AGMON-TYPE ESTIMATES FOR A CLASS OF JUMP PROCESSES

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ABSTRACT. In the limit $\varepsilon \to 0$ we analyze the generators H_ε of families of reversible jump processes in $\mathbb{R}^d$ associated with a class of symmetric non-local Dirichlet-forms and show exponential decay of the eigenfunctions. The exponential rate function is a Finsler distance, given as solution of a certain eikonal equation. Fine results are sensitive to the rate function being $\mathscr{C}^2$ or just Lipschitz. Our estimates are analog to the semiclassical Agmon estimates for differential operators of second order. They generalize and strengthen previous results on the lattice $\varepsilon\mathbb{Z}^d$.

1. INTRODUCTION

We derive exponential decay results on eigenfunctions of a family of self adjoint generators H_ε, $\varepsilon \in (0, \varepsilon_0]$, of (substochastic) jump processes in $\mathbb{R}^d$ in the limit $\varepsilon \to 0$. The jump processes are associated with non-local Dirichlet forms on the real Hilbert space $L^2(\mathbb{R}^d)$:

HYPOTHESIS 1.1 *Let $\mathcal{E}_\varepsilon$, $\varepsilon \in (0, \varepsilon_0]$, be a family of bilinear forms on $L^2(\mathbb{R}^d, dx)$ with domains $\mathscr{D}(\mathcal{E}_\varepsilon)$ given by*

$$\mathcal{E}_\varepsilon(u,v) := \frac{1}{2}\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d\setminus\{0\}} (u(x)-u(x+\varepsilon\gamma))(v(x)-v(x+\varepsilon\gamma))\, K_\varepsilon(x,d\gamma) + \int_{\mathbb{R}^d} V_\varepsilon(x)u(x)v(x)\,dx \tag{1.1}$$

$$\mathscr{D}(\mathcal{E}_\varepsilon) = \{u \in L^2(\mathbb{R}^d, dx)\,|\, \mathcal{E}_\varepsilon(u,u) < \infty\}\ ,$$

where for all $\varepsilon \in (0, \varepsilon_0]$

(a) *$V_\varepsilon(x)\,dx$ is a positive Radon measure on $\mathbb{R}^d$*
(b) *for $x \in \mathbb{R}^d$, $K_\varepsilon(x,\,.\,)$ is a positive Radon measure on the Borel sets $\mathcal{B}(\mathbb{R}^d\setminus\{0\})$ satisfying*
  (i) *$K_\varepsilon(x,E) < \infty$ for all $E \in \mathcal{B}(\mathbb{R}^d\setminus\{0\})$ with $\operatorname{dist}(E,0) \geq \delta > 0$*
  (ii) *$\int_{|\gamma|\leq 1} |\gamma|^2 K_\varepsilon(x,d\gamma) \leq C$ locally uniformly in $x \in \mathbb{R}^d$*
  (iii) *$K_\varepsilon(x,d\gamma)\,dx$ is a reversible measure on $Y := \mathbb{R}^d \times \mathbb{R}^d\setminus\{0\}$ in the sense that for all non-negative $\phi,\psi \in \mathscr{C}_0(\mathbb{R}^d)$*

$$\int_Y \phi(x+\varepsilon\gamma)\psi(x)K_\varepsilon(x,\,d\gamma)\,dx = \int_Y \phi(x)\psi(x+\varepsilon\gamma)K_\varepsilon(x,\,d\gamma)\,dx\ . \tag{1.2}$$

We shall formally denote the reversibility condition (1.2) as

$$K_\varepsilon(x,\,d\gamma)\,dx = K_\varepsilon(x+\varepsilon\gamma,\,-d\gamma)\,dx\ , \tag{1.3}$$

where the right hand side denotes the Radon measure on Y given by

$$\int_Y f(x,\gamma)K_\varepsilon(x+\varepsilon\gamma,-d\gamma)\,dx := \int_Y f(x,-\gamma)K_\varepsilon(x+\varepsilon\gamma,d\gamma)\,dx := \int_Y f(x-\varepsilon\gamma,-\gamma)K_\varepsilon(x,d\gamma)\,dx\ ,$$

and (abusing notation) we shall even cancel dx on both sides of (1.3).

Assuming Hypothesis 1.1, $\mathcal{E}_\varepsilon$ is a Dirichlet form (i.e. closed, symmetric and Markovian) and $\mathscr{C}_0^\infty(\mathbb{R}^d) \subset \mathscr{D}(\mathcal{E}_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0]$ (see Fukushima-Oshima-Takeda [6]).

The general theory of Dirichlet forms $\mathcal{E}$ analyzed in [6] covers the case, where $(\mathbb{R}^d, dx)$ is replaced by (X,m) if X is a locally compact separable metric space and m is a positive Radon measure on X with $\operatorname{supp} m = X$, provided that $\mathscr{D}(\mathcal{E})$ is dense in $L^2(X,m)$.

In particular, this is true for $X = (\varepsilon\mathbb{Z})^d$ and m being the counting measure on X. In this situation, we proved similar decay results in[10] and [14], with $K_\varepsilon(x, m(d\gamma)) = -a_{\varepsilon\gamma}(x;\varepsilon)m(d(\varepsilon\gamma))$ as a measure on $\mathbb{Z}^d\setminus\{0\}$ (in fact, we treated a slightly more general case where the form $\mathcal{E}_\varepsilon$ instead of being positive is only semibounded, $\mathcal{E}_\varepsilon(u,u) \geq -C\varepsilon$ for some $C > 0$ and $\varepsilon \in (0, \varepsilon_0]$).

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In this paper, we focus on the complementary and much more singular case $X = \mathbb{R}^d$. We remark that, combining the results of [10] with this paper, one could treat the case where X is an arbitrary abelian subgroup of $(\mathbb{R}^d, +)$ and m is Haar measure on X. Since our methods depend on some elements of Fourier analysis, this is a natural framework for our results.

The basic idea behind our estimates is due to Agmon [1]: The positivity of the quadratic form associated with a certain (weighted) operator H (on wavefunctions with support in a specific region) is related to decay of solutions of $Hu = f$ in that region in weighted L^2-sense, and the (optimal) rate function admits a geometric interpretation as a geodesic distance (which is Riemannian if H is a strongly elliptic differential operator of second order). A semiclassical version of the Agmon estimate (for the Schrödinger operator) was developed in [7] by Helffer and Sjöstrand who also applied such arguments in their analysis of Harper's equation [8] to a specific difference equation. In [10] a semiclassical Agmon estimate was proved for a classs of difference operators on the lattice $\varepsilon\mathbb{Z}^d$, identifying the rate function as a Finsler distance. We recall from [7] (for the Schrödinger operator) and from [11, 12, 13] that such estimates are an important first step to analyze the tunneling problem for a general multiwell problem. It is our main goal to develop this analysis in the context of jump processes as considered in this paper. Our motivation comes from previous work on metastability (see [2, 3]).

To control the limit $\varepsilon \to 0$, we shall impose stronger conditions on K_ε and V_ε.

HYPOTHESIS 1.2 (a) *The measure $K_\varepsilon(x, .)$ satisfies*

$$K_\varepsilon(x, .) = K^{(0)}(x, .) + R^{(1)}_\varepsilon(x, .) \qquad (x \in \mathbb{R}^d)\,, \tag{1.4}$$

where

(i) *for any $c > 0$ there exists $C > 0$ such that uniformly with respect to $x \in (\varepsilon\mathbb{Z})^d$ and $\varepsilon \in (0, \varepsilon_0]$*

$$\int_{|\gamma|\geq 1} e^{c|\gamma|} K^{(0)}(x, d\gamma) \leq C \qquad \text{and} \qquad \int_{|\gamma|\geq 1} e^{c|\gamma|} \big|R^{(1)}_\varepsilon(x, d\gamma)\big| \leq C\varepsilon \tag{1.5}$$

$$\int_{|\gamma|\leq 1} |\gamma|^2 K^{(0)}(x, d\gamma) \leq C \qquad \text{and} \qquad \int_{|\gamma|\leq 1} |\gamma|^2 \big|R^{(1)}_\varepsilon(x, d\gamma)\big| \leq C\varepsilon \tag{1.6}$$

(ii) *for all $x \in \mathbb{R}^d$ there exists $c_x > 0$ such that for all $v \in \mathbb{R}^d$*

$$\int_{\mathbb{R}^d\setminus\{0\}} (\gamma \cdot v)^2 K^{(0)}(x, d\gamma) \geq c_x \|v\|^2\,. \tag{1.7}$$

(b) (i) *The potential energy $V_\varepsilon \in \mathscr{C}^2(\mathbb{R}^d, \mathbb{R})$ satisfies*

$$V_\varepsilon(x) = V_0(x) + R_1(x; \varepsilon)\,,$$

where $V_0 \in \mathscr{C}^2(\mathbb{R}^d)$, $R_1 \in \mathscr{C}^2(\mathbb{R}^d \times (0, \varepsilon_0])$ and for any compact set $K \subset \mathbb{R}^d$ there exists a constant C_K such that $\sup_{x\in K} |R_1(x; \varepsilon)| \leq C_K\varepsilon$.

(ii) *$V_0(x) \geq 0$ and it takes the value 0 only at a finite number of non-degenerate minima x_j, i.e. $D^2V_0|_{x_j} > 0$, $j \in \mathcal{C} = \{1, \dots, r\}$, which we call potential wells.*

We remark that combining the positivity of the measure $K_\varepsilon(x, .)$ with Hypothesis 1.2(a), it follows that $K^{(0)}(x, .)$ is positive while $R^{(1)}_\varepsilon(x, .)$ is possibly signed.

It is well known (see e.g. [6]) that $\mathcal{E}_\varepsilon$ uniquely determines a self adjoint operator H_ε in $L^2(\mathbb{R}^d)$. To introduce Dirichlet boundary conditions for H_ε on some open set $\Sigma \subset \mathbb{R}^d$, one considers the form

$$\tilde{\mathcal{E}}^\Sigma_\varepsilon(u, v) = \mathcal{E}_\varepsilon(u, v) \quad \text{with domain} \quad \mathscr{D}(\tilde{\mathcal{E}}^\Sigma_\varepsilon) = \mathscr{C}^\infty_0(\Sigma)\,. \tag{1.8}$$

Then $\tilde{\mathcal{E}}^\Sigma_\varepsilon$ is Markovian (see [6], ex. 1.2.1) and closable. In fact, if we consider $L^2(\Sigma)$ as a subset of $L^2(\mathbb{R}^d)$ (extend $f \in L^2(\Sigma)$ to $\mathbb{R}^d$ by zero), the form

$$\widehat{\mathcal{E}}^\Sigma_\varepsilon(u, v) = \mathcal{E}_\varepsilon(u, v) \quad \text{with domain} \quad \mathscr{D}(\widehat{\mathcal{E}}^\Sigma_\varepsilon) = \{u \in L^2(\Sigma) \,|\, \widehat{\mathcal{E}}^\Sigma_\varepsilon(u, u) < \infty\} \tag{1.9}$$

- corresponding to Neumann boundary conditions - is a closed (Markovian) extension of $\tilde{\mathcal{E}}^\Sigma_\varepsilon$ (see [6], ex. 1.2.4), giving closability of $\tilde{\mathcal{E}}^\Sigma_\varepsilon$.

