overrightarrow{\mathsf{v}}^Q - \overleftarrow{\mathsf{v}}^Q)/2$$

and the *osmotic velocity* is

$$\mathsf{v}^{\mathrm{os},Q} := (\overrightarrow{\mathsf{v}}^Q + \overleftarrow{\mathsf{v}}^Q)/2.$$

We immediately observe that

$$\left\{\begin{array}{rcl} \overrightarrow{\mathsf{v}} &=& \mathsf{v}^{\mathrm{cu}} + \mathsf{v}^{\mathrm{os}}, \\ \overleftarrow{\mathsf{v}} &=& -\mathsf{v}^{\mathrm{cu}} + \mathsf{v}^{\mathrm{os}} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcl} \mathsf{v}_t^{\mathrm{cu},Q^*} &=& -\mathsf{v}_{T-t}^{\mathrm{cu},Q}, \\ \mathsf{v}_t^{\mathrm{os},Q^*} &=& \mathsf{v}_{T-t}^{\mathrm{os},Q}. \end{array}\right.$$

Entropy under time reversal. Next result is a central observation in Föllmer's approach to time reversal.

Proposition 4.6. *Under the Hypotheses 1.10, let P be a Markov probability measure such that $H(P|R) < \infty$.*
Then, there exist two measurable vector fields $\overrightarrow{\beta}^{P|R}$ and $\overleftarrow{\beta}^{P|R}$ such that

$$\begin{array}{rcl} \overrightarrow{\mathcal{L}}^P &=& \partial_t + \overrightarrow{\mathsf{v}}^P\cdot\nabla + \Delta_{\mathsf{a}}/2, \quad where \quad \overrightarrow{\mathsf{v}}^P = \mathsf{v}^{\mathsf{a},\mathsf{m}} + \mathsf{a}\overrightarrow{\beta}^{P|R} \\ \overleftarrow{\mathcal{L}}^P &=& -\partial_t + \overleftarrow{\mathsf{v}}^P\cdot\nabla + \Delta_{\mathsf{a}}/2, \quad where \quad \overleftarrow{\mathsf{v}}^P = \mathsf{v}^{\mathsf{a},\mathsf{m}} + \mathsf{a}\overleftarrow{\beta}^{P|R} \end{array}$$

with

$$E_P \int_{[0,T]} (|\overrightarrow{\beta}^{P|R}|_{\mathsf{a}}^2 + |\overleftarrow{\beta}^{P|R}|_{\mathsf{a}}^2)(\overline{X}_t)\, dt < \infty,$$

and

$$H(P|R) = H(P_0|R_0) + E_P \int_{[0,T]} \frac{1}{2}|\overrightarrow{\beta}^{P|R}|_{\mathsf{a}}^2(\overline{X}_t)\, dt$$

$$= E_P \int_{[0,T]} \frac{1}{2}|\overleftarrow{\beta}^{P|R}|_{\mathsf{a}}^2(\overline{X}_t)\, dt + H(P_T|R_T).$$

Proof. Since P is Markov, so is $P^* := (X^*)_\# P$. As the time reversal mapping X^* is one-one, we have $H(P|R) = H(P^*|R^*)$. Hence,

$$H(P|R) = H(P^*|R^*) = H(P^*|R) < \infty,$$

where last equality comes from the reversibility of R which implies $R^* = R$. Again, by Girsanov theory we know that there is some previsible vector field $\overrightarrow{\beta}^{P^*|R}$ such that P^* solves

the martingale problem $\mathrm{MP}(P_T, \mathsf{v}^{\mathsf{a,m}} + \mathsf{a}\overrightarrow{\beta}^{P^*|R}, \mathsf{a})$. Denoting $\overleftarrow{\beta}^{P|R}(t,z) := \overrightarrow{\beta}^{P^*|R}(T-t,z)$, we see that

$$H(P^*|R) = H(P_0^*|R_0) + E_{P^*} \int_{[0,T]} \frac{1}{2} |\overrightarrow{\beta}^{P^*|R}|_{\mathsf{a}}^2 (\overline{X}_t)\, dt$$

$$= H(P_T|R_T) + E_P \int_{[0,T]} \frac{1}{2} |\overleftarrow{\beta}^{P|R}|_{\mathsf{a}}^2 (\overline{X}_t)\, dt,$$

as announced. $\qquad\square$

Continuity equation. Proposition 4.8 below, which is the object of this subsection is not directly linked to time reversal (it is rather complementary). Nevertheless, we present its easy short proof because all the preliminary notions which are needed to its statement and proof appear in the last previous pages.

Lemma 4.7. *Under the Hypotheses 1.10, let $P \in \mathrm{P}(\Omega)$ be Markov and such that $H(P|R) < \infty$. Then, any compactly supported function $u \in C_c^{1,2}([0,T] \times \mathbb{R}^n)$ stands in the domain of both $\overrightarrow{\mathcal{L}}^P$ and $\overleftarrow{\mathcal{L}}^P$, and $E_{\overline{P}}|\overrightarrow{\mathcal{L}}^P u(\overline{X})|^2 < \infty$, $E_{\overline{P}}|\overleftarrow{\mathcal{L}}^P u(\overline{X})|^2 < \infty$.*
Moreover $u(\overline{X}_t) - u(\overline{X}_0) - \int_0^t \overrightarrow{\mathcal{L}}_s^P u[X_s]\, ds$ and $u(\overline{X}_t) - u(\overline{X}_T) - \int_t^T \overleftarrow{\mathcal{L}}_s^P u[X_s]\, ds$ are respectively genuine (rather than local) forward and backward P-martingales.

Proof. The proofs of the statements concerning $\overrightarrow{\mathcal{L}}^P$ and $\overleftarrow{\mathcal{L}}^P$ being similar, we focus on $\overrightarrow{\mathcal{L}}^P$. Take u in $C_c^{1,2}([0,T] \times \mathbb{R}^n)$. All we have to show is

$$E_{\overline{P}}|\overrightarrow{\mathcal{L}}^P u(\overline{X})|^2 < \infty.$$

By Proposition 4.6,

$$\overrightarrow{\mathcal{L}}^P u = \partial_t u + \mathsf{v}^{\mathsf{a,m}} \cdot \nabla u + \frac{1}{2}\Delta_{\mathsf{a}} u + \mathsf{a}\overrightarrow{\beta}^{P|R} \cdot \nabla u$$

and $E_{\overline{P}}\big(|\overrightarrow{\beta}^{P|R}|_{\mathsf{a}}^2(\overline{X})\big) < \infty$. Since a and $\mathsf{v}^{\mathsf{a,m}}$ are locally bounded, $\overrightarrow{\mathcal{L}}^R u = \partial_t u + \mathsf{v}^{\mathsf{a,m}} \cdot \nabla u + \frac{1}{2}\Delta_{\mathsf{a}} u$ is bounded. The last term is controlled by

$$E_{\overline{P}}\big(|\mathsf{a}\overrightarrow{\beta}^{P|R} \cdot \nabla u(\overline{X})|^2\big) \le E_{\overline{P}}\big(|\overrightarrow{\beta}^{P|R}|_{\mathsf{a}}^2(\overline{X})\,|\nabla u|_{\mathsf{a}}^2(\overline{X})\big) \le \sup |\nabla u|_{\mathsf{a}}^2 E_{\overline{P}}\big(|\overrightarrow{\beta}^{P|R}|_{\mathsf{a}}^2(\overline{X})\big) < \infty.$$

$\qquad\square$

For any measure m and vector field w on $\mathbb{R}^n$, we define $\mathrm{div}_m(\mathsf{w})$ by:

$$\int_{\mathbb{R}^n} u\,\mathrm{div}_m(\mathsf{w})\, dm := -\int_{\mathbb{R}^n} \nabla u \cdot \mathsf{w}\, dm, \quad u \in C_c^1(\mathbb{R}^n),$$

whenever the second integral is meaningful.

As a consequence of Proposition 4.6, we obtain

Proposition 4.8 (Continuity equation). *Under the Hypotheses 1.10, let $P \in \mathrm{P}(\Omega)$ be Markov and such that $H(P|R) < \infty$. Then for any t, $P_t \ll \mathsf{m}$ and*

$$\rho_t := \frac{dP_t}{d\mathsf{m}}$$

solves, in the sense of distributions, the continuity equation

$$\partial_t \rho + \mathrm{div}_{\mathsf{m}}(\rho \mathsf{v}^{\mathrm{cu},P}) = 0.$$

Similarly the density

$$\mu_t := \frac{dP_t}{d\mathrm{Leb}}$$

solves, in the sense of distributions, the continuity equation

$$\partial_t \mu + \mathrm{div}(\mu \mathsf{v}^{\mathrm{cu},P}) = 0.$$

Proof. By Lemma 4.7, for any $0 \le s \le t$, and any $u \in C_c^{1,2}((0,T) \times \mathbb{R}^n)$, we have

$$E_P[u(\overline{X}_t) - u(\overline{X}_s)] = \int_s^t E_P \overrightarrow{\mathcal{L}}^P u(\overline{X}_r)\, dr,$$

and similarly, with the definition (2.2) of $\overleftarrow{\mathcal{L}}^P$

$$E_P[u(\overline{X}_t) - u(\overline{X}_s)] = E_{P^*}[u^*(\overline{X}_{T-t}) - u^*(\overline{X}_{T-s})]$$

$$= E_{P^*} \int_{T-s}^{T-t} \overrightarrow{\mathcal{L}}^{P^*} u^*(\overline{X}_r)\, dr = E_P \int_s^t \overleftarrow{\mathcal{L}}^P u(\overline{X}_r)\, dr.$$

