

TRANSPORTATION-INFORMATION INEQUALITIES FOR MARKOV PROCESSES

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ABSTRACT. In this paper, one investigates the transportation-information $T_c I$ inequalities: $\alpha(T_c(\nu, \mu)) \leq I(\nu|\mu)$ for all probability measures ν on a metric space (\mathcal{X}, d) , where μ is a given probability measure, $T_c(\nu, \mu)$ is the transportation cost from ν to μ with respect to the cost function $c(x, y)$ on \mathcal{X}^2 , $I(\nu|\mu)$ is the Fisher-Donsker-Varadhan information of ν with respect to μ and $\alpha : [0, \infty) \rightarrow [0, \infty]$ is a left continuous increasing function. Using large deviation techniques, it is shown that $T_c I$ is equivalent to some concentration inequality for the occupation measure of a μ -reversible ergodic Markov process related to $I(\cdot|\mu)$.

The tensorization property of $T_c I$ and comparisons of $T_c I$ with Poincaré and log-Sobolev inequalities are investigated. Several easy-to-check sufficient conditions are provided for special important cases of $T_c I$ and several examples are worked out.

1. INTRODUCTION

Let $M_1(\mathcal{X})$ be the space of all probability measures on a complete separable metric space (\mathcal{X}, d) and consider the *cost function* $c(x, y) : \mathcal{X}^2 \rightarrow [0, +\infty]$ with $c(x, x) = 0$ (for all $x \in \mathcal{X}$), which is lower semicontinuous on \mathcal{X}^2 . Given $\mu, \nu \in M_1(\mathcal{X})$, the *transportation cost* $T_c(\nu, \mu)$ from ν to μ with respect to the cost function c is defined by

$$T_c(\nu, \mu) = \inf_{\pi \in M_1(\mathcal{X}^2) : \pi_0 = \nu, \pi_1 = \mu} \iint_{\mathcal{X}^2} c(x, y) \pi(dx, dy) \quad (1.1)$$

where $\pi_0(dx) = \pi(dx \times \mathcal{X})$, $\pi_1(dy) = \pi(\mathcal{X} \times dy)$ are the marginal distributions of π . When $c(x, y) = d^p(x, y)$ where $p \geq 1$, $(T_c(\nu, \mu))^{1/p} = W_p(\nu, \mu)$ is the L^p - Wasserstein distance between ν and μ .

The relative entropy (or Kullback information) of ν with respect to μ is given by

$$H(\nu|\mu) := \begin{cases} \int_{\mathcal{X}} f \log f d\mu, & \text{if } \nu \ll \mu \text{ and } f := \frac{d\nu}{d\mu} \\ +\infty, & \text{otherwise.} \end{cases} \quad (1.2)$$

The usual transportation inequalities for a given $\mu \in M_1(\mathcal{X})$, introduced by K. Marton [35] and M. Talagrand [40], compare the Wasserstein metric $W_p(\nu, \mu)$ with the relative entropy $H(\nu|\mu)$. The following extension of these inequalities:

$$\alpha(T_c(\nu, \mu)) \leq H(\nu|\mu), \quad \forall \nu \in M_1(\mathcal{X}), \quad (T_c H)$$

has recently been proposed and developed by Gozlan and Léonard [24]. Here $\alpha : [0, \infty) \rightarrow [0, +\infty]$ is some left continuous and increasing function with $\alpha(0) = 0$.

Let us denote

$$\alpha^{\otimes}(\lambda) := \sup_{r \geq 0} (\lambda r - \alpha(r)) \quad (1.3)$$

the monotone conjugate of α . With α as above, one sees that α^\circledast is the restriction to $[0, \infty)$ of the usual convex conjugate $\tilde{\alpha}^*(\lambda) = \sup_{r \in \mathbb{R}} (\lambda r - \tilde{\alpha}(r))$ of $\tilde{\alpha}(r) = \mathbf{1}_{r \geq 0} \alpha(r)$, $r \in \mathbb{R}$. We also denote $\mu(v) := \int_{\mathcal{X}} v d\mu$.

As an extension of the Bobkov-Götze criterion [6], we have

Theorem 1.1 (Gozlan-Léonard [24]). *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of \mathcal{X} -valued i.i.d. random variables with common law μ and α be moreover convex. Then the following properties are equivalent:*

- (a) *The transportation inequality $T_c H$ holds;*
- (b) *For any couple of bounded and measurable functions $u, v : \mathcal{X} \rightarrow \mathbb{R}$ such that $u(x) - v(y) \leq c(x, y)$ over \mathcal{X}^2 , $\log \int_{\mathcal{X}} e^{\lambda u} d\mu \leq \lambda \mu(v) + \alpha^\circledast(\lambda)$, $\forall \lambda \geq 0$;*
- (c) *For all $n \geq 1$ and $r > 0$ and for any couple of bounded and measurable functions $u, v : \mathcal{X} \rightarrow \mathbb{R}$ such that $u(x) - v(y) \leq c(x, y)$ over \mathcal{X}^2 , the following concentration inequality holds $\mathbb{P}(\frac{1}{n} \sum_{k=1}^n u(X_k) \geq \mu(v) + r) \leq e^{-n\alpha(r)}$.*

The main purpose of this paper. In this paper, instead of the transportation-entropy inequality $T_c H$, one investigates the following transportation-information inequality

$$\alpha(T_c(\nu, \mu)) \leq I(\nu|\mu), \quad \forall \nu \in M_1(\mathcal{X}) \quad (T_c I)$$

for some given probability measure μ . Here $I(\nu|\mu)$ is the Fisher-Donsker-Varadhan information of ν with respect to μ

$$I(\nu|\mu) = \begin{cases} \mathcal{E}(\sqrt{f}, \sqrt{f}) & \text{if } \nu = f\mu, \sqrt{f} \in \mathbb{D}(\mathcal{E}) \\ +\infty & \text{otherwise} \end{cases} \quad (1.4)$$

associated with the Dirichlet form \mathcal{E} on $L^2(\mu)$ with domain $\mathbb{D}(\mathcal{E})$.

Notation. In the special case where $c(x, y) = d^p(x, y)$, we use the notation $W_p I$ instead of $T_{d^p} I$. In particular, $W_1 I$ stands for $T_d I$.

Organization of the paper. This paper is organized as follows. In the next section we characterize $T_c I$ by means of concentration inequalities for the empirical means $L_t(u) = \frac{1}{t} \int_0^t u(X_s) ds$ of observables u , extending Theorem 1.1 from i.i.d. sequences to time-continuous Markov processes. The method of proof is borrowed from Gozlan and Léonard [24] who proved Theorem 1.1 by means of large deviations of the empirical measure of an i.i.d. sequence. Here, it relies on the large deviations of the occupation measure of (X_t) . The tensorization of $T_c I$ is proved, and the relations between $W_2 I$, Poincaré and log-Sobolev are exhibited with the help of Otto-Villani [37].