DEFINITION 1.3 *We denote by $\mathcal{E}^\Sigma_\varepsilon$ the closure of $\tilde{\mathcal{E}}^\Sigma_\varepsilon$ given in* (1.8). *The operator H^Σ_ε with Dirichlet boundary conditions on Σ is the unique self adjoint operator associated to $\mathcal{E}^\Sigma_\varepsilon$. The unique self adjoint operator $\widehat{H}^\Sigma_\varepsilon$ associated to $\widehat{\mathcal{E}}^\Sigma_\varepsilon$ defined in* (1.9) *represents Neumann boundary conditions on Σ.*

By [6], Thm. 3.1.1, $\mathcal{E}_\varepsilon^\Sigma$ is Markovian (as the closure of a Markovian form) and thus a Dirichlet form. In particular, since $\mathcal{E}_\varepsilon^\Sigma$ is a restriction of $\widehat{\mathcal{E}}_\varepsilon^\Sigma$, we have for $u, v \in \mathscr{D}(\mathcal{E}_\varepsilon^\Sigma)$,

$$\mathcal{E}_\varepsilon^\Sigma(u,v) = \mathcal{T}_\varepsilon^\Sigma(u,v) + \mathcal{V}_\varepsilon^\Sigma(u,v) \qquad \text{with} \tag{1.10}$$

$$\mathcal{T}_\varepsilon^\Sigma(u,v) := \frac{1}{2}\int_\Sigma dx \int_{\Sigma'(x)} (u(x) - u(x+\varepsilon\gamma))(v(x) - v(x+\varepsilon\gamma))\, K_\varepsilon(x, d\gamma) \quad \text{and} \tag{1.11}$$

$$\mathcal{V}_\varepsilon^\Sigma(u,v) := \int_\Sigma V_\varepsilon(x) u(x) v(x)\, dx \tag{1.12}$$

$$\text{where} \quad \Sigma'(x) := \{\gamma \in \mathbb{R}^d \setminus \{0\} \,|\, x + \varepsilon\gamma \in \Sigma\}. \tag{1.13}$$

Similarly, $\widehat{\mathcal{E}}_\varepsilon^\Sigma = \widehat{\mathcal{T}}_\varepsilon^\Sigma + \widehat{\mathcal{V}}_\varepsilon^\Sigma$. We remark that $\mathcal{T}_\varepsilon^\Sigma$, $\mathcal{V}_\varepsilon^\Sigma$. $\widehat{\mathcal{T}}_\varepsilon^\Sigma$ and $\widehat{\mathcal{V}}_\varepsilon^\Sigma$ are again Dirichlet forms, in particular they are positive.
We will use the notation $q[u] := q(u,u)$ for the quadratic form associated to any bilinear form q.

Concerning the operator H_ε associated to $\mathcal{E}_\varepsilon$ we remark that, even assuming Hypothesis 1.2 in addition to Hypothesis 1.1, it is far from trivial to characterize the domains $\mathscr{D}(H_\varepsilon)$ and $\mathscr{D}(H_\varepsilon^\Sigma)$. Without additional assumptions, $H_\varepsilon =: T_\varepsilon + V_\varepsilon$ (or H_ε^Σ) is not even defined on $\mathscr{C}_0^\infty(\mathbb{R}^d)$ (or $\mathscr{C}_0^\infty(\Sigma)$ resp.). However, there are some cases for which we can give formulae for $T_\varepsilon u$ on subsets of its domain.

(a) If the measure $K_\varepsilon(x,\ .\,)$ is finite uniformly with respect to $x \in \mathbb{R}^d$, one has

$$T_\varepsilon u(x) = \int_{\mathbb{R}^d\setminus\{0\}} (u(x) - u(x+\varepsilon\gamma)) K_\varepsilon(x,\, d\gamma)$$

and T_ε is bounded on $L^2(\mathbb{R}^d)$.

(b) If $K_\varepsilon(x, d\gamma) = k_\varepsilon(x,\gamma)\, d\gamma$, where $k_\varepsilon \in \mathscr{C}(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\})$ is Lipschitz in $x \in \mathbb{R}^d$, locally uniformly with respect to $\gamma \in \mathbb{R}^d \setminus \{0\}$, then $\mathscr{C}_0^2(\mathbb{R}^d) \subset \mathscr{D}(H_\varepsilon)$ and, for $u \in \mathscr{C}_0^2(\mathbb{R}^d)$,

$$T_\varepsilon u(x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(2u(x) - u(x+\varepsilon\gamma) - u(x-\varepsilon\gamma)\big) k_\varepsilon(x,\gamma)\, d\gamma \\ + \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x) - u(x-\varepsilon\gamma)\big)\big(k_\varepsilon(x-\varepsilon\gamma,\gamma) - k_\varepsilon(x,\gamma)\big)\, d\gamma\ . \tag{1.14}$$

(c) If $K_\varepsilon(x, d\gamma) = K_\varepsilon(x, -d\gamma)$, then $\mathscr{C}_0^2(\mathbb{R}^d) \subset \mathscr{D}(H_\varepsilon)$ and for $u \in \mathscr{C}_0^2(\mathbb{R}^d)$ one even has the simpler form

$$T_\varepsilon u(x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x) - u(x+\varepsilon\gamma) - \varepsilon\gamma\nabla u(x)\big) K_\varepsilon(x, d\gamma)\ . \tag{1.15}$$

(d) In the case of a Levy-process, i.e. if $K_\varepsilon(x,\, d\gamma) = K_\varepsilon(d\gamma)$ (which by reversibility, see (1.3), implies $K_\varepsilon(d\gamma) = K_\varepsilon(-d\gamma)$) one has both the representation (1.15) and (since the second term on the rhs of (1.14) formally vanishes)

$$T_\varepsilon u(x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(2u(x) - u(x+\varepsilon\gamma) - u(x-\varepsilon\gamma)\big) K_\varepsilon(d\gamma)\ .$$

Similar formulae hold for the operators with Dirichlet (and Neumann) boundary conditions. In this paper, we shall need none of them, since we shall directly work with the Dirichlet form (1.1).

We define $t_0 : \mathbb{R}^{2d} \to \mathbb{R}$ as

$$t_0(x,\xi) := \int_{\mathbb{R}^d\setminus\{0\}} \Big(1 - \cos\big(\eta\cdot\xi\big)\Big) K^{(0)}(x, d\gamma)\ , \tag{1.16}$$

which in view of Hypothesis 1.2(a),(ii) extends to an entire function in $\xi \in \mathbb{C}^d$, and we set

$$\tilde{t}_0(x,\xi) := -t_0(x, i\xi) = \int_{\mathbb{R}^d\setminus\{0\}} \Big(\cosh\big(\eta\cdot\xi\big) - 1\Big) K^{(0)}(x, d\gamma)\,, \qquad (x,\xi \in \mathbb{R}^d)\ . \tag{1.17}$$

We remark that t_0 formally is the principal symbol $\sigma_p(T_\varepsilon)$ - the leading order term in ε of the symbol - associated to the operator T_ε under semiclassical quantization (with ε as small parameter). Recall that for a symbol $b \in \mathscr{C}^\infty(\mathbb{R}^{2d} \times (0,\varepsilon_0))$, the corresponding operator is (formally) given by

$$\mathrm{Op}_\varepsilon(b)\, v(x) := (\varepsilon 2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{\frac{i}{\varepsilon}(y-x)\xi}\, b(x,\xi;\varepsilon) v(y)\, dy\, d\xi\,, \qquad v \in \mathscr{S}(\mathbb{R}^d)\,,$$

(for details on pseudo-differential operators see e.g. Dimassi-Sjöstrand [4]).

In particular, the translation operator $\tau_{\pm\varepsilon\gamma}$ acting as $\tau_{\pm\varepsilon\gamma}u(x) = u(x \pm \varepsilon\gamma)$, has the ε-symbol $e^{\mp i\gamma\xi}$. Thus, writing T_ε formally as

$$\frac{1}{2}\int_{\mathbb{R}^d\setminus\{0\}} \big(\mathbf{1} - \tau_{-\varepsilon\gamma}\big) K_\varepsilon(x, d\gamma)\big(\mathbf{1} - \tau_{\varepsilon\gamma}\big)$$

and using $\sigma_p(A \circ B) = \sigma_p(A)\sigma_p(B)$ for the principal symbols of operators A, B, immediately gives $t_0 = \sigma_p(T_\varepsilon)$ given in (1.16).

We emphasize, however, that under the weak regularity assumptions given in Hypotheses 1.1 and 1.2, T_ε is not an honest pseudo-differential operator (i.e. one with a $\mathscr{C}^\infty$-symbol, for which the symbolic calculus holds), but only a quantization of a singular symbol, giving a map $\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ (see [4]).

We shall now assume

Hypothesis 1.4 *Given Hypotheses 1.1 and 1.2, $\Sigma \subset \mathbb{R}^d$ is an open bounded set with $x_j \in \Sigma$ for exactly one $j \in \mathcal{C}$ and $x_k \notin \overline{\Sigma}$ for $k \in \mathcal{C}, k \neq j$. Moreover there is an open set $\Omega \subset \Sigma$ containing x_j and a Lipschitz-function $d : \overline{\Sigma} \to [0,\infty))$ satisfying, for $\tilde{t}_0$ defined in* (1.16),

(a) *$d(x_j) = 0$ and $d(x) \neq 0$ for $x \neq x_j$.*
(b) *$d \in \mathscr{C}^2(\overline{\Omega})$.*
(c) *the (generalized) eikonal equation holds in some neighborhood $U \subset \Omega$ of x_j, i.e.*

$$\tilde{t}_0(x, \nabla d(x)) = V_0(x) \qquad \text{for all} \quad x \in U\,. \tag{1.18}$$

(d) *the (generalized) eikonal inequality holds in Σ, i.e.*

$$\tilde{t}_0(x, \nabla d(x)) - V_0(x) \leq 0 \qquad \text{for all} \quad x \in \Sigma\,. \tag{1.19}$$

We remark that in a more regular setting, i.e. if $\tilde{h}_0 := \tilde{t}_0 - V_0 \in \mathscr{C}^\infty(\mathbb{R}^{2d})$, such a function d may be constructed as a distance in a certain Finsler metric associated with $\tilde{h}_0$ (see [10]), if Σ avoids the cut locus. We shall discuss the Finsler distance d in the case of low regularity and its relation to large deviation results for jump processes (see e.g. [9]) in a future publication.

Our central results are the following theorems on the decay of eigenfunctions of H_ε^Σ and $\widehat{H}_\varepsilon^\Sigma$.

Theorem 1.5 *Assume Hypotheses 1.1, 1.2 and 1.4 with $\Sigma = \Omega$. Let H_ε^Σ and $\widehat{H}_\varepsilon^\Sigma$ be the operators with Dirichlet and Neumann boundary conditions from Definition 1.3.*

Fix $R_0 > 0$ and let $E \in [0, \varepsilon R_0]$. Then there exist constants $\varepsilon_0, B, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and real $u \in \mathscr{D}(H_\varepsilon^\Sigma)$

$$\left\| \left(1 + \tfrac{d}{\varepsilon}\right)^{-B} e^{\frac{d}{\varepsilon}} u \right\|_{L^2(\Sigma)} \leq C\left[\varepsilon^{-1} \left\| \left(1 + \tfrac{d}{\varepsilon}\right)^{-B} e^{\frac{d}{\varepsilon}} \left(H_\varepsilon^\Sigma - E\right) u \right\|_{L^2(\Sigma)} + \|u\|_{L^2(\Sigma)}\right]. \tag{1.20}$$

In particular, let $u \in \mathscr{D}(H_\varepsilon^\Sigma)$ be a normalized eigenfunction of H_ε^Σ with respect to the eigenvalue $E \in [0, \varepsilon R_0]$. Then there exist constants $B, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$

$$\left\| \left(1 + \tfrac{d}{\varepsilon}\right)^{-B} e^{\frac{d}{\varepsilon}} u \right\|_{L^2(\Sigma)} \leq C\,. \tag{1.21}$$

The constants ε_0, B, C are uniform with respect to $E \in [0, \varepsilon R_0]$ and u with $\|u\|_{L^2(\Sigma)} \leq 1$.
Analog results hold for $u \in \mathscr{D}(\widehat{H}_\varepsilon^\Sigma)$ and u a normalized eigenfunction of $\widehat{H}_\varepsilon^\Sigma$ respectively.

The following theorem gives a weaker result in the case that d is only Lipschitz outside some small ball around x_j. Then we have to assume more regularity of $K^{(0)}$ with respect to x.