With the expressions of $\overrightarrow{\mathcal{L}}^P$ and $\overleftarrow{\mathcal{L}}^P$ stated at Proposition 4.6, this leads us to

$$0 = E_P \int_s^t \frac{1}{2}[\overrightarrow{\mathcal{L}}^P - \overleftarrow{\mathcal{L}}^P]u(\overline{X}_r)\, dr = E_P \int_s^t [\partial_r + \mathsf{v}^{\mathrm{cu},P}\cdot\nabla]u(\overline{X}_r)\, dr$$

$$= \int_{[s,t]\times\mathbb{R}^n} [\partial_r u + \mathsf{v}^{\mathrm{cu},P}\cdot\nabla u](r,x)\, \rho_r(x)\, \mathsf{m}(dx)dr$$

which is the first announced continuity equation. The second one follows replacing $\rho_r(x)\, \mathsf{m}(dx)$ by $\mu_r(x)dx$. $\qquad\square$

Time reversal formula. The main result of this section is the following

Theorem 4.9 (Time reversal formula). *Under the Hypotheses 1.10 on R given at (1.11), let $P \in \mathrm{P}(\Omega)$ be Markov and such that $H(P|R) < \infty$.*
Then, the time reversal P^* of P is a solution of the martingale problem

$$P^* \in \mathrm{MP}(\overrightarrow{\mathsf{v}}^{P^*}, \mathsf{a})$$

with

$$\overrightarrow{\mathsf{v}}_t^{P^*} = \overleftarrow{\mathsf{v}}_{T-t}^P(x) = -\overrightarrow{\mathsf{v}}_{T-t}^P(x) + \nabla\cdot(\mu_{T-t}\mathsf{a})(x)/\mu_{T-t}(x), \quad dtP_t(dx)\text{-a.e.} \qquad (4.10)$$

where the divergence is in the sense of distributions, $\mu_t := dP_t/d\mathrm{Leb}$ and $\overleftarrow{\mathsf{v}}_t^P$ is defined at almost all t.
Furthermore, P^* is the unique solution of $\mathrm{MP}(\overrightarrow{\mathsf{v}}^{P^*}, \mathsf{a})$ among the set of all $Q \in \mathrm{P}(\Omega)$ such that $H(Q|R) < \infty$.
Denoting $\rho_t := dP_t/d\mathsf{m}$ and $\beta^{\mathrm{os},P|R} := \beta^{\mathrm{os},P} - \beta^{\mathrm{os},R}$, (4.10) is equivalent to

$$\beta_t^{\mathrm{os},P|R}(x) = \nabla\log\sqrt{\rho_t}(x), \qquad dtP_t(dx)\text{-a.e.} \qquad (4.11)$$

where the derivative is distributional and

$$\int_{[0,T]\times\mathbb{R}^n} |\nabla\log\rho_t|_\mathsf{a}^2\, dP_t dt < \infty. \qquad (4.12)$$

Remarks 4.13 (about Theorem 4.9).
(a) As $H(P|R) < \infty$, $P_t \ll \mathsf{m} \ll \mathrm{Leb}$ for all t. Hence μ and ρ are well defined.
(b) With $\mathsf{v}^{\mathrm{os},P|R} := \mathsf{v}^{\mathrm{os},P} - \mathsf{v}^{\mathrm{os},R}$, this immediately implies that P_t-a.e., for almost all t,

$$\mathsf{v}_t^{\mathrm{os},P|R} = \mathsf{a}\nabla\log\sqrt{\rho_t}, \qquad (4.14)$$

$$\mu_t\, \mathsf{v}_t^{\mathrm{os},P} = \nabla\cdot(\mu_t\mathsf{a})/2, \qquad (4.15)$$

$$\rho_t\mathsf{v}_t^{\mathrm{os},P|R} = \mathsf{a}\nabla\rho_t/2, \qquad (4.16)$$

in the sense of distributions.

(c) Of course, (4.10) or (4.15) are equivalent to

$$\overleftarrow{\mathsf{v}}_t^P = -\overrightarrow{\mathsf{v}}_t^P + \nabla \cdot \mathsf{a} + \mathsf{a}\nabla \log \mu_t, \quad P_t\text{-a.e.} \tag{4.17}$$

or

$$\mathsf{v}_t^{\mathrm{os},P} = \nabla \cdot \mathsf{a}/2 + \mathsf{a}\nabla \log \sqrt{\mu_t}, \quad P_t\text{-a.e.,} \tag{4.18}$$

(d) The restriction "P_t-a.e." in (4.10), (4.14), (4.17) and (4.18) prevents ρ_t and μ_t from vanishing, so that the log is well defined.

(e) The reference measure

$$R^o \in \mathrm{MP}(\mathrm{Leb}, \nabla \cdot \mathsf{a}/2, \mathsf{a})$$

is the law of a stationary diffusion process with Lebesgue measure as stationary measure ($U = 0$). Its forward generator is

$$\partial_t u + (\nabla \cdot \mathsf{a}) \cdot \nabla u/2 + \Delta_\mathsf{a} u/2 = \partial_t u + \nabla \cdot (\mathsf{a}\nabla u)/2, \quad u \in C^{1,2}([0,T] \times \mathbb{R}^n).$$

Choosing this reference measure, we see that (4.18) writes as $\mathsf{v}^{\mathrm{os},P} = \overrightarrow{\mathsf{v}}^{R^o} + \mathsf{v}^{\mathrm{os},P|R^o}$ with $\overrightarrow{\mathsf{v}}^{R^o} = \nabla \cdot \mathsf{a}/2$, and $\mathsf{v}^{\mathrm{os},P|R^o} = \mathsf{a}\nabla \log \sqrt{\mu}$ which is (4.14) with $\rho_t = \mu_t$ for each t since $\mathsf{m}^o = \mathrm{Leb}$.

Proof of Theorem 4.9. Again, remark that the class of functions $\mathcal{U} = C_c^{1,2}([0,T] \times \mathbb{R}^n)$ for which Itô's formula is valid is an algebra, as required by the hypotheses of the IbP formula (Theorem 3.17).

By Proposition 4.6, we know that there exist two vector fields $\overrightarrow{\beta}^{P|R}$, $\overleftarrow{\beta}^{P|R}$ such that $\overrightarrow{\mathcal{L}}^P = \partial_t + (\mathsf{v}^{\mathsf{a},\mathsf{m}} + \mathsf{a}\overrightarrow{\beta}^{P|R}) \cdot \nabla + \Delta_\mathsf{a}/2$ and $\overleftarrow{\mathcal{L}}^P = -\partial_t + (\mathsf{v}^{\mathsf{a},\mathsf{m}} + \mathsf{a}\overleftarrow{\beta}^{P|R}) \cdot \nabla + \Delta_\mathsf{a}/2$ with

$$E_P \int_{[0,T]} |\beta^{\mathrm{os},P|R}|_\mathsf{a}^2(X_t)\, dt < \infty, \tag{4.19}$$

where

$$\beta^{\mathrm{os}} := (\overrightarrow{\beta} + \overleftarrow{\beta})/2.$$

Then, for any test function $w \in C_c^2(\mathbb{R}^n)$ and almost all t, we have

$$E_P\Big(w(X_t)\mathsf{v}_i^{\mathrm{os},P}(\overline{X}_t) + \mathsf{a}^i(\overline{X}_t) \cdot \nabla w(X_t)/2\Big) = 0, \quad 1 \le i \le n, \tag{4.20}$$

where $\mathsf{v}_i^{\mathrm{os},P}$ is the i-th component of $\mathsf{v}^{\mathrm{os},P}$, and a^i is the i-th column of a. This follows from an application of Theorem 3.17 with $u(t,x) = \mathrm{proj}_i(x)\chi(x)$, $1 \le i \le n$, (where $\mathrm{proj}_i(x) := x_i$ and $\chi \in C_c^2(\mathbb{R}^n)$ has a compact support and is equal to 1 on $\mathrm{supp}\, w$) and is allowed by Lemma 4.7 which ensures that $\overrightarrow{\mathsf{v}}_i^P := \overrightarrow{\mathcal{L}}^P(\mathrm{proj}_i) = (\mathsf{v}^{\mathsf{a},\mathsf{m}} + \mathsf{a}\overrightarrow{\beta}^{P|R})_i$ is in $L_{\mathrm{loc}}^2(\bar{\mathsf{p}})$. Similar estimates hold for $\overleftarrow{\mathsf{v}}^P$.

Integrating by parts in (4.20), we see that for any compactly supported test function w on $\mathbb{R}^n$ and almost every t,

$$0 = \int_{\mathbb{R}^n} w\,\mathsf{v}_i^{\mathrm{os},P}\mu\, d\mathrm{Leb} + \frac{1}{2}\int_{\mathbb{R}^n} \mathsf{a}^i \cdot \nabla w\, \mu\, d\mathrm{Leb} = \int_{\mathbb{R}^n} w\,\mathsf{v}_i^{\mathrm{os},P}\mu\, d\mathrm{Leb} - \frac{1}{2}\int_{\mathbb{R}^n} w\nabla \cdot (\mu\mathsf{a}^i)\, d\mathrm{Leb},$$

where we drop the time dependence. This proves (4.15).