In Section 3, $W_1 I$ is proved for the trivial metric $d(x, y) = \mathbf{1}_{x \neq y}$ with the sharp constant in terms of the spectral gap as well as a sharp Hoeffding concentration inequality for Markov processes.

For a general metric, using Lyons-Meyer-Zheng forward-backward martingale decomposition, we obtain in Section 4 a sharp $W_1 I$ inequality under the spectral gap existence of the Markov diffusion process in the space of Lipschitz functions. An explicit sharp constant is provided for one-dimensional diffusions.

Finally in Section 5 we propose a practical Lyapunov condition for $W_1 I$ (or a more general $T_\Phi I$) which, although not providing the sharp constant, yields a good order.

About the literature. Let us give some historical notes on the usual transportation inequality $W_p H$. K. Marton [33] first noticed that $W_1 H$ implies the concentration inequality for μ by a very elementary and neat argument, and she established $W_1 H$ for the law of a Dobrushin-contractive Markov chain in [35]. M. Talagrand [40] established $W_2 H$ for the Gaussian measure μ with $\alpha(r) = r/2C$ and provided the sharp constant C (this particular case of $T_c H$ is often called Talagrand's transportation inequality). Bobkov and Götze [6] obtained the characterization of $W_p H$ in Theorem 1.1 with [$p = 1$, α quadratic] and [$p = 2$, α linear]. Otto and Villani [37] proved that the log-Sobolev inequality is stronger than Talagrand's transportation inequality and presented a differential geometrical point of view on $M_1(\mathcal{X})$ equipped with the W_2 -metric. Bobkov, Gentil and Ledoux [5] shed light on a profound relation between Talagrand's transportation inequality, log-Sobolev inequality, inf-convolution and some Hamilton-Jacobi equation. Djellout, Guillin and Wu [15] obtained a necessary and sufficient condition for $W_1 H$ with a quadratic α by means of the Gaussian integrability of $d(x, x_0)$ under μ , and gave a direct proof of Talagrand's transportation inequality for the law of a diffusion process by means of Girsanov's formula, without appealing to log-Sobolev inequality. See X. Fernique [19], Feyel and Ustunel [20], Bogachev and Kolesnikov [7] for the approach of Girsanov's transform to optimal mass transportation and $T_c H$.

Bolley and Villani [8] and later Gozlan and Léonard [24] refined the result of [15] under a Gaussian integrability condition. Cattiaux and Guillin [11] constructed the first example for which Talagrand's transportation inequality holds but not log-Sobolev inequality, and Gozlan [23] found a necessary and sufficient condition for Talagrand's transportation inequality with $\mu(dx) = e^{-V(x)} dx$ on \mathbb{R} when the Bakry-Emery curvature V'' is lower bounded. Otto-Villani's differential geometrical point of view on $M_1(\mathcal{X})$ equipped with the W_2 -metric is very fruitful, as developed by the recent works of Sturm [38, 39] and Lott and Villani [32]. The reader is referred to the textbooks by Ledoux [29] and Villani [42, 41] for further references pertaining to this very active field.

The transportation-information inequalities $T_c I$ are new objects.

Convention and notation. Throughout this paper (\mathcal{X}, d) is a complete separable metric space with the associated Borel σ -field \mathcal{B} .

- The space of all real bounded and \mathcal{B} -measurable functions is denoted by $b\mathcal{B}$.
- The functions to be considered later are assumed to be measurable without warning.
- For $\mu, \nu \in M_1(\mathcal{X})$, $\|\nu - \mu\|_{TV} := \sup_{u: |u| \leq 1} \int u d(\nu - \mu)$ is the total variation norm.
- A cost function c is a nonnegative lower semicontinuous function on \mathcal{X}^2 such that $c(x, x) = 0$ for all $x \in \mathcal{X}$.

2. GENERAL RESULTS ON $T_c I$

2.1. Markov processes, Fisher-Donsker-Varadhan information and Feynman-Kac semigroup. The main probabilistic object to be considered in this paper is an \mathcal{X} -valued time-continuous Markov process $(\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathcal{X}})$ with an invariant probability measure μ . The transition semigroup is denoted $(P_t)_{t \geq 0}$.

Assumption: Ergodicity. It is assumed that the invariant probability measure μ is ergodic: if $f \in b\mathcal{B}$ satisfies $P_t f = f$, μ -a.e. for all $t \geq 0$, then f is constant μ -a.e. Denoting $\mathbb{P}_\beta(\cdot) := \int_{\mathcal{X}} \mathbb{P}_x(\cdot) \beta(dx)$ for any initial probability measure β , the previous condition on μ amounts to stating that $((X_t)_{t \geq 0}, \mathbb{P}_\mu)$ is a stationary ergodic process.

Assumption: Closability of the symmetrized Dirichlet form. It is assumed that (P_t) is strongly continuous on $L^2(\mu) := L^2(\mathcal{X}, \mathcal{B}, \mu)$. Let \mathcal{L} be its generator with domain $\mathbb{D}_2(\mathcal{L})$ on $L^2(\mu)$. It is also assumed that

$$\mathcal{E}(g, g) := \langle -\mathcal{L}g, g \rangle_\mu, \quad g \in \mathbb{D}_2(\mathcal{L})$$

is closable in $L^2(\mu)$. Its closure which is denoted by $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$ is a Dirichlet form: the symmetrized Dirichlet form associated with the Markov process (X_t) (or (P_t)). Notice that $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$ corresponds to a self-adjoint generator \mathcal{L}^σ (formally $\mathcal{L}^\sigma = (\mathcal{L} + \mathcal{L}^*)/2$), and $P_t^\sigma = e^{t\mathcal{L}^\sigma}$ is the symmetrized Markov semigroup of (P_t) . When P_t is symmetric on $L^2(\mu)$, the above closability assumption is always satisfied and the domain $\mathbb{D}(\mathcal{E})$ of the Dirichlet form coincides with the domain $\mathbb{D}_2(\sqrt{-\mathcal{L}})$ in $L^2(\mu)$.

These above assumptions of ergodicity and closability of the Dirichlet form prevail for the whole paper.

Fisher-Donsker-Varadhan information. The following definition is motivated by standard large deviation results.