Theorem 1.6 *Assume Hypotheses 1.1, 1.2 and 1.4 and let H_ε^Σ and $\widehat{H}_\varepsilon^\Sigma$ be the operators with Dirichlet and Neumann boundary conditions from Definition 1.3. Moreover assume that $K^{(0)}(\,.\,, d\gamma)$ is continuous in the sense that for all $c > 0$*

$$\int_{\gamma\in\mathbb{R}^d\setminus\{0\}} |\gamma|^2 e^{c|\gamma|}\big(K^{(0)}(x+h, d\gamma) - K^{(0)}(x, d\gamma)\big) = o(1) \qquad (|h| \to 0) \tag{1.22}$$

locally uniformly in $x \in \mathbb{R}^d$. Fix $R_0 > 0$ and a constant $D > 0$ such that the ball $K := \{x \in \mathbb{R}^d \,|\, d(x) \leq D\}$ is contained in Ω, and let $E \in [0, \varepsilon R_0]$. Then there exist constants $C, B > 0$ such that

(a) *for any fixed $\alpha \in (0, 1]$ there exists ε_α such that for all $\varepsilon \in (0, \varepsilon_\alpha]$ and real $u \in \mathscr{D}(H_\varepsilon^\Sigma)$*

$$\left\| e^{\frac{(1-\alpha)d}{\varepsilon}} u \right\|_{L^2(\Sigma)} \leq C\left[\varepsilon^{-1} \left\| e^{\frac{d}{\varepsilon}} \left(H_\varepsilon^\Sigma - E\right) u \right\|_{L^2(\Sigma)} + \|u\|_{L^2(\Sigma)}\right]. \tag{1.23}$$

(b) *there exists a constant* $\alpha_0 > 0$ *such that for any fixed* $\alpha \in (0, \alpha_0]$ *there exists* $\Phi_\alpha \in \mathscr{C}^2(\overline{\Sigma})$ *and* $\varepsilon_\alpha > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_\alpha]$ *and real* $u \in \mathscr{D}(H_\varepsilon^\Sigma)$

$$\left\| e^{\frac{\Phi_\alpha}{\varepsilon}} u \right\|_{L^2(\Sigma)} \leq C\left[\varepsilon^{-1} \left\| e^{\frac{\Phi_\alpha}{\varepsilon}} \left(H_\varepsilon^\Sigma - E\right) u \right\|_{L^2(\Sigma)} + \|u\|_{L^2(\Sigma)}\right] , \tag{1.24}$$

where for some $C' > 0$ *and for any fixed* $\alpha \in (0, 1]$

$$e^{\frac{d(x)}{\varepsilon}} \frac{1}{C'} \left(1 + \tfrac{d(x)}{\varepsilon}\right)^{-\frac{B}{2}} \leq e^{\frac{\Phi_\alpha(x)}{\varepsilon}} \leq e^{\frac{d(x)}{\varepsilon}} C' \left(1 + \tfrac{d(x)}{\varepsilon}\right)^{-\frac{B}{2}} \quad \text{for } x \in K \text{ and} \tag{1.25}$$

$$e^{\frac{(1-\alpha)d(x)}{\varepsilon}} \leq e^{\frac{\Phi_\alpha(x)}{\varepsilon}} \leq e^{\frac{d(x)}{\varepsilon}} \qquad \text{for } x \in \overline{\Sigma} \setminus K\,. \tag{1.26}$$

(c) *for any fixed* $\alpha \in (0, 1]$ *there exists* $\varepsilon_\alpha > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_\alpha]$ *and real* $u \in \mathscr{D}(H_\varepsilon^\Sigma)$

$$\frac{1}{C'} \left\| \left(1 + \tfrac{d}{\varepsilon}\right)^{-\frac{B}{2}} e^{\frac{d}{\varepsilon}} u \right\|^2_{L^2(K)} + \left\| e^{\frac{(1-\alpha)d(x)}{\varepsilon}} u \right\|^2_{L^2(\Sigma\setminus K)} \leq \left\| e^{\frac{\Phi_\alpha}{\varepsilon}} u \right\|^2_{L^2(\Sigma)} \tag{1.27}$$

and if u *is a normalized eigenfunction of* H_ε^Σ *with respect to the eigenvalue* $E \in [0, \varepsilon R_0]$, *then*

$$\left\| e^{\frac{\Phi_\alpha}{\varepsilon}} u \right\|_{L^2(\Sigma)} \leq C\,. \tag{1.28}$$

The constants $\alpha_0, \varepsilon_\alpha, B, C$ *are uniform with respect to* $E \in [0, \varepsilon R_0]$ *and* u *with* $\|u\|_{L^2(\Sigma)} \leq 1$.

Analog results hold for $\widehat{H}_\varepsilon^\Sigma$ *and for real* $u \in \mathscr{D}(\widehat{H}_\varepsilon^\Sigma)$ *respectively.*

REMARK 1.7 *All assertions of Theorem 1.5 and 1.6 remain true if* $\mathcal{E}_\varepsilon$ *is not necessarily positive, but only satisfies* $\mathcal{E}_\varepsilon(x) \geq -C\varepsilon$ *or, more special,* $\mathcal{E}_\varepsilon \geq 0$ *but* $V_\varepsilon \geq -C\varepsilon$. *In a stochastic context, such a situation could arise if e.g. one starts with a Dirichlet form* $\tilde{\mathcal{E}}_\varepsilon$ *on* $L^2(m_\varepsilon)$ *associated with a pure jump process (with* $V_\varepsilon = 0$*), given by a kernel* $\tilde{K}_\varepsilon(x, d\gamma)$, *which is integrable with respect to* $\gamma \in \mathbb{R}^d \setminus \{0\}$, *i.e. satisfies* $\int \tilde{K}_\varepsilon(x, d\gamma) < \infty$, *and reversible with respect to* $m_\varepsilon(dx) = e^{-\frac{F(x)}{\varepsilon}}dx$. *If* $K_\varepsilon(x, d\gamma) := e^{\frac{F(x+\varepsilon\gamma)-F(x)}{2\varepsilon}} \tilde{K}_\varepsilon(x, d\gamma)$ *is integrable with respect to* $\gamma \in \mathbb{R}^d \setminus \{0\}$, *then*

$$\mathcal{E}_\varepsilon(u, v) := \tilde{\mathcal{E}}_\varepsilon\big(e^{\frac{F}{2\varepsilon}}u, e^{\frac{F}{2\varepsilon}}v\big) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus\{0\}} (u(x+\varepsilon\gamma) - u(x))(v(x+\varepsilon\gamma) - v(x))\, K_\varepsilon(x, d\gamma)\, dx + \langle u\,, V_\varepsilon v\rangle_{L^2}\,,$$

is a Dirichlet form on $L^2(dx)$, *where*

$$V_\varepsilon(x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(e^{-\frac{F(x+\varepsilon\gamma)-F(x)}{2\varepsilon}} - 1\big) K_\varepsilon(x, d\gamma) = \int_{\mathbb{R}^d\setminus\{0\}} (\tilde{K}_\varepsilon - K_\varepsilon)(x, d\gamma)\,.$$

If F *is smooth and* K_ε *and* $\tilde{K}_\varepsilon$ *have an expansion as in Hypothesis 1.2(a), then one verifies that* $K^{(0)}(x, d\gamma) = K^{(0)}(x, -d\gamma)$ *and* $V_\varepsilon \geq -C\varepsilon$ *for some constant* $C > 0$. *If the integrability conditions for* K_ε *and* $\tilde{K}_\varepsilon$ *are not satisfied, the above transformation is more delicate and requires regularity of* $K_\varepsilon(x, d\gamma)$ *in* x.

We emphasize that the eigenvalue E in Theorem 1.5 and 1.6 need not be discrete (a priori, it could be of infinite multiplicity or be imbedded into the continuous (or essential) spectrum of H_ε). In this paper, H_ε need not have a spectral gap. However, to develop tunneling theory in analogy to [10, 11, 12, 13], one needs to impose further conditions on the jump kernel K_ε.

2. Preliminary Results

This section contains preparations for the proof of Theorem 1.5 and 1.6. Lemmata 2.1 - 2.3 contain our abstract approach to Agmon type estimates, while Lemmata 2.4 - 2.7 contain more specific estimates on $\tilde{t}_0(x, \xi), d(x)$ and the phasefunctions used in the proof of Theorem 1.5 and 1.6.

LEMMA 2.1 *Assume Hypotheses 1.1 and 1.2 and, for* $\Sigma \subset \mathbb{R}^d$ *open, let* $\mathcal{E}_\varepsilon^\Sigma$ *and* $\widehat{\mathcal{E}}_\varepsilon^\Sigma$ *denote the associated Dirichlet forms given in Definition 1.3 and* (1.9) *respectively. Let* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *be Lipschitz and bounded. Then for any real valued* v *with* $e^{\pm\frac{\varphi}{\varepsilon}} v \in \mathscr{D}(\mathcal{E}_\varepsilon^\Sigma)$ *(or* $\mathscr{D}(\widehat{\mathcal{E}}_\varepsilon^\Sigma)$ *resp.)*

$$\begin{aligned}\mathcal{E}_\varepsilon^\Sigma\big(e^{-\frac{\varphi}{\varepsilon}}v,\, e^{\frac{\varphi}{\varepsilon}}v\big) &= \Big\langle \big(V_\varepsilon + V^\varphi_{\varepsilon,\Sigma}\big)v\,,\, v\Big\rangle_{L^2(\Sigma)} \\ &\quad + \frac{1}{2}\int_\Sigma dx \int_{\Sigma'(x)} K_\varepsilon(x, d\gamma) \cosh\Big(\tfrac{1}{\varepsilon}\big(\varphi(x) - \varphi(x+\varepsilon\gamma)\big)\Big)\big(v(x) - v(x+\varepsilon\gamma)\big)^2\,, \end{aligned} \tag{2.1}$$

where $\Sigma'(x)$ *is defined in* (1.13) *and*

$$V^{\varphi}_{\varepsilon,\Sigma}(x) := \int_{\Sigma'(x)} \Big[1-\cosh\Big(\tfrac{1}{\varepsilon}\big(\varphi(x)-\varphi(x+\varepsilon\gamma)\big)\Big)\Big]\, K_\varepsilon(x,d\gamma)\;, \tag{2.2}$$

which ist bounded uniformly in ε*. An analog result holds for* $\widehat{\mathcal{E}}^\Sigma_\varepsilon$*.*

Proof. We have by (1.10)

$$\begin{aligned}
&\mathcal{E}^\Sigma_\varepsilon\big(e^{-\frac{\varphi}{\varepsilon}}v,\, e^{\frac{\varphi}{\varepsilon}}v\big) - \langle V_\varepsilon v\,,\, v\rangle_{L^2(\Sigma)}\\
&\quad= \frac12\int_\Sigma dx\int_{\Sigma'(x)}\Big(v(x)^2 - 2\cosh\big(\tfrac{1}{\varepsilon}(\varphi(x+\varepsilon\gamma)-\varphi(x))\big)v(x)v(x+\varepsilon\gamma) + v(x+\varepsilon\gamma)^2\Big)K_\varepsilon(x,\,d\gamma)\\
&\qquad= \frac12\int_\Sigma dx\int_{\Sigma'(x)}\big(v(x)^2+v(x+\varepsilon\gamma)^2\big)\Big(1-\cosh\big(\tfrac{1}{\varepsilon}(\varphi(x+\varepsilon\gamma)-\varphi(x))\big)\Big)K_\varepsilon(x,\,d\gamma)\\
&\qquad\qquad+\frac12\int_\Sigma dx\int_{\Sigma'(x)}\cosh\big(\tfrac{1}{\varepsilon}(\varphi(x+\varepsilon\gamma)-\varphi(x))\big)\big(v(x)-v(x+\varepsilon\gamma)\big)^2K_\varepsilon(x,\,d\gamma)\;.
\end{aligned}\tag{2.3}$$

Since $\cosh\xi$ is even with respect to ξ and by the reversibility (1.2) of $K_\varepsilon(x,d\gamma)$

$$\begin{aligned}
&\frac12\int_\Sigma dx\int_{\Sigma'(x)}\big(v(x)^2+v(x+\varepsilon\gamma)^2\big)\Big(1-\cosh\big(\tfrac{1}{\varepsilon}(\varphi(x+\varepsilon\gamma)-\varphi(x))\big)\Big)K_\varepsilon(x,\,d\gamma)\\
&\qquad= \int_\Sigma dx\int_{\Sigma'(x)} v(x)^2\Big(1-\cosh\big(\tfrac{1}{\varepsilon}(\varphi(x+\varepsilon\gamma)-\varphi(x))\big)\Big)K_\varepsilon(x,\,d\gamma)\;.
\end{aligned}\tag{2.4}$$

Thus inserting (2.4) into (2.3) and using the definition of $V^\varphi_{\varepsilon,\Sigma}$ gives (2.1).