Let us look at (4.16). Apply (4.15) to $P = R$ to obtain

$$m\, \mathsf{v}^{\mathrm{os},R} = \nabla \cdot (m\mathsf{a})/2, \tag{4.21}$$

where $m := d\mathsf{m}/d\mathrm{Leb} = e^{-U}$. Although R, unlike P, might not be a probability measure, it is easy to see that the proof of (4.15) directly works with R instead of P (in particular $\beta^{R|R} = 0$). Because $\mu = \rho m$, we obtain $\nabla \cdot (\mu\mathsf{a}) = \nabla \cdot (\rho m\mathsf{a}) = m\mathsf{a}\nabla\rho + \rho\nabla \cdot (m\mathsf{a})$.

It is important to note that both $\nabla \cdot (\mu \mathsf{a})$ and $\nabla \cdot (m \mathsf{a})$ are well defined in the sense of distributions (as divergence terms) and are functions by (4.15) and (4.21) (the existence of $\mathsf{v}^{\text{os},P}$ is a direct consequence of the assumption that $H(P|R) < \infty$). It follows that

$$m \mathsf{a} \nabla \rho = \nabla \cdot (\mu \mathsf{a}) - \rho \nabla \cdot (m \mathsf{a})$$

is also well defined in the sense of distributions and a function. Putting everything together,

$$\rho(\mathsf{v}^{\text{os},P} - \mathsf{v}^{\text{os},R}) = m^{-1}(\mu \mathsf{v}^{\text{os},P} - \rho m \mathsf{v}^{\text{os},R}) = m^{-1}(\nabla \cdot (\mu \mathsf{a}) - \rho \, \nabla \cdot (m \mathsf{a}))/2 = \mathsf{a} \nabla \rho / 2,$$

which is (4.16), and implies (4.11).

Finally, the estimate (4.12) is a rewriting of (4.19), and (4.10) follows directly from (4.11).
$\square$

5. Time reversal of a diffusion process. Abstract setting

We use the IbP formula (Theorem 3.17) again, to extend at Theorem 5.7 the time reversal formula of Theorem 4.9 to an abstract diffusion setting where the configuration space $\mathcal{X}$ is a non-specified Polish space. To our opinion, the main interest of this result is not the extension to an abstract space, but its set of assumptions which sheds light on the close to minimal hypotheses that are necessary for the time reversal formula to hold in a diffusion setting.

Stationary diffusion reference measure. Assume that the Markov measure $R \in \mathrm{M}(\Omega)$ is stationary (see page 21) and in addition that is a *diffusion* path measure with a Polish space $\mathcal{X}$ as its configuration space. In this abstract setting, being a diffusion means that the derivation identity

$$\Gamma(u, vw) = v\Gamma(u, w) + w\Gamma(u, v) \tag{5.1}$$

is valid, and that for any $P \in \mathrm{P}(\Omega)$ such that $P \ll R$, we have

$$\overrightarrow{\Gamma}^P = \overleftarrow{\Gamma}^P = \overrightarrow{\Gamma}^R = \overleftarrow{\Gamma}^R =: \Gamma. \tag{5.2}$$

These identities fail in presence of jumps.

Lemma 5.3. *Let $R \in \mathrm{M}(\Omega)$ be an m-stationary diffusion path measure with osmotic generator $\mathcal{L}^{\text{os},R}$. For any functions $\rho, u, v \in \mathcal{V}$ such that $\rho u, uv \in \mathcal{V}$, we have*

$$\int_{\mathcal{X}} \Gamma(\rho, u) v \, d\mathsf{m} = -\int_{\mathcal{X}} \left\{ \Gamma(u, v) + 2v\mathcal{L}^{\text{os},R} u \right\} \rho \, d\mathsf{m}. \tag{5.4}$$

Proof. The integration by parts formula is $\int_{\mathcal{X}} \Gamma(u, v) \, d\mathsf{m} = -2 \int_{\mathcal{X}} v\mathcal{L}^{\text{os},R} u \, d\mathsf{m}$. With $\Gamma(u, v) + 2v\mathcal{L}^{\text{os},R} u = \mathcal{L}^{\text{os},R}(uv) - u\mathcal{L}^{\text{os},R} v - v\mathcal{L}^{\text{os},R} u + 2v\mathcal{L}^{\text{os},R} u = \mathcal{L}^{\text{os},R}(uv) - u\mathcal{L}^{\text{os},R} v + v\mathcal{L}^{\text{os},R} u$, and the derivation identity (5.1), we obtain

$$-\int_{\mathcal{X}} \{ \Gamma(u, v) + 2v\mathcal{L}^{\text{os},R} u \} \rho \, d\mathsf{m} = \int_{\mathcal{X}} \left\{ -\rho \mathcal{L}^{\text{os},R}(uv) + u\rho \mathcal{L}^{\text{os},R} v - v\rho \mathcal{L}^{\text{os},R} u \right\} d\mathsf{m}$$

$$= \frac{1}{2} \int_{\mathcal{X}} \{ \Gamma(\rho, uv) - \Gamma(u\rho, v) + \Gamma(v\rho, u) \} \, d\mathsf{m} = \int_{\mathcal{X}} \Gamma(\rho, u) v \, d\mathsf{m},$$

as announced.
$\square$

Time reversal formula. The left hand side of (5.4) requires that for any $u \in \mathcal{V}$, the couple (ρ, u) stands in the domain of definition of Γ, while no regularity of ρ is needed for having a meaningful right hand side. This suggests the following notion, in the spirit of the definition of a distribution.

Hypothesis 5.5. *We assume that there exists some algebra $\mathcal{U} \subset \mathcal{V}$ which is total in $L^2(\mathsf{m})$. This means that for all $u, v \in \mathcal{U}$ we have $uv \in \mathcal{U}$ and that for any $w \in L^2(\mathsf{m})$, $\int_{\mathcal{X}} uw \, d\mathsf{m} = 0, \ \forall u \in \mathcal{U}$ implies that $w = 0$.*
In addition we suppose that for any $u, v \in \mathcal{V}$, $\Gamma(u, v)\,\mathsf{m}$ and $u\mathcal{L}^{\mathrm{os},R}v\,\mathsf{m}$ are bounded measures.

Recall Remark 3.4 for the relevance of this hypothesis.

Definition 5.6. *Let $\mathcal{U}$ be as in Assumption 5.5 and let ρ be a nonnegative measurable function which is defined m-a.e. We define the linear operator $\Gamma(\rho, \bullet)$ on $\mathcal{U}$ in the weak sense, by the identity (5.4), seeing $(u, v) \mapsto \int_{\mathcal{X}} \Gamma(\rho, u)v \, d\mathsf{m}$ as a bilinear form.*

Theorem 5.7. *Let $R \in \mathrm{M}(\Omega)$ be an m-stationary diffusion measure: i.e. (5.1) and (5.2) hold, such that the Hypothesis 5.5 is satisfied, and for any $u, v \in \mathcal{U}$, $\Gamma(u, v)$ is bounded. Let $P \in \mathrm{P}(\Omega)$ be a Markov measure such that $P \ll R$, $\mathcal{U} \subset \mathrm{dom}\,\overrightarrow{\mathcal{L}}^P \cap \mathrm{dom}\,\overleftarrow{\mathcal{L}}^P$, and for any $u \in \mathcal{U}$, $\overrightarrow{\mathcal{L}}^P u, \overleftarrow{\mathcal{L}}^P u \in L^2(\overline{P})$. Then, for any $u \in \mathcal{U}$,*

$$\mathcal{L}^{\mathrm{os},P}u = \mathcal{L}^{\mathrm{os},R}u + \frac{\Gamma(\rho, u)}{2\rho} = \mathcal{L}^{\mathrm{os},R}u + \frac{\Gamma(\sqrt{\rho}, u)}{\sqrt{\rho}}, \qquad dt dP_t\text{-a.e.,}$$

where $\rho_t := dP_t/d\mathsf{m}$, the linear operators $\Gamma(\rho, \bullet)$ and $\Gamma(\sqrt{\rho}, \bullet)$ are defined in the weak sense of Definition 5.6, and

$$\mathcal{L}^{\mathrm{os},R} = \mathcal{L}^{R,\mathrm{sym}} := (\overrightarrow{\mathcal{L}}^R + (\overrightarrow{\mathcal{L}}^R)^*)/2,$$

is the symmetrized extended generator of $\overrightarrow{\mathcal{L}}^R$, see (3.37). In other words,

$$P^* \in \mathrm{MP}(\overrightarrow{\mathcal{L}}^{P^*}, \mathcal{U})$$

where for any $u \in \mathcal{U}$,

$$\overrightarrow{\mathcal{L}}^{P^*}_t u = \mathcal{L}^{R,\mathrm{sym}}_{T-t}u + \frac{\Gamma(\sqrt{\rho_{T-t}}, u)}{\sqrt{\rho_{T-t}}}, \qquad dt dP_t\text{-a.e.}$$

Proof. The hypotheses of the IbP formula: Theorem 3.17, are fulfilled, allowing us to write for all $u, v \in \mathcal{U}$ and almost all t

$$\int_{\mathcal{X}} v_t \mathcal{L}^{\mathrm{os},P}_t u_t \, \rho_t \, d\mathsf{m} = -\frac{1}{2}\int_{\mathbb{R}^n} \Gamma(u_t, v_t) \, \rho_t \, d\mathsf{m}$$

$$= \int_{\mathcal{X}} [v_t \mathcal{L}^{\mathrm{os},R}u_t \, \rho_t + \Gamma(\rho_t, u_t)v_t/2] \, d\mathsf{m}$$

where last equality is (5.4). The second equality in the first displayed formula follows with (5.1) which implies that $\Gamma(\rho, u) = 2\sqrt{\rho}\,\Gamma(\sqrt{\rho}, u)$. The identification of the osmotic and symmetrized generators of a stationary path measure is (3.37). $\square$

As a direct corollary of this result, we see that any path measure $P \in \mathrm{P}(\Omega)$ verifying the hypotheses of Theorem 5.7 and such that $P_t = \mathsf{m}$ for all $0 \le t \le T$ (it might not be stationary), shares its osmotic generator with the m-stationary path measure R : $\mathcal{L}^{\mathrm{os},P} = \mathcal{L}^{\mathrm{os},R}$ in restriction to $\mathcal{U}$, because $\rho = 1$.