Definition 2.1. *Given the Dirichlet form \mathcal{E} with domain $\mathbb{D}(\mathcal{E})$ on $L^2(\mu)$, the Fisher-Donsker-Varadhan information of ν with respect to μ is defined by*

$$I(\nu|\mu) := \begin{cases} \mathcal{E}(\sqrt{f}, \sqrt{f}), & \text{if } \nu = f\mu, \sqrt{f} \in \mathbb{D}(\mathcal{E}) \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.1)$$

Remarks 2.2. (I as rate function) When (P_t) is μ -symmetric, $\nu \mapsto I(\nu|\mu)$ is exactly the Donsker-Varadhan entropy i.e. the rate function governing the large deviation principle of the empirical measure $L_t := \frac{1}{t} \int_0^t \delta_{X_s} ds$ for large time t . This was proved by Donsker and Varadhan [17] under some conditions of absolute continuity and regularity of $P_t(x, dy)$, and established in full generality by L. Wu [46, Corollary B.11].

Remarks 2.3. (Framework of Riemannian manifold) When $\mu = e^{-V(x)} dx / Z$ (Z is the normalization constant) with $V \in C^1$ on a complete connected Riemannian manifold $\mathcal{X} = M$, the diffusion (X_t) generated by $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ (Δ, ∇ are respectively the Laplacian and the gradient on M) is μ -reversible and the corresponding Dirichlet form is given by

$$\mathcal{E}_\mu(g, g) = \int_M |\nabla g|^2 d\mu, \quad g \in \mathbb{D}(\mathcal{E}_\mu) = H^1(\mathcal{X}, \mu)$$

where $H^1(\mathcal{X}, \mu)$ is the closure of $C_b^\infty(M)$ (the space of infinitely differentiable functions f on M with $|\nabla^n f|$ bounded for all n) with respect to the norm $\sqrt{\mu(|g|^2 + |\nabla g|^2)}$. It also matches with the space of these $g \in L^2(M)$ such that $\nabla g \in L^2(M \rightarrow TM; \mu)$ in distribution. In this case, if $\nu = f\mu$ with $0 < f \in C^1(M)$, then

$$I(\nu|\mu) = \int_{\mathcal{X}} |\nabla \sqrt{f}|^2 d\mu = \frac{1}{4} \int_{\mathcal{X}} \frac{|\nabla f|^2}{f} d\mu = \frac{1}{4} I_F(f|\mu) \quad (2.2)$$

where $I_F(f|\mu)$ is the classical Fisher information of the probability density f .

Feynman-Kac semigroup. The derivation of the large deviation results for L_t as t tends to infinity is intimately related to the Feynman-Kac semigroup

$$P_t^u g(x) := \mathbb{E}^x g(X_t) \exp \left(\int_0^t u(X_s) ds \right). \quad (2.3)$$

When u is bounded, (P_t^u) is a strongly continuous semigroup of bounded operators on $L^2(\mu)$ whose generator is given by $\mathcal{L}^u g = \mathcal{L}g + ug$, for all $g \in \mathbb{D}_2(\mathcal{L}^u) = \mathbb{D}_2(\mathcal{L})$.

It is no surprise that this semigroup also plays a role in the present investigation.

2.2. Characterizations of $T_c I$. Recall that Kantorovich's duality theorem (see [42]) states that for any $\nu, \mu \in M_1(\mathcal{X})$ so that $T_c(\nu, \mu) < +\infty$,

$$T_c(\nu, \mu) = \sup_{(u,v) \in \Phi_c} \int u d\nu - \int v d\mu \quad (2.4)$$

where $\Phi_c := \{(u, v) \in (b\mathcal{B})^2 : u(x) - v(y) \leq c(x, y), \forall (x, y) \in \mathcal{X}^2\}$. This motivates us to introduce as in [24]

$$T_\Phi(\nu, \mu) = \sup_{(u,v) \in \Phi} \int u d\nu - \int v d\mu \quad (2.5)$$

where $\Phi \subset (b\mathcal{B})^2$ (non-empty) satisfies

(A1) $u \leq v$ for all $(u, v) \in \Phi$;

(A2) For all $\nu_1, \nu_2 \in M_1(\mathcal{X})$, there exists $(u, v) \in \Phi$ such that $\int u d\nu_1 - \int v d\nu_2 \geq 0$.

Note that for (A1) and (A2) to be satisfied when $\Phi = \Phi_c$, it is enough that $c(x, x) = 0$ for all x . The main result of this section is the following generalization of Theorem 1.1.

Theorem 2.4. *Let $((X_t)_{t \geq 0}, \mathbb{P}_\mu)$ be a stationary ergodic Markov process with the symmetrized Dirichlet form $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$, Φ be as above and $\alpha : [0, \infty) \rightarrow [0, \infty]$ be a left continuous increasing function such that $\alpha(0) = 0$. Consider the following properties:*

(a) *The following transportation inequality holds*

$$\alpha(T_\Phi(\nu, \mu)) \leq I(\nu|\mu), \quad \forall \nu \in M_1(\mathcal{X}) \quad (T_\Phi I)$$

(b) *For all $(u, v) \in \Phi$ and all $\lambda, t \geq 0$*

$$\|P_t^{\lambda u}\|_{L^2(\mu)} \leq e^{t[\lambda\mu(v) + \alpha^*(\lambda)]} \quad (2.6)$$

where $P_t^{\lambda u}$ is the Feynman-Kac semigroup (2.3) and α^ is defined at (1.3).*

(b') *For all $(u, v) \in \Phi$ and all $\lambda \geq 0$*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_\mu \exp \left(\lambda \int_0^t u(X_s) ds \right) \leq \lambda\mu(v) + \alpha^*(\lambda)$$

(c) *For any initial measure $\beta \ll \mu$ with $d\beta/d\mu \in L^2(\mu)$ and for all $(u, v) \in \Phi$ and $r, t > 0$,*

$$\mathbb{P}_\beta \left(\frac{1}{t} \int_0^t u(X_s) ds \geq \mu(v) + r \right) \leq \left\| \frac{d\beta}{d\mu} \right\|_2 e^{-t\alpha(r)} \quad (2.7)$$

(c') *For all $(u, v) \in \Phi$ and for any $r \geq 0$, there exists $\beta \in M_1(\mathcal{X})$ such that $\beta \ll \mu$, $d\beta/d\mu \in L^2(\mu)$ and $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}_\beta \left(\frac{1}{t} \int_0^t u(X_s) ds \geq \mu(v) + r \right) \leq -\alpha(r)$*

We have

- (1) $(a) \Rightarrow (b) \Rightarrow (b')$ and $(a) \Rightarrow (c) \Rightarrow (c')$.
 (2) If α is convex, then $(a) \Leftrightarrow (b)$.
 (3) If (P_t) is symmetric on $L^2(\mu)$, then $(a) \Leftrightarrow (c) \Leftrightarrow (c')$.
 If furthermore α is convex, (a) , (b) , (b') , (c) and (c') are equivalent.