To show boundedness of the integral on the right hand side of (2.2), one observes that $\cosh t-1\leq |t|\sinh|t|$ for all $t\in\mathbb{R}$. Choosing $t=\frac{1}{\varepsilon}(\varphi(x)-\varphi(x+\varepsilon\gamma))$ and using that φ is Lipschitz with Lipschitz constant $L>0$ gives

$$\cosh\Big(\tfrac{1}{\varepsilon}\big(\varphi(x)-\varphi(x+\varepsilon\gamma)\big)\Big)-1\leq L^2|\gamma|^2\frac{\sinh(L|\gamma|)}{L|\gamma|}\;. \tag{2.5}$$

Inserting (2.5) into (2.2) proves the assertion, according to Hypothesis 1.2,(a),(i).

Since the formula (1.10) also holds for $\widehat{\mathcal{E}}^\Sigma_\varepsilon$, the same arguments give the analog result.

□

Lemma 2.2 *Assume Hypotheses 1.1 and 1.2 and for* $\Sigma\subset\mathbb{R}^d$ *open, let* $\mathcal{E}^\Sigma_\varepsilon$*,* $\widehat{\mathcal{E}}^\Sigma_\varepsilon$ *and* φ *be as in Lemma 2.1. Then*

$$v\in\mathscr{D}(\mathcal{E}^\Sigma_\varepsilon)\text{ or }\mathscr{D}(\widehat{\mathcal{E}}^\Sigma_\varepsilon)\text{ resp.}\quad\Rightarrow\quad e^{\frac{\varphi}{\varepsilon}}v\in\mathscr{D}(\mathcal{E}^\Sigma_\varepsilon)\text{ or }\mathscr{D}(\widehat{\mathcal{E}}^\Sigma_\varepsilon)\text{ resp.}\,.$$

Proof. We will use the notation (see (1.10))

$$\mathfrak{t}^\Sigma_\varepsilon[u] := \widehat{\mathcal{E}}^\Sigma_\varepsilon[u] + \|u\|^2_{L^2(\Sigma)} = \widehat{\mathcal{T}}^\Sigma_\varepsilon[u]+\widehat{\mathcal{V}}^\Sigma_\varepsilon[u]+\|u\|^2_{L^2(\Sigma)}\;. \tag{2.6}$$

We recall that a function $f\in\mathscr{D}(\widehat{\mathcal{E}}^\Sigma_\varepsilon)$ is in $\mathscr{D}(\mathcal{E}^\Sigma_\varepsilon)$, if and only if there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{D}(\tilde{\mathcal{E}}^\Sigma_\varepsilon)$ such that $\mathfrak{t}^\Sigma_\varepsilon[f_n-f]\to0$ as $n\to\infty$.

We notice that for some $C,L>0$

$$\|\varphi\|_\infty\leq C\quad\text{and}\quad|\varphi(x)-\varphi(y)|\leq L|x-y|\,,\quad x,y\in\mathbb{R}^d\;. \tag{2.7}$$

Step 1:
Let $v\in\mathscr{D}(\widehat{\mathcal{E}}^\Sigma_\varepsilon)$, then we shall show that for some $\tilde C>0$ uniformly with respect to $\varepsilon\in(0,\varepsilon_0]$

$$\mathfrak{t}^\Sigma_\varepsilon[e^{\frac{\varphi}{\varepsilon}}v]\leq e^{\frac{\tilde C}{\varepsilon}}\mathfrak{t}^\Sigma_\varepsilon[v]\;. \tag{2.8}$$

By (1.9), this implies $e^{\frac{\varphi}{\varepsilon}}v\in\mathscr{D}(\widehat{\mathcal{E}}^\Sigma_\varepsilon)$.

From (2.7) it follows at once that

$$\|e^{\frac{\varphi}{\varepsilon}}v\|^2_{L^2(\Sigma)}\leq e^{\frac{2C}{\varepsilon}}\|v\|^2_{L^2(\Sigma)}\;. \tag{2.9}$$

Using the definition (1.12) of $\widehat{\mathcal{V}}^\Sigma_\varepsilon$, we have by (2.7)

$$\widehat{\mathcal{V}}^\Sigma_\varepsilon[e^{\frac{\varphi}{\varepsilon}}v]\leq e^{\frac{2C}{\varepsilon}}\widehat{\mathcal{V}}^\Sigma_\varepsilon[v]\;. \tag{2.10}$$

It remains to analyze

$$\widehat{\mathcal{T}}_\varepsilon^\Sigma[e^{\frac{\varphi}{\varepsilon}}v] = \frac{1}{2}\int_\Sigma dx \int_{\Sigma'(x)} \Big(e^{\frac{\varphi(x+\varepsilon\gamma)}{\varepsilon}}v(x+\varepsilon\gamma) - e^{\frac{\varphi(x)}{\varepsilon}}v(x)\Big)^2 K_\varepsilon(x,d\gamma)\,. \tag{2.11}$$

Adding $f-f$ for $f = e^{\frac{\varphi(x+\varepsilon\gamma)}{\varepsilon}}v(x)$ inside the brackets on rhs(2.11) and then using $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$\text{rhs}(2.11) \leq A[v] + B[v] \qquad \text{where} \tag{2.12}$$

$$A[v] := \int_\Sigma dx \int_{\Sigma'(x)} e^{2\frac{\varphi(x+\varepsilon\gamma)}{\varepsilon}}\big(v(x+\varepsilon\gamma) - v(x)\big)^2 K_\varepsilon(x,d\gamma)$$

$$B[v] := \int_\Sigma dx \int_{\Sigma'(x)} v(x)^2\big(e^{\frac{\varphi(x+\varepsilon\gamma)}{\varepsilon}} - e^{\frac{\varphi(x)}{\varepsilon}}\big)^2 K_\varepsilon(x,d\gamma)\,. \tag{2.13}$$

By (2.7) we have

$$A[v] \leq e^{\frac{2C}{\varepsilon}}\widehat{\mathcal{T}}_\varepsilon^\Sigma[v]\,. \tag{2.14}$$

To estimate $B[v]$, observe that

$$|1-e^t| \leq e^{|t|} - 1\,, \qquad (t\in\mathbb{R})\,, \tag{2.15}$$

which by (2.7) leads to

$$\Big|e^{\frac{\varphi(x+\varepsilon\gamma)}{\varepsilon}} - e^{\frac{\varphi(x)}{\varepsilon}}\Big| \leq e^{\frac{\varphi(x+\varepsilon\gamma)}{\varepsilon}}\Big(e^{\frac{1}{\varepsilon}|\varphi(x)-\varphi(x+\varepsilon\gamma)|} - 1\Big) \leq e^{\frac{C}{\varepsilon}}L|\gamma|e^{L|\gamma|}\,. \tag{2.16}$$

Substituting (2.16) into (2.13) gives by Hypothesis 1.2(a)(i)

$$B[v] \leq e^{\frac{2C}{\varepsilon}}L^2\int_\Sigma dx|v(x)|^2\int_{\Sigma'(x)}|\gamma|^2e^{2L|\gamma|}K_\varepsilon(x,d\gamma) \leq e^{\frac{\tilde{C}}{\varepsilon}}\|v\|^2_{L^2(\Sigma)}\,, \tag{2.17}$$

where $\tilde{C}$ is uniform with respect to $\varepsilon\in(0,\varepsilon_0]$. Inserting (2.17) and (2.14) into (2.12) and the result in (2.11), and combining (2.11), (2.10) and (2.9) proves (2.8).

Step 2:
We prove

$$e^{\frac{\varphi}{\varepsilon}}v \in \mathscr{D}(\mathcal{E}_\varepsilon^\Sigma) \quad \text{for} \quad v\in\mathscr{C}_0^\infty(\Sigma)\subset\mathscr{D}(\mathcal{E}_\varepsilon^\Sigma)\,. \tag{2.18}$$

Let $j\in\mathscr{C}_0^\infty(\mathbb{R}^d)$ be non-negative with $\int_{\mathbb{R}^d} j(x)\,dx = 1$. For $\delta>0$ we set $j_\delta(x) := \delta^{-d}j(\frac{x}{\delta})$ and $\varphi_\delta := \varphi * j_\delta$, then $\varphi_\delta\in\mathscr{C}^\infty(\mathbb{R}^d)$ and $e^{\frac{\varphi_\delta}{\varepsilon}}v\in\mathscr{C}_0^\infty(\Sigma)\subset\mathscr{D}(\mathcal{E}_\varepsilon^\Sigma)$. Moreover

$$\|\varphi_\delta\|_\infty \leq \|\varphi\|_\infty \leq C\,, \qquad \big\|\varphi_\delta - \varphi\big\| \longrightarrow 0 \quad \text{as} \quad \delta\to 0 \tag{2.19}$$

and φ_δ has the same Lipschitz constant L as φ (see (2.7)), since

$$|\varphi_\delta(x) - \varphi_\delta(y)| = \Big|\int_{\mathbb{R}^d}\big(\varphi(x-z) - \varphi(y-z)\big)j_\delta(z)\,dz\Big| \leq L|x-y|\,. \tag{2.20}$$

Assume $v\in\mathscr{C}_0^\infty(\Sigma)$, then by Step 1, $e^{\frac{\varphi}{\varepsilon}}v\in\mathscr{D}(\widehat{\mathcal{E}}_\varepsilon^\Sigma)$. Thus it suffices to show that

$$\mathfrak{t}_\varepsilon^\Sigma\Big[\Big(e^{\frac{\varphi_\delta}{\varepsilon}} - e^{\frac{\varphi}{\varepsilon}}\Big)v\Big] \longrightarrow 0 \quad \text{as} \quad \delta\to 0\,. \tag{2.21}$$

By dominated convergence, using (2.19),

$$\Big\|\Big(e^{\frac{\varphi_\delta}{\varepsilon}} - e^{\frac{\varphi}{\varepsilon}}\Big)v\Big\|_{L^2(\Sigma)} \longrightarrow 0 \qquad \text{and} \qquad \widehat{\mathcal{V}}_\varepsilon^\Sigma\Big[\Big(e^{\frac{\varphi_\delta}{\varepsilon}} - e^{\frac{\varphi}{\varepsilon}}\Big)v\Big] \longrightarrow 0\,, \quad (\delta\to 0)\,. \tag{2.22}$$

To analyze $\widehat{\mathcal{T}}_\varepsilon^\Sigma$, we set $\Phi_\delta := e^{\frac{\varphi_\delta}{\varepsilon}} - e^{\frac{\varphi}{\varepsilon}}$, then

$$\widehat{\mathcal{T}}_\varepsilon^\Sigma\Big[\Big(e^{\frac{\varphi_\delta}{\varepsilon}} - e^{\frac{\varphi}{\varepsilon}}\Big)v\Big] = A'[v] + B'[v]\,, \quad \text{where} \tag{2.23}$$

$$A'[v] := \int_\Sigma dx\int_{\Sigma'(x)}\Phi_\delta^2(x+\varepsilon\gamma)\big(v(x+\varepsilon\gamma) - v(x)\big)^2K_\varepsilon(x,d\gamma)$$

$$B'[v] := \int_\Sigma dx\int_{\Sigma'(x)} v(x)^2\big(\Phi_\delta(x+\varepsilon\gamma) - \Phi_\delta(x)\big)^2K_\varepsilon(x,d\gamma)\,.$$

Since $\|\Phi_\delta\|_\infty \leq e^{\frac{C'}{\varepsilon}}$ by (2.19) uniformly with respect to $\delta > 0$, $e^{\frac{2C'}{\varepsilon}}(v(x+\varepsilon\gamma) - v(x))^2$ is a dominating function for the integrand of $A'[v]$, which is in $L^1(d\mu)$ for the measure $d\mu = K_\varepsilon(x, d\gamma)dx$ in $\Sigma \times \mathbb{R}^d \setminus \{0\}$. Thus by the dominated convergence theorem

$$A'[v] \to 0 \,, \qquad (\delta \to 0) \tag{2.24}$$

because $\|\Phi_\delta\|_\infty \to 0$ as $\delta \to 0$, by (2.19). Similarly,

$$B'[v] \to 0 \,, \qquad (\delta \to 0) \tag{2.25}$$

by the dominated convergence theorem (observe that using (2.16) for φ and φ_δ, uniformly with respect to δ in view of (2.19) and (2.20) one finds

$$|\Phi_\delta(x+\varepsilon\gamma) - \Phi_\delta(x)| \leq \left| e^{\frac{\varphi_\delta(x+\varepsilon\gamma)}{\varepsilon}} - e^{\frac{\varphi_\delta(x)}{\varepsilon}} \right| + \left| e^{\frac{\varphi(x+\varepsilon\gamma)}{\varepsilon}} - e^{\frac{\varphi(x)}{\varepsilon}} \right| \leq 2e^{\frac{C}{\varepsilon} + L|\gamma|} L|\gamma| \,,$$

which gives a dominating function for the integrand of $B'[v]$, which in view of Hypothesis 1.2(a) is integrable with respect to $d\mu$). Inserting (2.25) and (2.24) into (2.23) and combining the result with (2.22) proves (2.21) and (2.18).