6. Current-osmosis decomposition

This section presents an application of the time reversal formula for a diffusion to entropic optimal transport. The motivation for a decomposition of the relative entropy into the sum of current and osmotic terms was put forward in the introduction of the paper, see (1.3) and (1.4). This result is Proposition 6.2 below.

We go back to the setting of Section 4 and take the same reference path measure R satisfying the Hypotheses 1.10. For any $\mu_0 \in P(\mathbb{R}^n)$ such that $\mu_0 \ll$ Leb, we denote

$$R^{\mu_0}(\bullet) := \int_{\mathbb{R}^n} R(\bullet \mid X_0 = x_o)\, \mu_0(dx_o),$$

the Markov measure with the same forward dynamics as R, i.e. $\overrightarrow{\mathcal{L}}^{R^{\mu_0}} = \overrightarrow{\mathcal{L}}^R$, but with μ_0 as its initial marginal.

Definitions 6.1.

(1) (Free energy). The free energy *is defined by*

$$\mathcal{F}(\mu) := H(\mu|\mathsf{m})/2, \qquad \mu \in P(\mathbb{R}^n).$$

(2) (Fisher information). It is defined by

$$\mathcal{I}_{\mathsf{a}}(\mu|\mathsf{m}) := \int_{\mathbb{R}^n} |\nabla \log \sqrt{d\mu/d\mathsf{m}}|_{\mathsf{a}}^2/2 \; d\mu \in [0, \infty],$$

for any $\mu \in P(\mathbb{R}^n)$ such that $\nabla \log d\mu/d\mathsf{m}$ is well defined in the sense of distributions, and $+\infty$ otherwise.

Proposition 6.2. *Under the hypotheses of Theorem 4.9, for any $0 \le t \le T$,*

$$H(P_{[0,t]}|R^{P_0}_{[0,t]}) = \mathcal{F}(P_t) - \mathcal{F}(P_0) + \int_{[0,t]} \left\{ \left\langle |\mathsf{v}^{\mathrm{cu},P|R}|_{\mathsf{g}}^2/2, P_s \right\rangle + \mathcal{I}_{\mathsf{a}}(P_s|\mathsf{m}) \right\} ds.$$

Proof. Applying Proposition 4.6, we see that

$$H(P|R) = H(P_0|\mathsf{m}) + E_P \int_{[0,T]} |\overrightarrow{\beta}^{P|R}|_{\mathsf{a}}^2(X_t)/2\, dt = E_P \int_{[0,T]} |\overleftarrow{\beta}^{P|R}|_{\mathsf{a}}^2(X_t)/2\, dt + H(P_T|\mathsf{m})$$

$$= \frac{1}{2}\big(H(P_0|\mathsf{m}) + H(P_T|\mathsf{m})\big) + \frac{1}{2} E_P \int_{[0,T]} \big(|\overrightarrow{\beta}^{P|R}|_{\mathsf{a}}^2 + |\overleftarrow{\beta}^{P|R}|_{\mathsf{a}}^2\big)(\overline{X}_t)/2\, dt.$$

On the other hand, with the additive decomposition of the relative entropy

$$H(P|R) = H(P_0|R_0) + \int_{\mathbb{R}^n} H\big(P(\bullet|X_0 = x)\big|R(\bullet|X_0 = x)\big)\, P_0(dx),$$

we obtain

$$H(P|R^{P_0}) = H(P_0|P_0) + \int_{\mathbb{R}^n} H\big(P(\bullet|X_0 = x)\big|R(\bullet|X_0 = x)\big)\, P_0(dx)$$

$$= \int_{\mathbb{R}^n} H\big(P(\bullet|X_0 = x)\big|R(\bullet|X_0 = x)\big)\, P_0(dx).$$

Putting everything together, since $H(P_0|\mathsf{m}) \le H(P|R)$ is finite,

$$H(P|R^{P_0}) = H(P|R) - H(P_0|\mathsf{m})$$

$$= \frac{1}{2}\big(H(P_T|\mathsf{m}) - H(P_0|\mathsf{m})\big) + \frac{1}{2} E_P \int_{[0,T]} \big(|\overrightarrow{\beta}^{P|R}|_{\mathsf{a}}^2 + |\overleftarrow{\beta}^{P|R}|_{\mathsf{a}}^2\big)(\overline{X}_t)/2\, dt. \qquad (6.3)$$

From $\begin{cases} \beta^{\mathrm{cu}} & := \ (\overrightarrow{\beta} - \overleftarrow{\beta})/2 \\ \beta^{\mathrm{os}} & := \ (\overrightarrow{\beta} + \overleftarrow{\beta})/2 \end{cases}$, we derive the parallelogram identity

$$|\overrightarrow{\beta}|_{\mathsf{a}}^2/2 + |\overleftarrow{\beta}|_{\mathsf{a}}^2/2 = |\beta^{\mathrm{cu}}|_{\mathsf{a}}^2 + |\beta^{\mathrm{os}}|_{\mathsf{a}}^2,$$

leading to

$$H(P|R^{P_0}) = \mathcal{F}(P_T) - \mathcal{F}(P_0) + E_P \int_{[0,T]} \big(|\beta^{\mathrm{cu},P|R}|_{\mathsf{a}}^2 + |\beta^{\mathrm{os},P|R}|_{\mathsf{a}}^2 \big)(\overline{X}_t)/2\, dt.$$

We conclude with Theorem 4.9 and $H(P_{[0,t]}|R_{[0,t]}^{P_0}) \leq H(P|R^{P_0}) < \infty$, for all $0 \leq t \leq T$. $\qquad\square$

Heat flow. In this subsection, the reference measure R is defined with $T = \infty$, that is on $\Omega = C([0,\infty), \mathbb{R}^n)$.

Definition 6.4. *The time marginal flow* $\mu_t := R_t^{\mu_0}, t \geq 0$, *of* R^{μ_0} *is called the* heat flow *issued from* μ_0.

Next result is a direct consequence of Proposition 6.2 which tells us that the Fisher information is proportional to the rate of consumption of free energy along the heat flow.

Corollary 6.5. *If* $H(\mu_0|\mathsf{m}) < \infty$, *the heat flow* $(\mu_t)_{t \geq 0}$ *satisfies*

$$\mathcal{F}(\mu_t) - \mathcal{F}(\mu_0) = -2 \int_{[0,t]} \mathcal{I}_{\mathsf{a}}(\mu_s|\mathsf{m})\ ds, \quad \forall t \geq 0,$$

where all these quantities are finite.

Proof. We have $0 \leq H(\mu_t|\mathsf{m}) = H(R_t^{\mu_0}|R_t) \leq H(R^{\mu_0}|R) = H(\mu_0|\mathsf{m}) < \infty$. Applying (6.3) with $P = R^{P_0 = \mu_0}$ leads to

$$0 = H(R_{[0,t]}^{P_0}|R_{[0,t]}^{P_0}) = \mathcal{F}(\mu_t) - \mathcal{F}(\mu_0) + E_{R^{P_0}} \int_{[0,T]} \big(|\overrightarrow{\beta}^{R^{P_0}|R}|_{\mathsf{a}}^2 + |\overleftarrow{\beta}^{R^{P_0}|R}|_{\mathsf{a}}^2 \big)(\overline{X}_t)/4\, dt$$

$$= \mathcal{F}(\mu_t) - \mathcal{F}(\mu_0) + \int_{[0,t] \times \mathbb{R}^n} |\nabla \log \sqrt{d\mu_s/d\mathsf{m}}|_{\mathsf{a}}^2\ dsd\mu_s,$$

because $\overrightarrow{\beta}^{R^{P_0}|R} = 0$ implies that $\overleftarrow{\beta}^{R^{P_0}|R} = 2\beta^{\mathrm{os},R^{P_0}|R} = \nabla \log d\mu_s/d\mathsf{m}$. $\qquad\square$

7. RANDOM WALKS

In this section, the IbP formula is used to obtain at Theorem 7.11 a time reversal formula for a random walk on a graph under a finite entropy condition. This simple setting permits us to introduce Föllmer's guideline to derive time reversal formulas, with minimal technicalities.