Its proof is postponed to the end of the paper. From Theorem 2.4 we derive easily

Corollary 2.5 (The inequalities $W_1I(c)$ and $W_2I(c)$). *Let $c > 0$ and let (X_t) be a μ -reversible and ergodic Markov process such that $\int d^2(x, x_0) d\mu(x) < +\infty$.*

(1) *The statements below are equivalent:*

(a) *The following $W_1I(c)$ inequality holds true:*

$$W_1^2(\nu, \mu) \leq 4c^2 I(\nu|\mu), \quad \forall \nu \in M_1(\mathcal{X}); \quad (W_1I(c))$$

(b) *For all Lipschitz function u on \mathcal{X} with $\|u\|_{\text{Lip}} \leq 1$ and all $\lambda, t \geq 0$,*

$$\|P_t^{\lambda u}\|_{L^2(\mu)} \leq \exp(t[\lambda\mu(u) + c^2\lambda^2]);$$

(c) *For all Lipschitz function u on \mathcal{X} with $\|u\|_{\text{Lip}} \leq 1$, $\mu(u) = 0$ and all $\lambda \geq 0$,*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{E}_\mu \exp\left(\lambda \int_0^t u(X_s) ds\right) \leq c^2\lambda^2;$$

(d) *For all Lipschitz function u on \mathcal{X} , $r > 0$ and $\beta \in M_1(\mathcal{X})$ such that $d\beta/d\mu \in L^2(\mu)$,*

$$\mathbb{P}_\beta \left(\frac{1}{t} \int_0^t u(X_s) ds \geq \mu(u) + r \right) \leq \left\| \frac{d\beta}{d\mu} \right\|_2 \exp\left(-\frac{tr^2}{4c^2\|u\|_{\text{Lip}}^2} \right).$$

(2) *The statements below are equivalent:*

(a) *The following $W_2I(c)$ inequality holds true:*

$$W_2^2(\nu, \mu) \leq 4c^2 I(\nu|\mu), \quad \forall \nu \in M_1(\mathcal{X}); \quad (W_2I(c))$$

(b) *For any $v \in b\mathcal{B}$, $\|P_t^{\frac{1}{4c^2}Qv}\|_{L^2(\mu)} \leq e^{\frac{t}{4c^2}\mu(v)}$, $\forall t \geq 0$ where $Qv(x) = \inf_{y \in \mathcal{X}} \{v(y) + d^2(x, y)\}$ is the so-called “inf-convolution” of v ;*

(c) *For any $u \in b\mathcal{B}$, $\|P_t^{\frac{1}{4c^2}u}\|_{L^2(\mu)} \leq e^{\frac{t}{4c^2}\mu(Su)}$, $\forall t \geq 0$ where $Su(y) = \sup_{x \in \mathcal{X}} \{u(y) - d^2(x, y)\}$ is the so-called “sup-convolution” of u .*

Notation. The best constants $c > 0$ in $W_1I(c)$ and $W_2I(c)$ will be denoted respectively by $c_{W_1I}(\mu)$ and $c_{W_2I}(\mu)$.

Remarks 2.6.

- (i) The best constants $c_{W_1I}(\mu)$ and $c_{W_2I}(\mu)$ depend on the metric d and the Dirichlet form \mathcal{E} . Of course $c_{W_1I}(\mu) \leq c_{W_2I}(\mu)$.
 (ii) The above corollary may be seen as the counterpart of Bobkov-Götze’s characterizations of W_pH ($p = 1, 2$) for Markov processes.

The following simple example illustrates difference between W_1I and W_2I .

Example 2.7. (Bernoulli distribution). Let μ be the Bernoulli distribution on $\mathcal{X} = \{0, 1\}$ with $\mu(\{1\}) = p \in (0, 1)$. Consider the Dirichlet form $\mathcal{E}(g, g) = (g(1) - g(0))^2$. By Theorem 3.1-(a) in §3, we see that

$$pqW_1(\nu, \mu)^2 \leq I(\nu|\mu)$$

where W_1 is built with the trivial metric. The constant pg is sharp. However μ does not satisfy any $W_2I(c)$ as is easily seen with $\nu = \mu_\varepsilon = (1 + \varepsilon g)\mu$.

2.3. Relations between W_2I , Poincaré and log-Sobolev inequalities. In the rest of the paper we are interested in two particular cases of T_cI : $W_1I(c)$ and $W_2I(c)$ introduced at Corollary 2.5.

Notation (Spectral gap). As usual, one says that μ satisfies a Poincaré inequality if

$$\text{Var}_\mu(g) \leq c\mathcal{E}(g, g), \quad \forall g \in \mathbb{D}_2(\mathcal{L}) \quad (2.8)$$

for some *finite* $c \geq 0$ and a Dirichlet form \mathcal{E} which is closable in $L^2(\mu)$. We denote $c_P(\mu)$ the best constant c in the above Poincaré inequality. It is the inverse of the *spectral gap* of \mathcal{L} .

The following result is just a reformulation of the work of Otto-Villani [37].

Proposition 2.8. *In the framework of Riemannian manifold in Remarks 2.3, the followings hold.*

(a) *If the log-Sobolev inequality below*

$$H(\nu|\mu) \leq 2cI(\nu|\mu), \quad \forall \nu$$

is satisfied, then μ satisfies $W_2I(c)$.

(b) *$W_2I(c)$ implies the Poincaré inequality with constant c , i.e., $c_{W_2I}(\mu) \geq c_P(\mu)$.*

(c) *Assume that the Bakry-Emery curvature $\text{Ric} + \text{Hess}V$ is bounded from below by $K \in \mathbb{R}$, where Ric is the Ricci curvature and $\text{Hess}V$ is the Hessian of V . If $W_2I(c)$ holds with $cK \leq 1$ (this is possible by Part (a) and Bakry-Emery's criterion in the case $K > 0$), then we have the following log-Sobolev inequality*

$$H(\nu|\mu) \leq 2(2c - c^2K)I(\nu|\mu), \quad \forall \nu$$

Proof. Before the proof, let us remind the reader that $I = I_F/4$ where I_F is I in Otto-Villani's paper [37].

• (a). The proof is direct, as by [37] or [5] a logarithmic Sobolev inequality implies the

W_2H (sometimes called T_2) inequality so that $W_2(\nu, \mu) \leq \sqrt{2cH(\nu|\mu)} \leq 2c\sqrt{I(\nu|\mu)}$ which is the announced conclusion.