Step 3:
Assume $v \in \mathscr{D}(\mathcal{E}_\varepsilon^\Sigma)$, then by Definition 1.3, there are $v_n \in \mathscr{C}_0^\infty(\Sigma)$ with $\mathfrak{t}_\varepsilon^\Sigma[v_n - v] \to 0$ as $n \to \infty$. By Step 2, for all $n \in \mathbb{N}$, $e^{\frac{\varphi}{\varepsilon}} v_n \in \mathscr{D}(\mathcal{E}_\varepsilon^\Sigma)$, and

$$\mathfrak{t}_\varepsilon^\Sigma\big[e^{\frac{\varphi}{\varepsilon}}(v_n - v)\big] \to 0 \,, \qquad (n \to \infty)$$

by (2.8), proving $e^{\frac{\varphi}{\varepsilon}} v \in \mathscr{D}(\mathcal{E}_\varepsilon^\Sigma)$.

□

We will use Lemma 2.1 and Lemma 2.2 to prove the following norm estimate, which is a main ingredient in the proof of Theorem 1.5.

Lemma 2.3 *Assume Hypotheses 1.1, 1.2 and, for $\Sigma \subset \mathbb{R}^d$ open, let H_ε^Σ ($\widehat{H}_\varepsilon^\Sigma$) denote the operator with Dirichlet (Neumann) boundary conditions introduced in Definition 1.3. Let $\varphi : \Sigma \to \mathbb{R}$ be Lipschitz and bounded. For $E \geq 0$ fixed, let $F_\pm : \Sigma \to [0,\infty)$ be a pair of functions such that $F(x) := F_+(x) + F_-(x) > 0$ and*

$$F_+^2(x) - F_-^2(x) = V_\varepsilon(x) + V_{\varepsilon,\Sigma}^\varphi(x) - E \,, \qquad x \in \Sigma \,, \tag{2.26}$$

where $V_{\varepsilon,\Sigma}^\varphi(x)$ is given in (2.2). Then for $u \in \mathscr{D}(H_\varepsilon^\Sigma)$ (or $\mathscr{D}(\widehat{H}_\varepsilon^\Sigma)$) real-valued with $Fe^{\frac{\varphi}{\varepsilon}}u \in L^2(\Sigma)$, we have for some $C > 0$

$$\|Fe^{\frac{\varphi}{\varepsilon}}u\|^2_{L^2(\Sigma)} \leq 4\left\|\tfrac{1}{F} e^{\frac{\varphi}{\varepsilon}}(H_\varepsilon^\Sigma - E)u\right\|^2_{L^2(\Sigma)} + 8\|F_- e^{\frac{\varphi}{\varepsilon}}u\|^2_{L^2(\Sigma)} \,. \tag{2.27}$$

Proof. First observe that for $v := e^{\frac{\varphi}{\varepsilon}}u$

$$\|Fv\|^2_{L^2(\Sigma)} \leq 2\left(\|F_+v\|^2_{L^2(\Sigma)} + \|F_-v\|^2_{L^2(\Sigma)}\right) = 2\left(\|F_+v\|^2_{L^2(\Sigma)} - \|F_-v\|^2_{L^2(\Sigma)}\right) + 4\|F_-v\|^2_{L^2(\Sigma)} \,. \tag{2.28}$$

By (2.26) one has

$$\|F_+v\|^2_{L^2(\Sigma)} - \|F_-v\|^2_{L^2(\Sigma)} = \left\langle (V_\varepsilon + V_{\varepsilon,\Sigma}^\varphi - E)v \,,\, v \right\rangle_{L^2(\Sigma)} \,. \tag{2.29}$$

Since $v \in \mathscr{D}(\mathcal{E}_\varepsilon^\Sigma)$ (or $\mathscr{D}(\widehat{\mathcal{E}}_\varepsilon^\Sigma)$) by Lemma 2.2, it follows at once from Lemma 2.1 that

$$\left\langle (V_\varepsilon + V_{\varepsilon,\Sigma}^\varphi - E)v \,,\, v \right\rangle_{L^2(\Sigma)} \leq \mathcal{E}_\varepsilon^\Sigma\left(e^{-\frac{\varphi}{\varepsilon}}v, e^{\frac{\varphi}{\varepsilon}}v\right) - E\|v\|^2_{L^2(\Sigma)} \,. \tag{2.30}$$

(2.29) and (2.30) yield by use of the Cauchy-Schwarz inequality, since $u \in \mathscr{D}(H_\varepsilon^\Sigma)$,

$$\begin{aligned} 2\left(\|F_+v\|^2_{L^2(\Sigma)} - \|F_-v\|^2_{L^2(\Sigma)}\right) &\leq 2\big\langle \big(e^{\frac{\varphi}{\varepsilon}}(H_\varepsilon^\Sigma - E)\big)\, u \,,\, v \big\rangle_{L^2(\Sigma)} \\ &\leq 2\sqrt{2}\left\|\tfrac{1}{F}\left(e^{\frac{\varphi}{\varepsilon}}(H_\varepsilon^\Sigma - E)\right)u\right\|_{L^2(\Sigma)} \frac{1}{\sqrt{2}}\|Fv\|_{L^2(\Sigma)} \\ &\leq 2\left\|\tfrac{1}{F}\left(e^{\frac{\varphi}{\varepsilon}}(H_\varepsilon^\Sigma - E)\right)u\right\|^2_{L^2(\Sigma)} + \frac{1}{2}\|Fv\|^2_{L^2(\Sigma)} \,. \end{aligned} \tag{2.31}$$

Inserting (2.31) into (2.28) we get

$$\|Fv\|^2_{L^2(\Sigma)} \leq 2 \left\| \tfrac{1}{F} \left(e^{\frac{\varphi}{\varepsilon}} (H^\Sigma_\varepsilon - E) \right) u \right\|^2_{L^2(\Sigma)} + \frac{1}{2}\|Fv\|^2_{L^2(\Sigma)} + 4\|F_- v\|^2_{L^2(\Sigma)} \,,$$

which by definition of v gives (2.27).

□

Lemma 2.4 *Assume Hypotheses 1.1 and 1.2.*

(a) *For any $x \in \mathbb{R}^d$, the function $L_x : \mathbb{R}^d \ni \xi \mapsto \tilde{t}_0(x,\xi)$ is even and hyperconvex, i.e. $D^2 L_x|_{\xi_0} \geq \alpha > 0$, uniformly in ξ_0.*

(b) *At $\xi = 0$, for fixed $x \in \mathbb{R}^d$, the function $\tilde{t}_0$ has an expansion*

$$0 \leq \tilde{t}_0(x,\xi) - \langle \xi \,,\, B(x)\xi \rangle = O\left(|\xi|^4\right) \qquad \text{as } |\xi| \to 0\,, \tag{2.32}$$

where $B : \mathbb{R}^d \to \mathcal{M}(d\times d, \mathbb{R})$ is positive definite, symmetric and bounded.

Proof. (a): By Hypothesis 1.2(a)(ii) there exists $c_x > 0$ such that for all $\xi_0, v \in \mathbb{R}^d$

$$\left\langle v \,,\, D^2 L_x|_{\xi_0} v \right\rangle = \int_{\mathbb{R}^d\setminus\{0\}} (\gamma\cdot v)^2 \cosh(\gamma\cdot\xi_0) K^{(0)}(x,d\gamma) \geq \int_{\mathbb{R}^d\setminus\{0\}} (\gamma\cdot v)^2 K^{(0)}(x,d\gamma) \geq c_x \|v\|^2 \,.$$

(b): Since by Taylor expansion at $\xi = 0$

$$\cosh(\gamma\cdot\xi) - \left(1 + \tfrac{1}{2}(\gamma\cdot\xi)^2\right) \leq (\gamma\cdot\xi)^4 \,\frac{\sinh(\gamma\cdot\xi)}{(\gamma\cdot\xi)} \,,$$

one gets from (1.17) and Hypotheses 1.1 and 1.2

$$0 \leq \left|\tilde{t}_0(x,\xi) - \langle \xi \,,\, B(x)\xi\rangle\right| \leq \int_{\mathbb{R}^d\setminus\{0\}} (\gamma\cdot\xi)^4 \,\frac{\sinh(\gamma\cdot\xi)}{(\gamma\cdot\xi)} K^{(0)}(x,d\gamma) = O\left(|\xi|^4\right) \,,$$

as $|\xi| \to 0$, where the symmetric $d\times d$-matrix $B = (B_{\mu\nu})$ is given by

$$B_{\nu\mu}(x) = \frac{1}{2}\int_{\mathbb{R}^d} \gamma_\nu\gamma_\mu K^{(0)}(x,d\gamma) \qquad \text{for} \quad \mu,\nu \in \{1,\ldots,d\}\,, x \in \mathbb{R}^d \,.$$

By Hypothesis 1.2(a)(ii), B is strictly positive definite, by Hypothesis 1.2(a)(i), B is bounded.

□

Lemma 2.5 *Assume Hypotheses 1.1, 1.2 and 1.4, then*

(a) *$d(x) = \frac{1}{2}\left\langle x - x_j \,,\, D^2 d|_{x_j}(x-x_j)\right\rangle + o(|x-x_j|^2)$ as $|x-x_j| \to 0$, and $D^2 d|_{x_j}$ is positive definite.*

(b) *$\nabla d(x) = O(|x-x_j|)$ as $|x-x_j| \to 0$.*

Proof. (b): For $|x - x_j|$ sufficiently small, the eikonal equation (1.18) holds. Thus by Lemma 2.4 (b), we have, with $B(x)$ positive definite and bounded,

$$V_0(x) = \tilde{t}(x, \nabla d(x)) = \langle \nabla d(x) \,,\, B(x)\nabla d(x)\rangle + O(|\nabla d(x)|^4) \tag{2.33}$$

$$\geq C|\nabla d(x)|^2 \,. \tag{2.34}$$

(2.34) proves (b), since $V_0(x) = O(|x-x_j|^2)$ (Hypothesis 1.2(b)).

(a): Since $d \in \mathscr{C}^2(\Omega)$, $d(x_j) = 0$ and $\nabla d(x_j) = 0$ (use (b)), Taylor expansion gives

$$d(x) = \frac{1}{2}\left\langle x - x_j \,,\, D^2 d|_{x_j}(x-x_j)\right\rangle + o(|x-x_j|^2) \qquad \text{as} \quad |x-x_j| \to 0 \,.$$

Since $d(x) \geq 0$, the matrix $D^2 d|_{x_j}$ is non-negative. We shall now assume that 0 is an eigenvalue of $D^2 d|_{x_j}$ with eigenspace $\mathcal{N} \subset \mathbb{R}^d$ and derive a contradiction.