Graph. We consider continuous-time random walks on a countable graph $(\mathcal{X}, \sim)$ where $\mathcal{X}$ is the set of all vertices and the symmetric binary relation $x \sim y$, $x, y \in \mathcal{X}$ states that $\{x, y\}$ is a non-oriented edge of the graph. We assume without loss of generality that the graph is irreducible: $\mathcal{X}$ is the unique class of communication, and that it contains no elementary loop: $x \sim x$ is forbidden. We also assume that $(\mathcal{X}, \sim)$ is a locally finite graph meaning that each vertex $x \in \mathcal{X}$ admits finitely many neighbours. That is

$$n_x := \# \{y \in \mathcal{X}; y \sim x\} < \infty, \quad \forall x \in \mathcal{X}. \tag{7.1}$$

The countable set $\mathcal{X}$ is equipped with its discrete topology.

Random walk. A random walk on the graph $(\mathcal{X}, \sim)$ is a time-continuous Markov measure $Q \in \mathrm{M}(\Omega)$ which is specified by its initial distribution $Q_0 \in \mathrm{M}(\mathcal{X})$ and its forward generator acting on any real function in the class

$$\mathcal{U} := \{u : \mathcal{X} \to \mathbb{R}; \# \operatorname{supp}(u) < \infty\}$$

of all real functions with a finite support via the formula

$$\overrightarrow{\mathcal{L}}_t^Q u(x) = \sum_{y : y \sim x} [u(y) - u(x)] \, \overrightarrow{j}(t, x; y), \quad x \in \mathcal{X}, t \in [0, T], \quad u \in \mathcal{U}, \qquad (7.2)$$

where for any adjacent neighbours $x \sim y$, $\overrightarrow{j}(t, x; y) \geq 0$ is the average frequency of jumps from x to y at time t. The jump kernel associated with this generator is

$$\sum_{y : y \sim x} \overrightarrow{j}(t, x; y) \delta_y \in \mathrm{M}(\mathcal{X}), \ x \in \mathcal{X}, t \in [0, T]$$

where δ_y stands for the Dirac measure at y. For any pair of functions u, v in $\mathcal{U}$, the carré du champ is

$$\overrightarrow{\Gamma}_t^Q(u, v)(x) = \sum_{y : y \sim x} [u(y) - u(x)][v(y) - v(x)] \, \overrightarrow{j}(t, x; y).$$

Note that the class of functions $\mathcal{U}$ is an algebra. See Remark 3.4 for the significance of this property.

A first time reversal formula. We start by applying part (b) of Theorem 3.17 as a first step of a more general result.

Proposition 7.3. *If for any $x \sim y$ the function $t \mapsto \overrightarrow{j}(t, x; y)$ is continuous, then $\mathcal{U} \subset \operatorname{dom} \overrightarrow{\mathcal{L}}^Q \cap \operatorname{dom} \overleftarrow{\mathcal{L}}^Q$, the backward generator is*

$$\overleftarrow{\mathcal{L}}_t^Q u(x) = \sum_{y : y \sim x} [u(y) - u(x)] \, \overleftarrow{j}(t, x; y), \quad x \in \mathcal{X}, t \in [0, T], \quad u \in \mathcal{U},$$

where for all $t \in [0, T]$ and all $x, y \in \mathcal{X}, x \neq y$,

$$\mathsf{q}_t(x) \overleftarrow{j}(t, x; y) = \mathsf{q}_t(y) \overrightarrow{j}(t, y; x).$$

Proof. Let us apply Theorem 3.17-(b). Under the assumptions (7.1) and

$$\int_{[0,T]} \overrightarrow{j}(t, x; y) \, dt < \infty, \qquad \forall x, y : x \sim y,$$

the processes $\overrightarrow{\mathcal{L}}^Q u(X)$ and $\overrightarrow{\Gamma}^Q u(X)$ are in $L^1(\overline{Q})$, and

$$M_t^{Q, [u, v]} = \sum_{0 \leq s \leq t} [u(X_s) - u(X_{s^-})][v(X_s) - v(X_{s^-})] - \int_{[0,t]} \overrightarrow{\Gamma}_s^Q(u, v)(X_s) \, ds.$$

With our notation, this means that: $\mathcal{U}^Q = \mathcal{U}$.

The class $\mathcal{U}^Q = \mathcal{U}$ determines the weak convergence of measures and our assumption about the continuity of $\overrightarrow{j}$ implies (3.20). Denoting $\overrightarrow{\mathsf{q}j}(t, x; y) := \mathsf{q}_t(x) \overrightarrow{j}(t, x; y)$, we see

that for any $w \in \mathcal{U}^{\overline{Q}}$,

$$\int_{[0,T] \times \mathcal{X}} \overrightarrow{\Gamma}_t^Q(u, w_t)[x]\, \mathsf{q}_t(dx)dt = \int_{[0,T]} dt \sum_{(x,y):x \sim y} [u(y) - u(x)][w_t(y) - w_t(x)]\, \overrightarrow{\mathsf{q}j}(t, x; y)$$

$$= -\int_{[0,T]} dt \sum_{(x,y):x \sim y} w_t(x)[u(y) - u(x)] \left(\overrightarrow{\mathsf{q}j}(t, x; y) + \overrightarrow{\mathsf{q}j}(t, y; x)\right).$$

This proves that $\int_{[0,T] \times \mathcal{X}} \overrightarrow{\Gamma}^Q(u, \bullet)\, d\overline{\mathsf{q}}$ is a finite measure, showing that the hypotheses of Theorem 3.17-(b) are satisfied. Hence, $u \in \mathrm{dom}\, \overleftarrow{\mathcal{L}}^Q$, $\overleftarrow{\mathcal{L}}^Q u$ is integrable and for almost every t the IbP formula (3.19) holds, that is

$$\sum_{x \in \mathcal{X}} v(x) \overleftarrow{\mathcal{L}}_t^Q u(x)\, \mathsf{q}_t(x)$$

$$= -\sum_{(x,y):x \sim y} \{u(y) - u(x)\} v(x) \overrightarrow{\mathsf{q}j}(t, x; y) - \sum_{(x,y):x \sim y} \{u(y) - u(x)\}\{v(y) - v(x)\} \overrightarrow{\mathsf{q}j}(t, x; y)$$

$$= -\sum_{(x,y):x \sim y} \{u(y) - u(x)\} v(y) \overrightarrow{\mathsf{q}j}(t, x; y) = \sum_{(x,y):x \sim y} \{u(y) - u(x)\} v(x) \overrightarrow{\mathsf{q}j}(t, y; x),$$

for any $u, v \in \mathcal{U}$.

On the other hand, with [29, Proposition 3.4] we know that for almost all t and for every x,

$$\begin{aligned}\overleftarrow{\mathcal{L}}_t^Q u(x) &= \lim_{h \to 0^+} h^{-1} E_Q\left[u(X_{t-h}) - u(X_t) \mid X_t = x\right] \\ &= \sum_{y \in \mathcal{X}} \{u(y) - u(x)\} \lim_{h \to 0^+} h^{-1} Q(X_{t-h} = y \mid X_t = x),\end{aligned} \tag{7.4}$$

proving that the backward generator writes as $\overleftarrow{\mathcal{L}}_t^Q u(x) = \sum_{y \in \mathcal{X}} \{u(y) - u(x)\} \overleftarrow{j}(t, x; y)$ for some function $\overleftarrow{j}$. Plugging this into the expression $\sum_{x \in \mathcal{X}} v(x) \overleftarrow{\mathcal{L}}_t^Q u(x)\, \mathsf{q}_t(x)$, we arrive at

$$\sum_{(x,y):x \sim y} \{u(y) - u(x)\} v(x) \mathsf{q}_t(x) \overleftarrow{j}(t, x; y) = \sum_{(x,y):x \sim y} \{u(y) - u(x)\} v(x) \overrightarrow{\mathsf{q}j}(t, y; x),$$

and conclude remarking that the family of functions $(x, y) \mapsto \{u(y) - u(x)\} v(x)$ when u and v describe $\mathcal{U}$ is measure-determining off the diagonal of $\mathcal{X}^2$. $\qquad\square$

Reversible random walk. Saying that $Q \in \mathrm{M}(\Omega)$ is reversible means that there is a (possibly unbounded) positive measure $\mathsf{m} \in \mathrm{M}(\mathcal{X})$ on $\mathcal{X}$ such that, not only Q is m-stationary i.e. $\mathsf{q}_t = \mathsf{m}$, $\forall 0 \le t \le T$, but also that Q is invariant with respect to time reversal i.e.: for any subinterval $[r, t] \in [0, T]$,

$$(X_{(r+t-s)^-}; r \le s \le t)_\# Q = (X_s; r \le s \le t)_\# Q.$$

This implies that the forward and backward transition mechanisms do not depend on the time variable t and are the same: $\overrightarrow{j} = \overleftarrow{j} =: j$. In view of Proposition 7.3, we obtain the detailed balance condition

$$\mathsf{m}(x) j_x(y) = \mathsf{m}(y) j_y(x), \quad \forall x, y \in \mathcal{X} : x \sim y. \tag{7.5}$$

Without loss of generality, we assume that $x \sim y \iff j_x(y), j_y(x) > 0$ and that the graph is irreducible. It follows that $\mathsf{m}(x) > 0$ for all $x \in \mathcal{X}$. The general solution of (7.5)

is

$$j_x(y) = s(x,y)\sqrt{\mathsf{m}(y)/\mathsf{m}(x)}$$

where s is any symmetric function such that $x \sim y \iff s(x,y) > 0$.