• (b). The proof follows from the usual linearization procedure. Set $\mu_\varepsilon = (1 + \varepsilon g)\mu$ for some smooth and compactly supported g with $\int g d\mu = 0$, we easily get

$\lim_{\varepsilon \rightarrow 0} I(\mu_\varepsilon|\mu)/\varepsilon^2 = \frac{1}{4}\mathcal{E}_\nabla(g, g)$ and by Otto-Villani [37, p.394], there exists r such that $\int g^2 d\mu \leq \sqrt{\mathcal{E}_\nabla(g, g)} \frac{W_2(\mu_\varepsilon, \mu)}{\varepsilon} + \frac{r}{\varepsilon} W_2^2(\mu_\varepsilon, \mu)$. Using now $W_2I(c)$ we get

$$\int g^2 d\mu \leq 2c\sqrt{\mathcal{E}_\nabla(g, g)} \sqrt{\frac{I(\mu_\varepsilon|\mu)}{\varepsilon^2}} + \frac{4rc^2}{\varepsilon} I(\mu_\varepsilon|\mu).$$

Letting $\varepsilon \rightarrow 0$ gives the result.

• (c) is a direct application of the HWI inequality [37, Th.3] in the Euclidean case and [5] for a general Riemannian manifold: $H(\nu|\mu) \leq 2W_2(\nu, \mu)\sqrt{I(\nu|\mu)} - \frac{K}{2}W_2^2(\nu, \mu)$. \square

2.4. Tensorization of T_cI . Assume that $\mu_i \in M_1(\mathcal{X}_i)$ satisfies

$$\alpha_i(T_{c_i}(\nu, \mu_i)) \leq I_i(\nu|\mu_i), \quad \forall \nu \in M_1(\mathcal{X}_i) \quad (2.9)$$

where $I_i(\nu|\mu_i)$ is the Fisher-Donsker-Varadhan information related to the Dirichlet form $(\mathcal{E}_i, \mathbb{D}(\mathcal{E}_i))$, and α_i is moreover convex. On the product space $\mathcal{X}^{(n)} := \prod_{i=1}^n \mathcal{X}_i$ equipped with the product measure $\mu := \otimes_{i=1}^n \mu_i$, consider the sum-cost function

$$\oplus_i c_i(x, y) := \sum_{i=1}^n c(x_i, y_i), \quad \forall x, y \in \mathcal{X}^{(n)} \quad (2.10)$$

and the *inf-convolution* of (α_i)

$$\alpha_1 \square \cdots \square \alpha_n(r) := \inf \left\{ \sum_{i=1}^n \alpha(r_i); r_i \geq 0, \sum_{i=1}^n r_i = r \right\}. \quad (2.11)$$

It also shares the following properties of every α_i : it is increasing, left continuous and convex on \mathbb{R}^+ with $\alpha(0) = 0$ (see [24]). Define the sum-Dirichlet form of $\oplus_i \mathcal{E}_i$ by

$$\begin{aligned} \mathbb{D}(\oplus_i \mathcal{E}_i) &:= \left\{ g \in L^2(\mu) : g_i \in \mathbb{D}(\mathcal{E}_i), \text{ for } \mu\text{-a.e. } \hat{x}_i \text{ and } \int_{\mathcal{X}^{(n)}} \sum_{i=1}^n \mathcal{E}_i(g_i, g_i) d\mu < +\infty \right\} \\ \oplus_i \mathcal{E}_i(g, g) &:= \int_{\mathcal{X}^{(n)}} \sum_{i=1}^n \mathcal{E}_i(g_i, g_i) d\mu, \quad g \in \mathbb{D}(\mathcal{E}) \end{aligned} \quad (2.12)$$

where $g_i(x_i) := g(x_1, \dots, x_i, \dots, x_n)$ with $\hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ fixed.

Theorem 2.9. *Assume (2.9) for each $i = 1, \dots, n$ with α_i moreover convex. Define c, α, \mathcal{E} respectively by (2.10), (2.11) and (2.12). Let $I_{\oplus_i \mathcal{E}_i}(\nu|\mu)$ be the Fisher-Donsker-Varadhan information associated with $(\oplus_i \mathcal{E}_i, \mathbb{D}(\oplus_i \mathcal{E}_i))$. Then*

$$\alpha_1 \square \cdots \square \alpha_n(T_{\oplus_i c_i}(\nu, \mu)) \leq I_{\oplus_i \mathcal{E}_i}(\nu|\mu), \quad \forall \nu \in M_1(\mathcal{X}^{(n)}). \quad (2.13)$$

This result is similar to [24, Corollary 5], but the proof will be different. It is based on the following sub-additivity result for the transportation cost of a product measure, which is different from Marton's original result [33] where an ordering of sites is required.

Lemma 2.10. *Let $\mu = \otimes_{i=1}^n \mu_i$. Given a probability measure ν on $\prod_{i=1}^n \mathcal{X}_i$, let ν_i be the regular conditional distribution of x_i knowing \hat{x}_i . Then with the cost function c given at (2.10), $T_{\oplus_i c_i}(\mu, \nu) \leq \mathbb{E}^\nu \sum_{i=1}^n T_{c_i}(\mu_i, \nu_i)$.*

Notation. The expectation \mathbb{E}^ν simply means integration with respect to ν .

Proof. Let $(Z_i = (X_i, Y_i))_{i=1, \dots, n}$ be a sequence of random variables valued in $\prod_{i=1}^n \mathcal{X}_i^2$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, realizing $T_{\oplus_i c_i}(\mu, \nu)$, i.e., the law of $X = (X_i)_{i=1, \dots, n}$ is $\mu = \otimes_{i=1}^n \mu_i$, the law of $Y = (Y_i)_{i=1, \dots, n}$ is ν and $\mathbb{E} \sum_i c_i(X_i, Y_i) = T_{\oplus_i c_i}(\mu, \nu)$.

For each i fixed, construct a couple of r.v. $(\tilde{X}_i, \tilde{Y}_i)$ so that its conditional law given $(Z_j)_{j \neq i}$ is a coupling of $(\mu_i(dx_i), \nu_i(dx_i|Y_j, j \neq i))$ and \mathbb{P} -a.s.,

$$\mathbb{E}[c_i(\tilde{X}_i, \tilde{Y}_i) | Z_j, j \neq i] = T_{c_i}(\mu_i, \nu_i(\cdot | Y_j, j \neq i)).$$

Obviously $(X_j, j \neq i; \tilde{X}_i)$ and $(Y_j, j \neq i; \tilde{Y}_i)$ (more precisely their joint law) constitute a coupling of (μ, ν) . Thus $\mathbb{E} \sum_j c_j(X_j, Y_j) \leq \mathbb{E}[\sum_{j \neq i} c_j(X_j, Y_j) + c_i(\tilde{X}_i, \tilde{Y}_i)]$ or $\mathbb{E} c_i(X_i, Y_i) \leq \mathbb{E} c_i(\tilde{X}_i, \tilde{Y}_i) = \mathbb{E} T_{c_i}(\mu_i, \nu_i(\cdot | Y_j, j \neq i))$. Consequently

$$T_{\oplus_i c_i}(\mu, \nu) = \mathbb{E} \sum_{i=1}^n T_{c_i}(X_i, Y_i) \leq \mathbb{E} \sum_{i=1}^n T_{c_i}(\mu_i, \nu_i(\cdot | Y_j, j \neq i)) = \int \sum_{i=1}^n T_{c_i}(\mu_i, \nu_i) d\nu.$$

□

The following additivity property of the Fisher information will be needed. It holds even in the dependent case.