By the mean value theorem and the continuity of $D^2 d|_x$

$$\nabla d(x) = \int_0^1 D^2 d|_{x_j + t(x-x_j)}(x-x_j)\,dt = D^2 d|_{x_j}(x-x_j) + o(|x-x_j|) \qquad (|x-x_j| \to 0).$$

Thus

$$\nabla d(x) = o(|x-x_j|) \qquad (|x-x_j| \to 0, (x-x_j) \in \mathcal{N}) \,.$$

By (2.33) this gives $V_0(x) = o(|x-x_j|^2)$ as $x - x_j \to 0$ in $\mathcal{N}$, which contradicts $D^2 V(x_j) > 0$ (Hypothesis 1.2(b)). Thus $D^2 d|_{x_j}$ is positive definite. □

Lemma 2.6 *Assume Hypotheses 1.1, 1.2 and 1.4 and let $\chi \in \mathscr{C}^\infty(\mathbb{R}_+, [0,1])$ such that $\chi(r) = 0$ for $r \leq \frac{1}{2}$ and $\chi(r) = 1$ for $r \geq 1$. In addition we assume that $0 \leq \chi'(r) \leq \frac{2}{\log 2}$. For $B > 0$ we define $g : \overline{\Sigma} \to [0,1]$ by*

$$g(x) := \chi\left(\frac{d(x)}{B\varepsilon}\right) , \qquad x \in \overline{\Sigma} \tag{2.35}$$

and set

$$\Phi(x) := d(x) - \frac{B\varepsilon}{2} \ln\left(\frac{B}{2}\right) - g(x)\frac{B\varepsilon}{2} \ln\left(\frac{2d(x)}{B\varepsilon}\right) , \qquad x \in \overline{\Sigma} . \tag{2.36}$$

Then $\Phi \in \mathscr{C}^2(\overline{\Omega})$ and there exists a constant $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$

$$|\partial_\nu \partial_\mu \Phi(x)| \leq C , \qquad x \in \Sigma , \mu, \nu \in \{1, \dots d\} . \tag{2.37}$$

Furthermore, for any $B > 0$ there is $C' > 0$ such that

$$e^{\frac{d(x)}{\varepsilon}} \frac{1}{C'} \left(1 + \frac{d(x)}{\varepsilon}\right)^{-\frac{B}{2}} \leq e^{\frac{\Phi(x)}{\varepsilon}} \leq e^{\frac{d(x)}{\varepsilon}} C' \left(1 + \frac{d(x)}{\varepsilon}\right)^{-\frac{B}{2}} . \tag{2.38}$$

Proof. Using the estimates of Lemma 2.5, the proof follows word by word the proof of Lemma 3.3 in [10]. □

Lemma 2.7 *Let $j \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ be non-negative with $\int_{\mathbb{R}^d} j(x)\, dx = 1$ and $\operatorname{supp} j \subset B_1(0) := \{x \in \mathbb{R}^d \,|\, |x| < 1\}$. For $\delta > 0$ we introduce the Friedrichs mollifier $j_\delta(x) := \delta^{-d} j(\frac{x}{\delta})$. Under the assumptions of Theorem 1.6, setting $d_\delta := d * j_\delta$, we have, locally uniformly in $x \in \mathbb{R}^d$,*

$$V_0(x) \geq \tilde{t}_0(x, \nabla d_\delta(x)) + o(1) \qquad (\delta \to 0) . \tag{2.39}$$

We emphasize that ∇d_δ does *not* converge to ∇d in $||.||_\infty$. The estimate (2.39) compensates. This is crucial to obtain the positivity needed in our Agmon estimate.

Proof. First observe that by (1.22), (1.5) and (1.6)

$$\tilde{t}_0(x - y, \xi) - \tilde{t}_0(x, \xi) = \int_{R^d \setminus \{0\}} \big(\cosh \gamma \cdot \xi - 1\big)\big(K^{(0)}(x - y, d\gamma) - K^{(0)}(x, d\gamma)\big) = o(1) \tag{2.40}$$

as $|y| \to 0$ locally uniformly in $(x, \xi) \in \mathbb{R}^{2d}$ (since $|\cosh \gamma \cdot \xi - 1| \leq C|\gamma|^2 e^{C|\gamma|}$).

We remark that

$$\nabla d_\delta(x) = \int_{\mathbb{R}^d} \nabla d(x - y) j_\delta(y)\, dy = \mathbb{E}_\delta\big[\nabla d(x - \,.\,)\big] , \tag{2.41}$$

where $\mathbb{E}_\delta$ denotes expectation with respect to the probability measure $d\mu_\delta(y) = j_\delta(y)\, dy$ (supported in the ball $B_\delta(0)$). Recall the multidimensional Jensen inequality (see e.g. [5])

$$\mathbb{E}\big[f(X)\big] \geq f\big(\mathbb{E}[X]\big) \tag{2.42}$$

for any convex function $f : \mathbb{R}^d \to \mathbb{R}$ and random variable X with values in $\mathbb{R}^d$. Choosing $X(\,.\,) = \nabla d(x - \,.\,)$ and using the convexity of $\tilde{t}_0(x, \,.\,)$ (see Lemma 2.4), we get by (2.41) and (2.42)

$$\begin{aligned} \tilde{t}_0(x, \nabla d_\delta(x)) &\leq \int_{\mathbb{R}^d} \tilde{t}_0(x, \nabla d(x - y)) d\mu_\delta(y) \\ &= \int_{\mathbb{R}^d} \tilde{t}_0(x - y, \nabla d(x - y)) d\mu_\delta(y) + o(1) \qquad (\delta \to 0) , \end{aligned} \tag{2.43}$$

where the last equality follows from (2.40) and $\operatorname{supp} j_\delta \subset B_\delta(0)$. Thus, by (2.43) and the eikonal inequality (1.19)

$$\tilde{t}_0(x, \nabla d_\delta(x)) \leq \int_{\mathbb{R}^d} V_0(x - y) d\mu_\delta(y) + o(1) \leq V_0(x) + o(1) \qquad (\delta \to 0)$$

□

3. Proof of Theorem 1.5 and 1.6

Proof of Theorem 1.6. We partly follow the ideas in the proof of Theorem 1.7 in [10].

Proof of (b):

For $\Sigma'(x)$ given in (1.13), let

$$\tilde{t}_0^{\Sigma}(x,\xi) := \int_{\Sigma'(x)} (\cosh(\gamma\cdot\xi) - 1)\, K^{(0)}(x, d\gamma), \qquad (x,\xi)\in\Sigma\times\mathbb{R}^d\,,$$

then by the positivity of the integrand

$$\tilde{t}_0^{\Sigma}(x,\xi) \le \tilde{t}_0(x,\xi)\,. \tag{3.1}$$

For any $B>0$ we choose $\varepsilon_B>0$ such that $d^{-1}([0,\varepsilon_B B))\subset U$, then by Hypothesis 1.4 for all $\varepsilon<\varepsilon_B$

$$V_0(x) - \tilde{t}_0(x,\nabla d(x)) = 0\,, \qquad x\in\Sigma\cap d^{-1}([0,B\varepsilon))\,. \tag{3.2}$$

Let Φ be given in (2.36), then by (2.35)

$$\nabla\Phi(x) = \nabla d(x)(1 - f_1(x) - f_2(x))\,, \tag{3.3}$$

where

$$f_1(x) := \frac{B\varepsilon}{2d(x)}\chi\left(\frac{d(x)}{B\varepsilon}\right) \quad\text{and}\quad f_2(x) := \frac{1}{2}\chi'\left(\frac{d(x)}{B\varepsilon}\right)\log\left(\frac{2d(x)}{B\varepsilon}\right).$$

Choose $\eta>0$ such that $\tilde{K} := d^{-1}([0,D+2\eta])\subset\Omega$ and let $\widehat{\chi},\tilde{\chi}\in\mathscr{C}^\infty(\mathbb{R}_+,[0,1])$ be monotone with

$$\tilde{\chi}(x) = \begin{cases} 0\,, & x\le D+\eta \\ 1\,, & x\ge D+2\eta \end{cases} \qquad \widehat{\chi}(x) = \begin{cases} 0\,, & x\le D \\ 1\,, & x\ge D+\eta \end{cases}\,.$$

Then we define

$$\tilde{g}(x) := \tilde{\chi}(d(x)) \qquad\text{and}\qquad \widehat{g}(x) := \widehat{\chi}(d(x)) \tag{3.4}$$

and we set for $\delta>0$

$$\Phi_{\alpha,\delta}(x) = (1-\widehat{g}(x))\Phi(x) + \widehat{g}(x)\big(1-\tfrac{\alpha}{2}\big)\big((1-\tilde{g}(x))d(x) + \tilde{g}(x)d_\delta(x)\big)\,,$$

where $d_\delta = d * j_\delta$ is defined in Lemma 2.7. Then $\Phi_{\alpha,\delta}\in\mathscr{C}^2(\overline{\Sigma})$ for any $\delta>0$.

Step 1: We show that there is $\delta(\alpha)$ such that for any $\delta<\delta(\alpha)$ the function $\Phi_\alpha := \Phi_{\alpha,\delta}$ satisfies (1.25) and (1.26).

Clearly, $\Phi_{\alpha,\delta}$ satisfies (1.25) for all $\delta>0$ in view of (2.38), since $\Phi_{\alpha,\delta}(x)=\Phi(x)$ for $x\in K$. Now, by (1.18), for $x\in\Sigma\setminus K$

$$\Phi_{\alpha,\delta}(x) = d(x) - \widehat{g}(x)\tfrac{\alpha}{2}d(x) - (1-\widehat{g})(x)\Big(\frac{B\varepsilon}{2}\ln\Big(\frac{d(x)}{\varepsilon}\Big)\Big) + \tilde{g}(x)\big(1-\tfrac{\alpha}{2}\big)\big(d_\delta - d\big)(x) \tag{3.5}$$

Choosing $B\ge 2$, all logarithms in (3.5) are positive (using $\frac{2d(x)}{B\varepsilon}\ge 1$ on the support of g). Since $\|d_\delta - d\|_\infty\to 0$ as $\delta\to 0$ and using that for some C, by Hypothesis 1.4,

$$\inf\{d(x)\,|\,x\in\Sigma\setminus K\}\ge C>0\,, \tag{3.6}$$

it follows that there is a $\delta(\alpha)$ such that for all $\delta<\delta(\alpha)$

$$\big(1-\tfrac{\alpha}{2}\big)\big|d_\delta(x) - d(x)\big| \le \tfrac{\alpha}{2}d(x)\,, \quad x\in\Sigma\setminus K\,, \tag{3.7}$$

proving the upper bound in (1.26) for Φ_α.

Now observe that there is an $\varepsilon_\alpha>0$ such that for all $\varepsilon\in(0,\varepsilon_\alpha)$

$$\frac{B\varepsilon}{2}\ln\Big(\frac{d(x)}{\varepsilon}\Big)\le\frac{\alpha}{4}d(x)\,, \qquad x\in\Sigma\setminus K\,. \tag{3.8}$$

This follows from the fact that lhs(3.8)$=o(1)$ as $\varepsilon\to 0$ uniformly in x together with (3.6). Inserting (3.8) and (3.7) into (3.5) proves the lower bound of (1.26).

Step 2: We shall show that there are constants $\alpha_0, C_0, C_1 > 0$ independent of B and E and $\varepsilon_\alpha, \delta(\alpha) > 0$ such that for all $\delta < \delta(\alpha)$, $\varepsilon < \varepsilon_\alpha$ and for any fixed $\alpha \in (0, \alpha_0]$

$$V_0(x) - \tilde{t}_0^\Sigma(x, \nabla\Phi_{\alpha,\delta}(x)) \geq \begin{cases} 0\,, & x \in \Sigma \cap d^{-1}([0, B\varepsilon)) \\ \frac{B}{C_0}\varepsilon\,, & x \in \Sigma \cap d^{-1}([B\varepsilon, D+\eta)) \\ C_1\,, & x \in \Sigma \cap d^{-1}([D+\eta, \infty)) \end{cases} \tag{3.9}$$

Case 1: $d(x) \leq \frac{B\varepsilon}{2}$
Since $\Phi_{\alpha,\delta}(x) = d(x) - \frac{B\varepsilon}{2}\ln\big(\frac{B}{2}\big)$ and the eikonal equation (1.18) holds, we get

$$V_0(x) - \tilde{t}_0(x, \nabla\Phi_{\alpha,\delta}(x)) = V_0(x) - \tilde{t}_0(x, \nabla d(x)) = 0\,, \qquad x \in \Sigma \cap d^{-1}([0, \tfrac{B\varepsilon}{2}])\,.$$

which by (3.1) leads to (3.9).