Counting random walk. If the waiting time at x is an exponential random variable $\mathcal{E}(n_x)$ and the jump occurs uniformly onto each neighbour, we obtain the jump kernel

$$J_x^o := \sum_{y:y\sim x} \delta_y, \qquad x \in \mathcal{X}, \tag{7.6}$$

which admits the counting measure

$$\mathsf{m}^o = \sum_{x\in\mathcal{X}} \delta_x \in \mathrm{M}(\mathcal{X}) \tag{7.7}$$

as a reversing measure. We denote $R \in \mathrm{M}(\Omega)$ this reversible random walk with $R_0 = \mathsf{m}^o$ and call it the counting random walk. It will be the reference path measure for the rest of this section.

The remainder of this section is devoted to the proof of an extension of Proposition 7.3, stated at Theorem 7.11, where the hypothesis on the continuity of $\overrightarrow{j}$ is removed and replaced by a finite entropy assumption.

Finite entropy assumption. Let $P \in \mathrm{P}(\Omega)$ be a Markov probability measure such that

$$H(P|R) < \infty,$$

with $R \in \mathrm{M}(\Omega)$ the counting random walk. This finite entropy property implies (Girsanov's theory) that there exists some measurable function $\overrightarrow{j}^P : [0,T] \times \mathcal{X}^2 \to [0,\infty)$ which is defined $dt P_t(dx) J_x^o(dy)$-almost everywhere such that P is the unique solution of the martingale problem $\mathrm{MP}(P_0, \overrightarrow{J}^P)$ associated to the initial marginal P_0 and the jump kernel $\overrightarrow{J}^P = \overrightarrow{j}^P J^o$, that is

$$\overrightarrow{J}_{t,x}^P = \sum_{y:y\sim x} \overrightarrow{j}_{t,x}^P(y)\,\delta_y, \quad (t,x) \in [0,T]\times\mathcal{X}.$$

Moreover,

$$H(P|R) = H(P_0|\mathsf{m}^o) + \int_{[0,T]\times\mathcal{X}^2} \mathsf{h}\left(\overrightarrow{j}_{t,x}^P(y)\right) dt\mathsf{p}_t(dx)J_x^o(dy) < \infty, \tag{7.8}$$

where

$$\mathsf{h}(a) := \begin{cases} a\log a - a + 1, & \text{if } a > 0, \\ 1, & \text{if } a = 0, \\ \infty, & \text{if } a < 0. \end{cases}$$

Lemma 7.9. *Let u be any function in $\mathcal{U}$ and $P \in \mathrm{P}(\Omega)$ satisfy $H(P|R) < \infty$.*

(a) The function u stands in dom $\overrightarrow{\mathcal{L}}^P$, $\overrightarrow{\mathcal{L}}^P u(\overline{X}) \in L\log L(\overline{P})$ and

$$\overrightarrow{\mathcal{L}}_t^P u(x) = \sum_{y:y\sim x} [u(y) - u(x)]\, \overrightarrow{j}_{t,x}^P(y).$$

(b) There exists some measurable function $\overleftarrow{j}^P : [0,T] \times \mathcal{X}^2 \to [0,\infty)$ which is defined $dt\mathsf{p}_t(dx)J_x^o(dy)$-almost everywhere such that u stands in $\mathrm{dom}\,\overleftarrow{\mathcal{L}}^P$, with $\overleftarrow{\mathcal{L}}^P u(\overline{X}) \in L\log L(\overline{P})$ and

$$\overleftarrow{\mathcal{L}}_t^P u(x) = \sum_{y:y\sim x} [u(y) - u(x)]\, \overleftarrow{j}\,_{t,x}^P(y). \tag{7.10}$$

Moreover

$$H(P|R) = H(P_T|\mathsf{m}^o) + \int_{[0,T]\times\mathcal{X}^2} \mathsf{h}\Big(\overleftarrow{j}\,_{t,x}^P(y)\Big)\, dt\mathsf{p}_t(dx)J_x^o(dy) < \infty.$$

Proof. • Proof of (a). Let us denote the right hand side of the desired identity by: $A(t,x) := \sum_{y:y\sim x}[u(y) - u(x)]\,\overrightarrow{j}\,_{t,x}^P(y)$. With (7.8), our assumption (7.1), the finiteness of the support of u, and

$$|A(t,x)| \leq 2\sup|u|\mathbf{1}_{\{x\in\mathrm{supp}(u)\}} \sum_{y:y\sim x} \overrightarrow{j}\,_{t,x}^P(y),$$

we see that $A(\overline{X})$ is in $L\log L(\overline{P})$. This implies that $\int_{[0,T]} |A(\overline{X}_t)|\, dt$ is finite P-a.e., so that $A(t,x) = \overrightarrow{\mathcal{L}}^P u_t(x)$, $\bar{\mathsf{p}}(dtdx)$-a.e. and $\overrightarrow{\mathcal{L}}^P u(\overline{X}) \in L\log L(\overline{P})$.

• Proof of (b). Time reversal being a bijective mapping: $H(P^*|R^*) = H(P|R)$, see Proposition B.2. Since R is chosen to be reversible, we also have $R^* = R$, leading to:

$$H(P^*|R) = H(P|R) < \infty.$$

Hence we are allowed to apply (a) which tells us that $u \in \mathrm{dom}\,\overrightarrow{\mathcal{L}}^{P^*}$, $\overrightarrow{\mathcal{L}}^{P^*}u(\overline{X}) \in L\log L(\overline{P^*})$, and there is some measurable function $\overrightarrow{j}^{P^*}$ such that $\overrightarrow{\mathcal{L}}_t^{P^*}u(x) = \sum_{y:y\sim x}[u(y) - u(x)]\,\overrightarrow{j}\,_{t,x}^{P^*}(y)$. We conclude taking $\overleftarrow{j}\,_{t,x}^P(y) := \overrightarrow{j}\,_{T-t,x}^{P^*}(y)$. $\qquad\square$

Time reversal formula. The main theorem of this section is

Theorem 7.11. *Let $P \in \mathrm{P}(\Omega)$ be a Markov random walk with forward generator*

$$\overrightarrow{\mathcal{L}}_t^P u(x) = \sum_{y:y\sim x} [u(y) - u(x)]\, \overrightarrow{j}\,_{t,x}^P(y), \qquad x \in \mathcal{X},\ u \in \mathcal{U},$$

where the forward intensity of jump $\overrightarrow{j}^P$ is measurable. If $H(P|R) < \infty$, i.e.

$$H(\mathsf{p}_0|\mathsf{m}^o) + \int_{[0,T]} dt \sum_{(x,y):x\sim y} \mathsf{h}\Big(\overrightarrow{j}\,_{t,x}^P(y)\Big)\, \mathsf{p}_t(x) < \infty,$$

then, $\mathcal{U} \subset \mathrm{dom}\,\overleftarrow{\mathcal{L}}^P$ and

$$\overleftarrow{\mathcal{L}}_t^P u(x) = \sum_{y:y\sim x} [u(y) - u(x)]\, \overleftarrow{j}\,_{t,x}^P(y), \qquad x \in \mathcal{X},\ u \in \mathcal{U},$$

where the backward intensity of jump $\overleftarrow{j}^P$ verifies for almost all $0 \leq t \leq T$,

$$\mathsf{p}_t(x)\overrightarrow{j}\,_{t,x}^P(y) = \mathsf{p}_t(y)\overleftarrow{j}\,_{t,y}^P(x), \quad \forall x,y \sim x \in \mathcal{X}. \tag{7.12}$$

Proof. With (7.8) we see that $H(P|R) < \infty$, i.e. the hypothesis of Lemma 7.9 is satisfied. This lemma tells us that the assumptions of Theorem 3.17-(a) are satisfied. Therefore, for almost all $t \in [0, T]$, and any $u, v \in \mathcal{U}$, the IbP formula (3.19) holds, i.e.

$$
\begin{aligned}
0 &= \int_{\mathcal{X}^2} [u(y) - u(x)] v(x) (\overrightarrow{j}^P + \overleftarrow{j}^P)_{t,x}(y)\, \mathsf{p}_t(dx) J_x^o(dy) \\
&\quad + \int_{\mathcal{X}^2} [u(y) - u(x)][v(y) - v(x)] \overrightarrow{j}^P_{t,x}(y)\, \mathsf{p}_t(dx) J_x^o(dy) \\
&= \int_{\mathcal{X}^2} [u(y) - u(x)][v(x) \overleftarrow{j}^P_{t,x}(y) + v(y) \overrightarrow{j}^P_{t,x}(y)] \mathsf{p}_t(x)\, \mathsf{m}^o(dx) J_x^o(dy).
\end{aligned}
$$

As the counting random walk is m^o-reversible, by (7.5) $\mathsf{m}^o(dx)J_x^o(dy)$ is a symmetric measure on $\mathcal{X}^2$ (obvious by direct inspection). It follows that

$$
0 = \int_{\mathcal{X}^2} [u(y) - u(x)]v(y)[\mathsf{p}_t(x)\overrightarrow{j}^P_{t,x}(y) - \mathsf{p}_t(y)\overleftarrow{j}^P_{t,y}(x)]\, \mathsf{m}^o(dx) J_x^o(dy),
$$

from which the result follows.   □

Appendix A. Stochastic derivatives and extended generators

After recalling the definitions of Markov measures, extended generators and stochastic derivatives, we state a couple of technical results obtained in [29].