Lemma 2.11. *Let ν, μ be probability measures on $\prod_{i=1}^n \mathcal{X}_i$ such that $I(\nu|\mu) < +\infty$, let μ_i, ν_i be the regular conditional distributions of x_i knowing \hat{x}_i under μ, ν . Then*

$$I_{\oplus_i \mathcal{E}_i}(\nu|\mu) = \mathbb{E}^\nu \sum_i I_i(\nu_i|\mu_i). \quad (2.14)$$

Proof. Let $f = d\nu/d\mu$. Then $d\nu_i/d\mu_i = f/\mu_i(f) = f_i/\mu_i(f_i)$, ν -a.s. (recalling that f_i is the function f of x_i with \hat{x}_i fixed). For ν -a.e. \hat{x}_i fixed,

$$I_i(\nu_i|\mu_i) = \mathcal{E}_i \left(\sqrt{\frac{f_i}{\mu_i(f_i)}}, \sqrt{\frac{f_i}{\mu_i(f_i)}} \right) = \frac{1}{\mu_i(f_i)} \mathcal{E}_i(\sqrt{f_i}, \sqrt{f_i})$$

(for $\mu_i(f_i)$ is constant with \hat{x}_i fixed). We obtain

$$\begin{aligned} \mathbb{E}^\nu \sum_{i=1}^n I_i(\nu_i|\mu_i) &= \mathbb{E}^\mu f \sum_{i=1}^n \frac{1}{\mu_i(f_i)} \mathcal{E}_i(\sqrt{f_i}, \sqrt{f_i}) = \mathbb{E}^\mu \sum_{i=1}^n \mathcal{E}_i(\sqrt{f_i}, \sqrt{f_i}) \\ &= \oplus_i \mathcal{E}_i(\sqrt{f}, \sqrt{f}) = I_{\oplus_i \mathcal{E}_i}(\nu|\mu), \end{aligned}$$

which completes the proof. □

The above additivity is different from the super-additivity of the Fisher information for product measure obtained by E. Carlen [9].

Proof of Theorem 2.9. Without loss of generality we may assume that $I(\nu|\mu) < +\infty$. For simplicity write $\alpha = \alpha_1 \square \cdots \square \alpha_n$. By Lemma 2.10, Jensen's inequality and the definition of α ,

$$\begin{aligned} \alpha(T_{\oplus_{c_i}}(\nu, \mu)) &\leq \alpha \left(\mathbb{E}^\nu \sum_{i=1}^n T_{c_i}(\nu_i, \mu_i) \right) \leq \mathbb{E}^\nu \alpha \left(\sum_{i=1}^n T_{c_i}(\nu_i, \mu_i) \right) \\ &\leq \mathbb{E}^\nu \sum_{i=1}^n \alpha_i(T_{c_i}(\nu_i, \mu_i)) \leq \mathbb{E}^\nu \sum_{i=1}^n I_i(\nu_i|\mu_i). \end{aligned}$$

The last quantity is equal to $I_{\oplus_i \mathcal{E}_i}(\nu|\mu)$, by Lemma 2.11. □

As an example of application, let $(X_t^i)_{t \geq 0}, i = 1, \dots, n$ be n Markov processes with the same transition semigroup (P_t) and the same symmetrized Dirichlet form \mathcal{E} on $L^2(\mu)$, and conditionally independent once $(X_0^i)_{i=1, \dots, n}$ is fixed. Then $X_t := (X_t^1, \dots, X_t^n)$ is a Markov process with the symmetrized Dirichlet form given by

$$\oplus_n \mathcal{E}(g, g) = \int \sum_{i=1}^n \mathcal{E}(g_i, g_i) \mu(dx_1) \cdots \mu(dx_n)$$

which is the n -fold sum-Dirichlet form of \mathcal{E} .

Corollary 2.12. *Assume that μ satisfies $T_c I$ on \mathcal{X} with α convex. Then $\mu^{\otimes n}$ satisfies*

$$n\alpha \left(\frac{T_{\oplus_n c}(\nu, \mu^{\otimes n})}{n} \right) \leq I_{\oplus_n \mathcal{E}}(\nu|\mu^{\otimes n}), \quad \forall \nu \in M_1(\mathcal{X}^n). \quad (2.15)$$

In particular for all $(u, v) \in \Phi_c$, for all initial measure β on \mathcal{X}^n with $d\beta/d\mu^{\otimes n} \in L^2(\mu^{\otimes n})$ and for any $t, r > 0$,

$$\mathbb{P}_\beta \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{t} \int_0^t u(X_s^i) ds \geq \mu(v) + r \right) \leq \left\| \frac{d\beta}{d\mu^{\otimes n}} \right\|_2 e^{-nt\alpha(r)}. \quad (2.16)$$

Proof. As $\alpha^{\square n}(r) = n\alpha(r/n)$, (2.15) follows from Theorem 2.9. Noting that for $u, v \in \Phi_c$, $(\sum_{i=1}^n u(x_i), \sum_{i=1}^n v(x_i))$ as a couple of functions on \mathcal{X}^n belongs to $\Phi_{\oplus n c}$, we obtain (2.16) by Theorem 2.4. \square

The tensorization of $W_p I$ in the dependent Gibbs measure case is carried out in Gao and Wu [21].