Case 2: $\frac{B\varepsilon}{2} < d(x) < B\varepsilon$
Here $\Phi_{\alpha,\delta}(x) = \Phi(x)$. Since $1 < \frac{2d(x)}{B\varepsilon} < 2$, f_1 and f_2 in (3.3) are non-negative. In addition $0 \leq f_j(x) \leq 1, j = 1, 2$ (use assumption $\chi'(r) \leq \frac{2}{\log 2}$). Therefore

$$|1 - f_1(x) - f_2(x)| =: |\lambda(x)| \leq 1. \tag{3.10}$$

By Lemma 2.4, $\tilde{t}_0(x,\xi)$ is convex with respect to ξ, therefore

$$\tilde{t}_0(x, \lambda\xi + (1-\lambda)\eta) \leq \lambda\tilde{t}_0(x,\xi) + (1-\lambda)\tilde{t}_0(x,\eta) \quad \text{for} \quad 0 \leq \lambda \leq 1,\ \xi, \eta \in \mathbb{R}^d\,. \tag{3.11}$$

and, since $\tilde{t}_0(x, 0) = 0$ and $\tilde{t}_0(x,\xi) = \tilde{t}_0(x,-\xi)$, it follows by choosing $\eta = 0$ that

$$\tilde{t}_0(x, \lambda\xi) \leq |\lambda|\tilde{t}_0(x,\xi)\,, \qquad \text{for} \quad \lambda \in \mathbb{R}, |\lambda| \leq 1, \xi \in \mathbb{R}^d, x \in \Sigma\,. \tag{3.12}$$

Combining (3.10), (3.12) and (3.1) it follows that

$$V_0(x) - \tilde{t}_0^\Sigma(x, \nabla\Phi_{\alpha,\delta}(x)) \geq V_0(x) - |\lambda(x)|\tilde{t}_0(x, \nabla d(x)) \geq V_0(1 - |\lambda(x)|)\,, \tag{3.13}$$

where for the second step we used (3.2). Since $|\lambda(x)| \leq 1$ and $V_0 \geq 0$, (3.13) gives (3.9).

Case 3: $B\varepsilon \leq d(x) < D$
In this region, we have $\Phi_{\alpha,\delta}(x) = \Phi(x) = d(x) - \frac{B\varepsilon}{2}\ln\big(\frac{d(x)}{\varepsilon}\big)$, thus

$$\nabla\Phi_{\alpha,\delta}(x) = \nabla d(x)\left(1 - \frac{B\varepsilon}{2d(x)}\right)\,. \tag{3.14}$$

Since $\frac{1}{2} \leq (1 - \frac{B\varepsilon}{2d(x)}) < 1$, by (3.14) and (3.12) we get the estimate

$$\begin{aligned} V_0(x) - \tilde{t}_0(x, \nabla\Phi_{\alpha,\delta}(x)) &\geq V_0(x) - \left(1 - \frac{B\varepsilon}{2d(x)}\right)\tilde{t}_0(x, \nabla d(x)) \\ &\geq V_0(x)\frac{B\varepsilon}{2d(x)}\,, \end{aligned} \tag{3.15}$$

where for the second estimate we used that by Hypothesis 1.4 the eikonal inequality $\tilde{t}_0(x, \nabla d(x)) \leq V_0(x)$ holds. We now claim that there exists a constant $C_0 > 0$ such that

$$\frac{V_0(x)}{2d(x)} \geq C_0^{-1}\,, \qquad x \in \Sigma \cap d^{-1}([B\varepsilon, \infty))\,. \tag{3.16}$$

Then, combining (3.1), (3.15) and (3.16), we finally get (3.9).

To see (3.16), we split the region $W = \Sigma \cap d^{-1}([B\varepsilon, \infty))$ into two parts. Clearly, for any $\delta > 0$, (3.16) holds for $x \in W \cap \{|x - x_j| > \delta\}$ (since Σ is bounded, $d \in \mathscr{C}^2(\overline{\Sigma})$ and $V_0(x) \geq C > 0$ for $|x - x_j| > \delta$ by Hypothesis 1.2,(b)).

To discuss the region $W \cap \{|x - x_j| \leq \delta\}$, we remark that for some $C > 0$ by Hypothesis 1.2,(b) $V_0(x) \geq C|x - x_j|^2$ if $|x - x_j| \leq \delta$. Thus it suffices to show that for some $\tilde{C} > 0$

$$d(x) \leq C|x - x_j|^2\,, \qquad |x - x_j| \leq \delta\,.$$

This follows from Lemma 2.5(a).

Case 4: $D \le d(x) < D + \eta$
Since $\Phi_{\alpha,\delta}(x) = (1-\widehat{g}(x))\Phi(x) + \widehat{g}(x)\big(1-\frac{\alpha}{2}\big)d(x)$ and $\nabla\Phi(x)$ is given by (3.14) in this region, we have

$$\begin{aligned}\nabla\Phi_{\alpha,\delta}(x) &= \nabla d(x)\Big[\Big(1-\frac{B\varepsilon}{2d(x)}\Big)(1-\widehat{g}(x)) - \widehat{\chi}'(d(x))\Big(\Big(d(x)-\frac{B\varepsilon}{2}\ln\Big(\frac{d(x)}{\varepsilon}\Big)\Big)+ \\ &\qquad + \widehat{\chi}'(d(x))\big(1-\tfrac{\alpha}{2}\big)d(x) + \widehat{g}(x)\big(1-\tfrac{\alpha}{2}\big)\Big] \\ &= \lambda\nabla d(x)\,, \end{aligned} \tag{3.17}$$

where

$$\lambda = 1 + h_\alpha(x) - (1-\widehat{g}(x))\frac{B\varepsilon}{2d(x)} - \widehat{g}(x)\frac{\alpha}{2}, \qquad h_\alpha(x) := \widehat{\chi}'(d(x))\Big(-\frac{\alpha}{2}d(x) + \frac{B\varepsilon}{2}\ln\Big(\frac{d(x)}{\varepsilon}\Big)\Big).$$

Since $\widehat{\chi}'(y) \ge 0$ it follows from the upper bound in (3.8) that $h_\alpha \le 0$, proving for α sufficiently small

$$0 \le \lambda \le 1 - t, \qquad t = t(x,\alpha,\varepsilon) = (1-\widehat{g}(x))\frac{B\varepsilon}{2d(x)} + \widehat{g}(x)\frac{\alpha}{2}\,. \tag{3.18}$$

Combining (3.1), (3.12), (3.17) and(3.18) gives, for all $\varepsilon \le \varepsilon_\alpha$ sufficiently small

$$V_0(x) - \tilde{t}_0^{\Sigma}(x,\nabla\Phi_{\alpha,\delta}) \ge V_0(x)t(x,\alpha,\varepsilon) \ge \frac{B}{C_0}\varepsilon\,,$$

where we used (1.19) and, for the last estimate, (3.16).

Case 5: $D + \eta \le d(x) < D + 2\eta$
We have $\Phi_{\alpha,\delta}(x) = \big(1-\frac{\alpha}{2}\big)((1-\tilde{g}(x))d(x) + \tilde{g}(x)d_\delta(x)$ and thus

$$\begin{aligned}\nabla\Phi_{\alpha,\delta}(x) &= \big(1-\tfrac{\alpha}{2}\big)\big[(1-\tilde{g}(x))\nabla d(x) + \tilde{\chi}'(d(x))(d_\delta(x)-d(x))\nabla d(x) + \tilde{g}(x)\nabla d_\delta(x)\big] \\ &= \big(1-\tfrac{\alpha}{2}\big)\big[(1-\tilde{g}(x))\nabla d(x) + \tilde{g}(x)\nabla d_\delta(x)\big] + \tfrac{\alpha}{2}\tfrac{2}{\alpha}f_\delta(x)\nabla d(x)\,,\end{aligned} \tag{3.19}$$

where we set

$$f_\delta(x) := \tilde{\chi}'(d(x))(d_\delta(x)-d(x)) \quad \text{and thus} \quad |f_\delta(x)| = o(1) \quad (\delta\to 0)\,.$$

Using (3.11) twice, we get by (3.19)

$$\tilde{t}_0(x,\nabla\Phi_{\alpha,\delta}(x)) \le \big(1-\tfrac{\alpha}{2}\big)\Big[(1-\tilde{g}(x))\tilde{t}_0(x,\nabla d(x)) + \tilde{g}(x)\tilde{t}_0(x,\nabla d_\delta(x))\Big] + \tfrac{\alpha}{2}\tilde{t}_0\big(x,\tfrac{2}{\alpha}f_\delta(x)\nabla d(x)\big)\,.$$

Combining Lemma 2.7 with (1.19) yields, as $\delta\to 0$,

$$V_0(x) - \tilde{t}_0\big(x,\nabla\Phi_{\alpha,\delta}(x)\big) \ge V_0(x)\Big[\frac{\alpha}{2} + o(1)\Big] \ge C_1\,, \tag{3.20}$$

since $V_0(x) \ge C > 0$ in this region. Combining (3.1) with (3.20) gives (3.9).

Case 6: $d(x) \ge D + 2\eta$
Since $\Phi_{\alpha,\delta}(x) = \big(1-\frac{\alpha}{2}\big)d_\delta(x)$, we have by (3.12)

$$\tilde{t}_0\big(x,\nabla\Phi_{\alpha,\delta}(x)\big) \le \Big(1-\frac{\alpha}{2}\Big)\tilde{t}_0\big(x,\nabla d_\delta(x)\big) \tag{3.21}$$

Combining Lemma 2.7 with (3.21) gives (3.20), as in *Case 5*.

Step 3: We shall show

$$V_\varepsilon(x) + V^{\Phi_\alpha}(x) \ge \begin{cases} -C_2\,\varepsilon & \text{for} \quad x\in\Sigma\cap d^{-1}([0,B\varepsilon]) \\ \Big(\frac{B}{C_0} - C_3\Big)\varepsilon & \text{for} \quad x\in\Sigma\cap d^{-1}([B\varepsilon, D+\eta)) \\ C_4 & \text{for} \quad x\in\Sigma\cap d^{-1}([D+\eta,\infty)) \end{cases} \tag{3.22}$$

for some $C_2, C_3, C_4 > 0$ independent of B and E, where $V^{\Phi_\alpha} := V^{\Phi_\alpha}_{\varepsilon,\Sigma}$ is defined in (2.2) and $\Phi_\alpha = \Phi_{\alpha,\delta}$ for any $\delta < \delta(\alpha)$.