Definition A.1 (Markov measure). *A path measure Q such that Q_t is σ-finite for all t is called a conditionable path measure. A path measure $Q \in \mathrm{M}(\Omega)$ is said to be Markov if it is conditionable and for any $0 \le t \le T$, $Q(X_{[t,T]} \in \bullet \mid X_{[0,t]}) = Q(X_{[t,T]} \in \bullet \mid X_t)$.*

The reason for requiring Q to be conditionable is that it allows for defining the conditional expectations $E_Q(\bullet \mid X_{\mathcal{T}})$ for any $\mathcal{T} \subset [0, T]$ even in the case where Q is an unbounded measure, see [31, Def. 1.10].

The notion of extended generator was introduced by H. Kunita [27] and extensively used by P. A. Meyer and his collaborators, see [9]. Here is a variant of this definition.

Definition A.2 (Extended forward generator of a Markov measure). *Let Q be a Markov measure. A measurable function u on $[0, T] \times \mathcal{X}$ is said to be in the domain of the extended forward generator of Q if there exists a measurable function v on $[0, T] \times \mathcal{X}$ such that $\int_{[0,T]} |v(\overline{X}_t)|\, dt < \infty$, Q-a.e. and the process*

$$
M_t^u := u(\overline{X}_t) - u(\overline{X}_0) - \int_{[0,t]} v(\overline{X}_s)\, ds, \quad 0 \le t \le T,
$$

is a local Q-martingale. We denote

$$
\overrightarrow{\mathcal{L}}^Q u(t, x) := v(t, x)
$$

and call $\overrightarrow{\mathcal{L}}^Q$ the extended forward generator of Q. Its domain is denoted by $\mathrm{dom}\,\overrightarrow{\mathcal{L}}^Q$.

Remarks A.3.

(a) In the case where Q is the law of a Markov process associated with some semigroup with generator $\mathcal{G}$ and $u : [0, T] \times \mathcal{X} \to \mathbb{R}$ is a t-differentiable function such that for each t, $u(t, \bullet)$ belongs to the domain of $\mathcal{G}$, then u belongs to $\mathrm{dom}\,\overrightarrow{\mathcal{L}}^Q$ and

$$
\overrightarrow{\mathcal{L}}^Q u = (\partial_t + \mathcal{G})u.
$$

(b) The notation $v = \mathcal{L}u$ almost rightly suggests that v is a function of u. Indeed, when u is in dom $\overrightarrow{\mathcal{L}}^Q$, the Doob-Meyer decomposition of the special semimartingale $u(\overline{X}_t)$ into its predictable bounded variation part $\int v_s\, ds$ and its local martingale part is unique. But one can modify $v = \overrightarrow{\mathcal{L}}^Q u$ on a small (zero-potential) set without breaking the martingale property. As a consequence, $u \mapsto \overrightarrow{\mathcal{L}}^Q u$ is a multivalued operator and $u \mapsto \overrightarrow{\mathcal{L}}^Q u$ is an almost linear operation.

Extended generators are connected with martingale problems which were introduced by Stroock and Varadhan [41].

Definition A.4 (Martingale problem). *Let $\mathcal{C}$ be a class of measurable real functions u on $[0,T] \times \mathcal{X}$ and for each $u \in \mathcal{C}$, let $\mathcal{L}u : [0,T] \times \mathcal{X} \to \mathbb{R}$ be a measurable function such that $\int_{[0,T]} |\mathcal{L}u(t, \omega_t)|\, dt < \infty$ for all $\omega \in \Omega$. Take also a positive σ-finite measure $\mu_0 \in \mathrm{M}(\mathcal{X})$. One says that $Q \in \mathrm{M}(\Omega)$ is a solution to the martingale problem $\mathrm{MP}(\mathcal{L}, \mathcal{C}; \mu_0)$ if $Q_0 = \mu_0 \in \mathrm{M}(\mathcal{X})$ and for all $u \in \mathcal{C}$, the process $u(\overline{X}_t) - u(\overline{X}_0) - \int_{[0,t]} \mathcal{L}_s u[X_s]\, ds$ is a local Q-martingale.*

Proposition A.10 below states that the extended generator can be computed by means of a stochastic derivative. Nelson's definition [36] of the stochastic derivative is the following.

Definition A.5 (Stochastic forward derivative of a Markov measure). *Let Q be a Markov measure and u be a measurable real function on $[0,T] \times \mathcal{X}$ such that $E_Q|u(\overline{X}_s)| < \infty$ for all $0 \le s \le T$.*

(1) We say that u admits a stochastic forward derivative under Q at time $t \in [0,T)$ if the following limit

$$\overrightarrow{L}^Q u(t,x) := \lim_{h \to 0^+} E_Q\left(\frac{1}{h}[u(\overline{X}_{t+h}) - u(t,x)] \mid X_t = x\right)$$

exists in $L^1(\mathbb{R}^n, Q_t)$.
In this case, $\overrightarrow{L}^Q u(t, \bullet)$ is called the stochastic forward derivative of u at time t.
(2) If u admits a stochastic forward derivative for almost all t, we say that u belongs to the domain dom $\overrightarrow{L}^Q$ of the stochastic forward derivative $\overrightarrow{L}^Q$ of Q.
(3) If u does not depend on the time variable t, we denote $\overrightarrow{L}^Q_t u(x) := \overrightarrow{L}^Q u(t,x)$.

Reversing time. If $Q \in \mathrm{M}(\Omega)$ is Markov, so is its time reversal Q^*, and one can consider the extended generators and stochastic derivatives of both Q and Q^*. More generally, we introduce the following notions.

As a notation, the σ-field generated by $X_{[t^-,T]}$ is $\sigma(X_{[t^-,T]}) := \cap_{h>0}\sigma(X_{[t-h,T]}) = \sigma(X_{t^-}) \vee \sigma(X_{[t,T]})$.

Definition A.6 (Extended backward generator). *Let Q be a conditionable path measure. A process u adapted to the predictable backward filtration $(\sigma(X_{[t^-,T]}); 0 \le t \le T)$ is said to be in the domain of the extended backward generator of Q if there exists a process v also adapted to the predictable backward filtration such that $\int_{[0,T]} |v(t, X_{[t^-,T]})|\, dt < \infty$, Q-a.e. and the process*

$$u(t, X_{[t^-,T]}) - u(T, X_T) - \int_{[t,T]} v(s, X_{[s^-,T]})\, ds, \quad 0 \le t \le T,$$

is a local backward Q-martingale. We denote

$$\overleftarrow{\mathcal{L}}^Q_t u := v_t$$

and call $\overleftarrow{\mathcal{L}}^Q$ the extended backward generator of Q. Its domain is denoted by $\operatorname{dom} \overleftarrow{\mathcal{L}}^Q$.

Definition A.7 (Stochastic backward derivative). *Let Q be a conditionable path measure and a measurable function u on $[0,T] \times \mathcal{X}$ such that $E_Q|u(s, X_s)| < \infty$ for all $0 \le s \le T$.*

(1) *We say that u admits a stochastic backward derivative under Q at time $t \in (0,T]$ if the following limit*

$$\overleftarrow{L}^Q u(t, X_{[t^-,T]}) := \lim_{h \to 0^+} E_Q \left(\frac{1}{h}[u(\overline{X}_{t-h}) - u(\overline{X}_t)] \mid X_{[t^-,T]} \right)$$

if this limit exists in $L^1(Q)$.

In this case, $\overleftarrow{L}^Q u(t, \bullet)$ is called the stochastic backward derivative of u at time t.

(2) *If u admits a stochastic backward derivative for almost all t, we say that u belongs to the domain $\operatorname{dom} \overleftarrow{L}^Q$ of the stochastic backward derivative $\overleftarrow{L}^Q$ of Q.*

Convergence results. A useful technical result for our purpose is the following convolution result.

Lemma A.8. *For all $h > 0$, let k^h be a measurable nonnegative convolution kernel such that $\operatorname{supp} k^h \subset [-h, h]$ and $\int_{\mathbb{R}} k^h(s)\, ds = 1$. Let Q be a σ-finite positive measure on Ω and v be a process in $L^p(\overline{Q})$ with $1 \le p < \infty$.*
Define for all $h > 0, t \in [0,T]$ and $\omega \in \Omega$, $k^h * v(t, \omega) := \int_{[0,T]} k^h(t-s)v_s(\omega)\, ds$.
Then, $k^h * v$ is in $L^p(\overline{Q})$ and $\lim_{h \to 0^+} k^h * v = v$ in $L^p(\overline{Q})$.

We see that $k^h(s)\, ds$ is a probability measure on $\mathbb{R}$ which converges narrowly to the Dirac measure δ_0 as h tends down to zero. We shall invoke this lemma with $p = 1$ or 2.