3. POINCARÉ INEQUALITY IMPLIES Hoeffding's DEVIATION INEQUALITY

The purpose of this section is to establish

Theorem 3.1. *Let $((X_t), \mathbb{P}_\mu)$ be a stationary ergodic Markov process.*

(a) *The Poincaré inequality*

$$\text{Var}_\mu(g) \leq c_P \mathcal{E}(g, g), \quad \forall g \in \mathbb{D}_2(\mathcal{L}) \quad (3.1)$$

implies

$$\|\nu - \mu\|_{TV}^2 \leq 4c_P I(\nu|\mu), \quad \forall \nu \in M_1(\mathcal{X}) \quad (3.2)$$

and for $u \in b\mathcal{B}$ so that $\|u\|_\infty \leq 1, \mu(u) = 0$,

$$\int u d(\nu - \mu) \leq \sqrt{4c_P I(\nu|\mu) \left(\text{Var}_\mu(u) + \sqrt{2c_P I(\nu|\mu)} \right)}. \quad (3.3)$$

In particular for every initial probability measure $\beta \ll \mu$ with $d\beta/d\mu \in L^2(\mu)$ and for all $u \in b\mathcal{B}$ with $\mu(u) = 0$ and $\text{Var}_\mu(u) = \sigma^2$, $t, r, \varepsilon > 0$,

$$\begin{aligned} & \mathbb{P}_\beta \left(\frac{1}{t} \int_0^t u(X_s) ds \geq r \right) \\ & \leq \left\| \frac{d\beta}{d\mu} \right\|_2 \exp \left(-\frac{t}{c_P} \max \left[\frac{r^2}{\delta(u)^2}, \varepsilon(\varepsilon + \sigma^2) \left(\sqrt{1 + \frac{r^2}{2\varepsilon(\varepsilon + \sigma^2)^2 \delta(u)^2}} - 1 \right) \right] \right) \end{aligned} \quad (3.4)$$

where $\delta(u) := \sup_{x, y \in \mathcal{X}} |u(x) - u(y)|$ is the oscillation of u .

(b) *Conversely, assume that (X_t, \mathbb{P}_μ) is reversible and $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$ is quasi-regular ([34]), if there is some left-continuous and increasing $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\alpha(1/2) > 0$ such that*

$$\alpha(\|\nu - \mu\|_{TV}) \leq I(\nu|\mu), \quad \forall \nu \in M_1(\mathcal{X}), \quad (3.5)$$

then the Poincaré inequality (3.1) holds with $c_P \leq 4/\alpha(1/2)$.

Remarks 3.2. (i) Let $d(x, y) = \mathbf{1}_{x \neq y}$ (the trivial metric) and $\Phi = \{(u, u); \delta(u) \leq 1\}$.

Then $\frac{1}{2}\|\nu - \mu\|_{TV} = W_1(\nu, \mu) = T_\Phi(\nu, \mu)$. Hence (3.2) is exactly the inequality $T_\Phi I$ or $W_1 I(c)$ with $4c^2 = c_P$. (3.4) is a direct consequence of (3.2) and Theorem 2.4.

(ii) Lezaud [30] proved a better deviation inequality using the asymptotic variance $V(u) = \lim_{t \rightarrow \infty} \frac{1}{t} \text{Var}_{\mathbb{P}_\mu} \left(\int_0^t u(X_s) ds \right)$ in the central limit theorem instead of $\text{Var}_\mu(u)$, which is sharp in the moderate deviation scale (r very small), nevertheless his proof involves

Kato's theory of perturbation of operators and difficult combinatorial techniques. Our result (3.3) is similar to Cattiaux-Guillin [10].

Proof of Theorem 3.1. • (a). (3.4) follows by (3.2) and (3.3) by Theorem 2.4. We first prove the transportation inequality (3.2). To this end recall the following known inequality in statistics (see Gibbs and Su [22]) $\frac{1}{4}\|\nu - \mu\|_{\text{TV}}^2 \leq d_H^2(\nu, \mu)[2 - d_H^2(\nu, \mu)]$ where $d_H^2(\nu, \mu) = \frac{1}{2} \int (1 - \sqrt{f})^2 d\mu = 1 - \mu(\sqrt{f})$ is the square of the Hellinger distance between $\nu = f\mu$ and μ . The above right-hand side is exactly $1 - [\mu(\sqrt{f})]^2 = \text{Var}_\mu(\sqrt{f})$. In other words we have for every μ -probability density f , i.e. $f \geq 0$ and $\mu(f) = 1$,

$$\|f\mu - \mu\|_{\text{TV}}^2 \leq 4\text{Var}_\mu(\sqrt{f}). \quad (3.6)$$

Now for every probability density f so that $\sqrt{f} \in \mathbb{D}(\mathcal{E})$, we have by (3.6) and the assumed Poincaré inequality, $\frac{1}{4}\|\nu - \mu\|_{\text{TV}}^2 \leq \text{Var}_\mu(\sqrt{f}) \leq c_{\text{P}}\mathcal{E}(\sqrt{f}, \sqrt{f}) = c_{\text{P}}I(\nu|\mu)$ which is (3.2).

For (3.3) it is enough to prove that if $\|u\|_\infty \leq 1$, $\mu(u^2) \leq \sigma^2(\leq 1)$, then for every probability density f

$$\int u(f d\mu - d\mu) \leq \sqrt{4\text{Var}_\mu(\sqrt{f}) \left(\sigma^2 + \sqrt{2\text{Var}_\mu(\sqrt{f})} \right)} \quad (3.7)$$

Indeed by Cauchy-Schwarz inequality

$$\begin{aligned} \int u(f d\mu - d\mu) &\leq \sqrt{\int (\sqrt{f} - 1)^2 d\mu \int u^2 (\sqrt{f} + 1)^2 d\mu} \\ &= \sqrt{4\text{Var}_\mu(\sqrt{f}) \int u^2 (\sqrt{f} + 1)^2 d\mu / \mu[(\sqrt{f} + 1)^2]} \end{aligned}$$

which gives us (3.6) (again). By (3.6) and the fact that $\delta(u^2) \leq 1$,

$$\begin{aligned} \int u^2 (\sqrt{f} + 1)^2 d\mu / \mu[(\sqrt{f} + 1)^2] &\leq \mu(u^2) + \sqrt{4\text{Var}_\mu \left(\sqrt{(\sqrt{f} + 1)^2 / \mu[(\sqrt{f} + 1)^2]} \right)} \\ &\leq \sigma^2 + \sqrt{2\text{Var}_\mu(\sqrt{f})} \end{aligned}$$

which yields (3.7).

• (b). This converse part is based on the capacity-measure method, recalled below.