We write

$$V_\varepsilon(x) + V^{\Phi_\alpha}(x) = (V_\varepsilon(x) - V_0(x)) + \big(V^{\Phi_\alpha}(x) + \tilde{t}_0^{\Sigma}(x,\nabla\Phi_\alpha(x))\big) + \big(V_0(x) - \tilde{t}_0^{\Sigma}(x,\nabla\Phi_\alpha(x))\big) \tag{3.23}$$

By Hypothesis 1.1 and since Σ is bounded, there exists a constant $C_1 > 0$ such that

$$V_\varepsilon(x) - V_0(x) \ge -C_1\varepsilon\,, \qquad x\in\Sigma\,. \tag{3.24}$$

We shall show that
$$\left|V^{\Phi_\alpha}(x) + \tilde{t}_0^{\Sigma}(x, \nabla\Phi_\alpha(x))\right| \leq \varepsilon C_2 \,. \tag{3.25}$$
Then inserting (3.25), (3.24) and (3.9) into (3.23) proves (3.22). Setting (see (2.2))
$$V_0^{\Phi_\alpha}(x) := \int_{\Sigma'(x)} \left[1 - \cosh\left(F_\alpha(x)\right)\right] K^{(0)}(x, d\gamma), \quad F_\alpha(x) = F_\alpha(x,\gamma,\varepsilon) = \tfrac{1}{\varepsilon}(\Phi_\alpha(x) - \Phi_\alpha(x+\varepsilon\gamma))$$
we write
$$V^{\Phi_\alpha}(x) + \tilde{t}_0^{\Sigma}(x, \nabla\Phi_\alpha(x)) = \Big(V^{\Phi_\alpha}(x) - V_0^{\Phi_\alpha}(x)\Big) + \Big(V_0^{\Phi_\alpha}(x) + \tilde{t}_0^{\Sigma}(x, \nabla\Phi_\alpha(x))\Big) =: D_1(x) + D_2(x)$$
and analyze the two summands on the right hand side separately. Since $\Phi_\alpha \in \mathscr{C}^2(\overline{\Sigma})$, it follows from Hypotheses 1.1 and 1.2(a), using (2.5), that for some $\tilde{C} > 0$
$$|D_1(x)| = \Big| \int_{\Sigma'(x)} \left[1 - \cosh\left(F_\alpha(x)\right)\right] \Big(\varepsilon K^{(1)} + R_\varepsilon^{(2)}\Big)(x, d\gamma) \Big| \leq \tilde{C}\varepsilon \,. \tag{3.26}$$
uniformly with respect to x. We have for $x \in \Sigma$
$$|D_2(x)| \leq \int_{\Sigma'(x)} \Big|\cosh\big(\gamma\nabla\Phi_\alpha(x)\big) - \cosh\big(F_\alpha(x)\big)\Big| K^{(0)}(x, d\gamma) \,. \tag{3.27}$$
By the mean value theorem for $\cosh z$ and since $|\sinh x| \leq e^{|x|}$
$$\Big|\cosh\big(F_\alpha(x)\big) - \cosh\big(\gamma\nabla\Phi_\alpha(x)\big)\Big| \leq \sup_{t\in[0,1]} \exp\big(\big|F_\alpha(x)t + \gamma\nabla\Phi_\alpha(x)(1-t)\big|\big) \big|F_\alpha(x) + \gamma\nabla\Phi_\alpha(x)\big| \,. \tag{3.28}$$
Since $\Phi_\alpha \in \mathscr{C}^2(\overline{\Sigma})$ there exist constants $c_1, c_2 > 0$ such that
$$|F_\alpha(x)| \leq c_1|\gamma| \quad \text{and} \quad |\gamma\nabla\Phi_\alpha(x)| \leq c_2|\gamma| \,, \qquad x \in \Sigma,\ \gamma \in \Sigma'(x) \,. \tag{3.29}$$
(3.29) gives a constant $D > 0$ such that
$$\exp\big(\big|F_\alpha(x)t + \gamma\nabla\Phi_\alpha(x)(1-t)\big|\big) \leq e^{D|\gamma|} \,. \tag{3.30}$$
By second order Taylor-expansion, using (2.37)
$$\big|(F_\alpha(x) + \gamma\nabla\Phi_\alpha(x)\big| \leq \tilde{C}_3\varepsilon|\gamma|^2 \,. \tag{3.31}$$
for all $\varepsilon \in (0, \varepsilon_0]$ and some $\tilde{C}_3 > 0$ independent of the choice of B. By (1.5), inserting (3.30) and (3.31) into (3.28) and this in (3.27), using Hypothesis 1.2(a), we get
$$|D_2(x)| = \Big|V_0^{\Phi_\alpha}(x) - t_0^{\Sigma}(x, -i\nabla\Phi_\alpha)\Big| \leq \varepsilon C' \,.$$
This and (3.26) give (3.25).

Step 4: We prove (1.24) and (1.21) by use of Lemma 2.3.

Choosing $B \geq C_0(1 + R_0 + C_3)$ (depending only on R_0, but independent of u and E), we have
$$\left(\frac{B}{C_0} - C_3\right)\varepsilon - E \geq \varepsilon \,, \qquad E \in [0, \varepsilon R_0] \,. \tag{3.32}$$
Let
$$\Omega_- := \{x \in \Sigma \,|\, V_\varepsilon(x) + V^{\Phi_\alpha}(x) - E < 0\} \qquad \text{and} \qquad \Omega_+ := \Sigma \setminus \Omega_- \,, \tag{3.33}$$
then from (3.32) combined with (3.22) it follows that $\Omega_- \subset \{d(x) < \varepsilon B\}$ and by (3.22)
$$|V_\varepsilon(x) + V^{\Phi_\alpha}(x)| \leq \varepsilon \max\{C_3, R_0\} \qquad \text{for all} \quad x \in \Omega_- \,. \tag{3.34}$$
We define the functions $F_\pm : \Sigma \to [0, \infty)$ by
$$F_+(x) := \sqrt{\varepsilon \mathbf{1}_{\{d(x) < B\varepsilon\}}(x) + (V_\varepsilon(x) + V^{\Phi_\alpha}(x) - E)\mathbf{1}_{\Omega_+}(x)} \tag{3.35}$$
and
$$F_-(x) := \sqrt{\varepsilon \mathbf{1}_{\{d(x) < B\varepsilon\}}(x) + (E - V_\varepsilon(x) - V^{\Phi_\alpha}(x))\mathbf{1}_{\Omega_-}(x)} \,. \tag{3.36}$$
Then $F_\pm$ are well defined and furthermore there exists a constant $C, \tilde{C} > 0$ depending only of R_0 and B such that
$$F := F_+ + F_- \geq C\sqrt{\varepsilon} > 0 \,, \qquad |F_-| \leq \tilde{C}\sqrt{\varepsilon} \qquad \text{and} \qquad F_+^2 - F_-^2 = V_\varepsilon + V^{\Phi_\alpha} - E \,. \tag{3.37}$$

The first inequality uses (3.22) combined with (3.32). By (2.38) and (3.37)

$$\left\| F e^{\frac{\Phi_\alpha}{\varepsilon}} u \right\|^2_{L^2(\Sigma)} \geq C\varepsilon \left\| e^{\frac{\Phi_\alpha}{\varepsilon}} u \right\|^2_{L^2(\Sigma)} \tag{3.38}$$

and

$$\left\| \tfrac{1}{F} e^{\frac{\Phi_\alpha}{\varepsilon}} \left(H^\Sigma_\varepsilon - E\right) u \right\|^2_{L^2(\Sigma)} \leq C\varepsilon^{-1} \left\| e^{\frac{\Phi_\alpha}{\varepsilon}} \left(H^\Sigma_\varepsilon - E\right) u \right\|^2_{L^2(\Sigma)} . \tag{3.39}$$

Since $\operatorname{supp} F_- \subset \{d(x) < B\varepsilon\}$, by (2.38) and (3.37) there exists a constant $C > 0$ such that

$$\left\| F_- e^{\frac{\Phi}{\varepsilon}} u \right\|^2_{L^2(\Sigma)} \leq C\varepsilon \, \|u\|^2_{L^2(\Sigma)} . \tag{3.40}$$

Inserting (3.38), (3.39) and (3.40) in (2.27) yields (1.24) uniformly with respect to $E \in (0, \varepsilon R_0)$ and u.

Proof of (c):

(1.27) follows at once from (1.24) together with (1.25) and (1.26).
If u is an eigenfunction of H^Σ_ε with eigenvalue E, then the first summand on rhs(1.24) vanishes. The normalization of u therefore leads to (1.28).

Proof of (a):

For $\widehat{g}$ defined in (3.4) and $d_\delta = d * j_\delta$ defined in Lemma 2.7, we set

$$\tilde{\Phi}_{\alpha,\delta}(x) = \Big(1 - \frac{\alpha}{2}\Big)\Big(\big(1 - \widehat{g}\big)\, d(x) + \widehat{g}\, d_\delta(x)\Big) .$$

Then for all $x \in \Sigma$ and $\delta < \delta_\alpha$

$$(1-\alpha)\, d(x) \leq \tilde{\Phi}_{\alpha,\delta}(x) \leq d(x) . \tag{3.41}$$

In fact, if $x \in K$ (3.41) is trivial, so let $x \in \Sigma \setminus K$. Then, writing $\tilde{\Phi}_{\alpha,\delta} = (1 - \frac{\alpha}{2})(d + \widehat{g}(d_\delta - d))$, (3.41) follows directly from (3.7).

For η as in the definition of $\widehat{g}$, we now claim that for any fixed $\alpha \in (0,1]$ and for all $\delta < \delta_\alpha$ and $\varepsilon < \varepsilon_0$

$$V_0(x) - \tilde{t}^\Sigma_0(x, \nabla \tilde{\Phi}_{\alpha,\delta}(x)) \geq \begin{cases} 0\,, & x \in \Sigma \cap d^{-1}([0, D+\eta) \\ C\,, & x \in \Sigma \cap d^{-1}([D+\eta, \infty)) \,. \end{cases} \tag{3.42}$$

If $x \geq D$, this follows as in the proof of (b) (Case 5 and 6 of Step 2).

If $x < D$, we have $\nabla \tilde{\Phi}_{\alpha,\delta}(x) = (1 - \frac{\alpha}{2}) \nabla d(x)$ and thus by the convexity of $\tilde{t}_0$ and the eikonal inequality (1.19), analog to Step 2, Case 3 in the proof of (b),

$$V_0(x) - \tilde{t}^\Sigma_0(x, \nabla \tilde{\Phi}_{\alpha,\delta}(x)) \geq \frac{\alpha}{2} V_0(x) \geq 0 \,.$$

Similar to Step 3 in the proof of (b), it follows that for some $C_1, C_2 > 0$

$$V_\varepsilon(x) + V^{\tilde{\Phi}_\alpha}(x) \geq \begin{cases} -\varepsilon C_1\,, & x \in \Sigma \cap d^{-1}([0, D+\eta) \\ C_2\,, & x \in \Sigma \cap d^{-1}([D+\eta, \infty)) \,, \end{cases} \tag{3.43}$$

where we set $\tilde{\Phi}_\alpha := \tilde{\Phi}_{\alpha,\delta}$ for any $\delta < \delta_\alpha$. If F_+, F_- are defined as in (3.35) and (3.36) with Φ_α replaced by $\tilde{\Phi}_\alpha$, arguments similar to those in (3.37) and below lead to

$$\left\| e^{\frac{\tilde{\Phi}_\alpha}{\varepsilon}} u \right\|_{L^2(\Sigma)} \leq C \Big[\varepsilon^{-1} \left\| e^{\frac{\tilde{\Phi}_\alpha}{\varepsilon}} \left(H^\Sigma_\varepsilon - E\right) u \right\|_{L^2(\Sigma)} + \|u\|_{L^2(\Sigma)} \Big] ,$$

which combined with (3.41) proves (1.23). □

Proof of Theorem 1.5. This is a consequence of (the proof of) Theorem 1.6,(b). Since $d \in \mathscr{C}^2(\overline{\Sigma})$, we can use Φ defined in (2.36) instead of Φ_α. The arguments in Step 2, Case 1 - 3, show that there are constants $C_0, C_1 > 0$ independent of B, E and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$

$$V_0(x) - \tilde{t}^\Sigma_0(x, \nabla \Phi(x)) \geq \begin{cases} 0\,, & x \in \Sigma \cap d^{-1}([0, B\varepsilon]) \\ \frac{B}{C_0}\varepsilon\,, & x \in \Sigma \cap d^{-1}([B\varepsilon, \infty)) \end{cases}$$

Since $\Phi \in \mathscr{C}^2(\overline{\Sigma})$, the same arguments as in Step 3 of the proof of Thm. 1.6,(b), show

$$V_\varepsilon(x) + V^\Phi(x) \geq \begin{cases} -C_2\,\varepsilon & \text{for} \quad x \in \Sigma \cap d^{-1}([0, B\varepsilon]) \\ \left(\frac{B}{C_0} - C_3\right)\varepsilon & \text{for} \quad x \in \Sigma \cap d^{-1}([B\varepsilon, \infty)) \end{cases}$$

for some $C_2, C_3 > 0$ independent of B and E, where $V^\Phi := V^\Phi_{\varepsilon,\Sigma}$ is defined in (2.2). Defining F_- and F_+ by (3.35) and (3.36) with Φ_α replaced by Φ, we get (1.20) by use of Lemma 2.3 and Lemma 2.6. Note that Lemma 2.7 is not needed and neither is the continuity assumption (1.22).

□

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