Corollary A.9. *Assume that in addition to the hypotheses of Lemma A.8, for any $0 \le t \le T$, the random variable $v(t, \bullet)$ is $\mathcal{A}_t$-measurable where $\mathcal{A}_t$ is some sub-σ-field. Then, the process v^h defined by $v^h_t := E_Q[k^h * v(t) \mid \mathcal{A}_t]$, is in $L^p(\overline{Q})$ and $\lim_{h \to 0^+} v^h = v$ in $L^p(\overline{Q})$.*

Proof. By Jensen's inequality

$$\|v^h - v\|^p_{p,\overline{Q}} = \int_{\overline{\mathcal{X}}} |E_Q[k^h * v(t) \mid \mathcal{A}_t] - v(t)|^p\, d\overline{Q} = \int_{\overline{\mathcal{X}}} |E_Q[k^h * v(t) - v(t) \mid \mathcal{A}_t]|^p\, d\overline{Q}$$

$$\le \int_{\overline{\mathcal{X}}} E_Q[|k^h * v(t) - v(t)|^p \mid \mathcal{A}_t]\, d\overline{Q} = \int_{\overline{\mathcal{X}}} E_Q|k^h * v(t) - v(t)|^p\, d\overline{Q}$$

$$= \|k^h * v - v\|^p_{p,\overline{Q}} \xrightarrow[h \to 0^+]{} 0,$$

where the vanishing limit is the content of Lemma A.8. $\qquad\qquad\square$

Next proposition states that extended generators and stochastic derivatives are essentially the same.

Proposition A.10. *Let Q be a conditionable measure.*
(a) If u is in $\operatorname{dom} \overrightarrow{\mathcal{L}}^Q$ and satisfies $E_Q \int_{[0,T]} |\overrightarrow{\mathcal{L}}^Q u(t, X_{[0,t]})|^p\, dt < \infty$ for some $p \ge 1$, then

$$\lim_{h \to 0^+} E_Q \int_{[0,T-h]} \left| \frac{1}{h} E_Q \Big[u(\overline{X}_{t+h}) - u(\overline{X}_t) \mid X_{[0,t]} \Big] - \overrightarrow{\mathcal{L}}^Q u(t, X_{[0,t]}) \right|^p dt = 0.$$

In particular, this implies that $u \in \operatorname{dom} \overrightarrow{L}^Q$, and the limit

$$\overrightarrow{\mathcal{L}}^Q u(\bullet, X_{[0,\bullet]}) = \overrightarrow{L}^Q u(\bullet, X_{[0,\bullet]}) := \lim_{h \to 0^+} \frac{1}{h} E_Q \Big[u(\overline{X}_{\bullet+h}) - u(\overline{X}_\bullet) \mid X_{[0,\bullet]} \Big]$$

takes place in $L^p(\overline{Q})$.

(b) If u is in dom $\overleftarrow{\mathcal{L}}^Q$ is such that $E_Q \int_{[0,T]} |\overleftarrow{\mathcal{L}}^Q u(t, X_{[t,T]})|^p \, dt < \infty$ for some $p \geq 1$, then

$$\lim_{h \to 0^+} E_Q \int_{[h,T]} \left| \frac{1}{h} E_Q \left[u(\overline{X}_{t-h}) - u(\overline{X}_t) \mid X_{[t^-,T]} \right] - \overleftarrow{\mathcal{L}}^Q u(t, X_{[t^-,T]}) \right|^p \, dt = 0.$$

In particular, this implies that $u \in$ dom $\overleftarrow{L}^Q$, and the limit

$$\overleftarrow{\mathcal{L}}^Q u(\bullet, X_{[\bullet,T]}) = \overleftarrow{L}^Q u(\bullet, X_{[\bullet,T]}) := \lim_{h \to 0^+} \frac{1}{h} E_Q \left[u(\overline{X}_{\bullet-h}) - u(\overline{X}_\bullet) \mid X_{[\bullet,T]} \right]$$

takes place in $L^p(\overline{Q})$.

Proposition A.11.

(a) Let u be a measurable real function on $\overline{\mathcal{X}}$, and v be a forward-adapted process such that $u(\overline{X})$ and v are $\overline{Q}$-integrable, $t \mapsto u(\overline{X}_t)$ is right continuous (for instance u might be continuous) and

$$\lim_{h \to 0^+} E_Q \int_{[0,T-h]} \left| \frac{1}{h} E_Q[u(\overline{X}_{t+h}) - u(\overline{X}_t) \mid X_{[0,t]}] - v_t \right| \, dt = 0. \tag{A.12}$$

Then, u belongs to dom $\overrightarrow{\mathcal{L}}^Q$ and dom L^Q, and $\overrightarrow{\mathcal{L}}^Q u = L^Q u = v$, $\overline{Q}$-a.e.

(b) Let u be a measurable real function on $\overline{\mathcal{X}}$ and v a backward-predictable process, such that $u(\overline{X}), v$ are $\overline{Q}$-integrable, $t \mapsto u(\overline{X}_t^*)$ is right continuous (for instance u might be continuous) and

$$\lim_{h \to 0^+} E_Q \int_{[h,T]} \left| \frac{1}{h} E_Q[u(\overline{X}_{t-h}) - u(\overline{X}_t) \mid X_{[t^-,T]}] - v(t, X_{[t^-,T]}) \right| \, dt = 0.$$

Then, u belongs to dom $\overleftarrow{\mathcal{L}}^Q$ and dom $\overleftarrow{L}^Q$, and $\overleftarrow{\mathcal{L}}^Q u = \overleftarrow{L}^Q u = v$, $\overline{Q}$-a.e.

APPENDIX B. RELATIVE ENTROPY WITH RESPECT TO AN UNBOUNDED MEASURE

Let r be some σ-finite positive measure on some measurable space Y. The relative entropy of the probability measure p with respect to r is loosely defined by

$$H(p|r) := \int_Y \log(dp/dr) \, dp \in (-\infty, \infty], \qquad p \in \mathrm{P}(Y)$$

if $p \ll r$ and $H(p|r) = \infty$ otherwise. More precisely, when r is a probability measure, we have $H(p|r) = \int_Y h(dp/dr) \, dr \in [0, \infty]$ with $h(a) = a \log a - a + 1 \geq 0$ for all $a \geq 0$, (take $h(0) = 1$). Hence this definition is meaningful and it follows from the strict convexity of h that $H(\bullet|r)$ is also strictly convex.

If r is unbounded, one must restrict the definition of $H(\bullet|r)$ to some subset of $\mathrm{P}(Y)$ as follows. As r is assumed to be σ-finite, there exist measurable functions $W : Y \to [0, \infty)$ such that

$$z_W := \int_Y e^{-W} \, dr < \infty. \tag{B.1}$$

Define the probability measure $r_W := z_W^{-1} e^{-W} r$ so that $\log(dp/dr) = \log(dp/dr_W) - W - \log z_W$. It follows that for any $p \in \mathrm{P}(Y)$ satisfying $\int_Y W \, dp < \infty$, the formula

$$H(p|r) := H(p|r_W) - \int_Y W \, dp - \log z_W \in (-\infty, \infty]$$

is a meaningful definition of the relative entropy which is coherent in the following sense. If $\int_Y W' \, dp < \infty$ for another measurable function $W' : Y \to [0, \infty)$ such that $z_{W'} < \infty$,

then $H(p|r_W) - \int_Y W \, dp - \log z_W = H(p|r_{W'}) - \int_Y W' \, dp - \log z_{W'} \in (-\infty, \infty]$.
Therefore, $H(p|r)$ is well-defined for any $p \in \mathrm{P}(Y)$ such that $\int_Y W \, dp < \infty$ for some measurable non-negative function W verifying (B.1).

It is well known that the relative entropy with respect to a *probability* measure r is invariant with respect to the push-forward by an injective mapping. This is still true if r is unbounded.

Proposition B.2. *Let r and $H(\bullet|r)$ be as above, and let $f : Y \to Z$ be a measurable mapping. For any $p \in \mathrm{P}(Y)$ satisfying $\int_Y W \, dp < \infty$, we have: $H(f_\# p | f_\# r) \leq H(p|r)$. If in addition f is injective, then $H(f_\# p | f_\# r) = H(p|r)$.*

Proof. It is a direct consequence of the variational formula

$$H(p|r) = \sup_{u \in B_W(Y)} \left\{ \int_Y u \, dp - \int_Y e^{u-1} \, dr \right\},$$

where $B_W(Y) := \{ u : Y \to \mathbb{R}, \sup_Y |u|/(1+W) < \infty \}$. Indeed

$$H(f_\# p | f_\# r) = \sup_{v \in B_{W \circ f^{-1}}(f(Y))} \left\{ \int_{f(Y)} v \, d(f_\# p) - \int_{f(Y)} e^{v-1} \, d(f_\# r) \right\}$$

$$= \sup_{v \in B_{W \circ f^{-1}}(f(Y))} \left\{ \int_Y v \circ f \, dp - \int_Y e^{v \circ f^{-1}} \, dr \right\}$$

$$\leq \sup_{u \in B_W(Y)} \left\{ \int_Y u \, dp - \int_Y e^{u-1} \, dr \right\} = H(p|r),$$

because $\{ v \circ f ; v \in B_{W \circ f^{-1}}(f(Y)) \} \subset B_W(Y)$.
If f is injective, $u = v \circ f$ describes $B_W(Y)$ when v describes $B_{W \circ f^{-1}}(f(Y))$, leading to an equality. $\square$

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