We begin with the definition of the capacity of a set: given $D \in \mathcal{B}$ and a measurable $A \subset D$, the capacity $\text{Cap}_\mu(A, D)$ (w.r.t. the Dirichlet form \mathcal{E} with the Dirichlet boundary condition on D^c) is defined as

$$\text{Cap}_\mu(A, D) := \inf\{\mathcal{E}(g, g), 1_A \leq \tilde{g} \leq 1_D \text{ quasi-everywhere, } g \in \mathbb{D}(\mathcal{E})\},$$

(see [34] for the notion ‘‘quasi-everywhere’’ and the quasi-continuous version \tilde{g} of $g \in \mathbb{D}(\mathcal{E})$); if now A satisfies $\mu(A) \leq 1/2$,

$$\text{Cap}_\mu(A) := \inf\{\text{Cap}_\mu(A, D); A \subset D, \mu(D) \leq 1/2\}.$$

The capacity-measure method says that the Poincaré inequality is equivalent to

$$\mu(A) \leq \kappa \text{Cap}_\mu(A), \quad \forall A \in \mathcal{B}, \mu(A) \leq 1/2.$$

and the best constant κ verifies (see for example [4] for a nice review of capacity-measure characterization of various functional inequalities)

$$\kappa \leq c_{\mathbb{P}} \leq 4\kappa. \quad (3.8)$$

Notice that (3.5) applied to $\nu = \frac{g^2}{\mu(g^2)}\mu$ for $g \in \mathbb{D}(\mathcal{E})$) reads

$$\mu(g^2)\alpha \left(\int \left| \frac{g^2}{\mu(g^2)} - 1 \right| d\mu \right) \leq \mathcal{E}(g, g). \quad (3.9)$$

If now $A \subset D$ and $\mu(D) \leq 1/2$, we have for any $g \in \mathbb{D}(\mathcal{E})$ satisfying $1_A \leq g \leq 1_D$

$$\int \left| \frac{g^2}{\mu(g^2)} - 1 \right| d\mu \geq \int_{D^c} \left| \frac{g^2}{\mu(g^2)} - 1 \right| d\mu = \mu(D^c) \geq 1/2$$

so that (3.9) implies that if $\mu(A) \leq 1/2$, $\mu(A) \leq \text{Cap}_{\mu}(A)/\alpha(1/2)$. By (3.8), this yields the desired result: $c_{\mathbb{P}} \leq 4/\alpha(1/2)$. \square

Note that W_1I for the trivial metric implies W_1I for any bounded metric. So our next purpose is to obtain W_1I for unbounded metrics. Our study is naturally separated into two sections. Next Section 4 is concerned with estimating sharply c_{W_1I} under strong dissipative conditions. In Section 5, Lyapunov function conditions for W_1I or more general $T_{\Phi}I$ are taken into consideration.

4. SPECTRAL GAP IN THE SPACE OF LIPSCHITZ FUNCTIONS IMPLIES W_1I FOR DIFFUSION PROCESSES

4.1. General observations. Let $((X_t), \mathbb{P}_{\mu})$ be a reversible ergodic Markov process with generator \mathcal{L} , Dirichlet form $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$ and with continuous sample paths valued in some separable complete metric space (\mathcal{X}, d) (called Markov diffusion). We assume that $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$ is given by the carré-du-champs $\Gamma : \mathbb{D}(\mathcal{E}) \times \mathbb{D}(\mathcal{E}) \rightarrow L^1(\mu)$ (symmetric, bilinear definite nonnegative form):

$$\mathcal{E}(h, h) = \int_{\mathcal{X}} \Gamma(h, h) d\mu, \quad \forall h \in \mathbb{D}(\mathcal{E}). \quad (4.1)$$

The continuity of sample paths of (X_t) implies that Γ is a differentiation (cf. Bakry [1]), that is: for all $(h_k)_{1 \leq k \leq n} \subset \mathbb{D}(\mathcal{E})$, $g \in \mathbb{D}(\mathcal{E})$ and $F \in C_b^1(\mathbb{R}^n)$,

$$\Gamma(F(h_1, \dots, h_n), g) = \sum_{i=1}^n \partial_i F(h_1, \dots, h_n) \Gamma(h_i, g).$$

Theorem 4.1. *Assume that $\int_{\mathcal{X}} d^2(x, x_0) d\mu(x) < +\infty$ and for any $g \in C_{\text{Lip}}(\mathcal{X}, d)$ bounded with $\mu(g) = 0$, then $g \in \mathbb{D}(\mathcal{E})$ and*

$$\sqrt{\overline{\Gamma(g, g)}} \leq \sigma \|g\|_{\text{Lip}}, \quad \mu\text{-a.s.} \quad (4.2)$$

and there is some $h \in \mathbb{D}_2(\mathcal{L})$ such that $-\mathcal{L}h = g$ (μ -a.e.) and a μ -continuous version \tilde{h} of h satisfying

$$\|\tilde{h}\|_{\text{Lip}} \leq C \|g\|_{\text{Lip}} \quad (4.3)$$

where $\sigma, C > 0$ are fixed constants. Then for any $g \in C_{\text{Lip}}(\mathcal{X}, d)$ and any convex function ϕ on \mathbb{R} ,

$$\mathbb{E}_{\mu} \phi \left(\int_0^t g(X_s) ds \right) \leq \mathbb{E}_{\mu} \phi(B_{2\sigma^2 C^2 \|g\|_{\text{Lip}}^2 t}) \quad (4.4)$$

where B is a standard Brownian Motion. In particular

$$\mathbb{E}_\beta \exp \left(\lambda \int_0^t g(X_s) ds \right) \leq \left\| \frac{d\beta}{d\mu} \right\|_2 e^{\lambda^2 (\sigma C)^2 \|g\|_{\text{Lip}}^2 t}, \quad \forall \lambda \in \mathbb{R}, t > 0. \quad (4.5)$$

and μ satisfies $W_1 I(\sigma C)$ on (\mathcal{X}, d) . Furthermore let $V(g) = \lim_{t \rightarrow \infty} \frac{1}{t} \text{Var}_{\mathbb{P}_\mu} \left(\int_0^t g(X_s) ds \right)$ and $c_P = c_P(\mu) \leq C$, we have for all $\lambda \in \mathbb{R}, t > 0, p > 1$ (and $1/p + 1/q = 1$),

$$\mathbb{E}_\beta \exp \left(\lambda \int_0^t g(X_s) ds \right) \leq \left\| \frac{d\beta}{d\mu} \right\|_2 \exp \left(t \left[\frac{\lambda^2}{2} p V(g) + \frac{\lambda^4}{4} p^2 q c_P (\sigma C)^4 \|g\|_{\text{Lip}}^4 \right] \right). \quad (4.6)$$

Remarks 4.2.

- (i) Let C_{Lip}^0 be the Banach space of those $g \in C_{\text{Lip}}$ with $\mu(g) = 0$, equipped with $\|\cdot\|_{\text{Lip}}$. Hence the best constant C in (4.3) is exactly

$$\|(-\mathcal{L})^{-1}\|_{C_{\text{Lip}}^0}.$$

By the spectral decomposition we always have (cf. [48, Proof of Lemma 4.3])

$$C = \|(-\mathcal{L})^{-1}\|_{C_{\text{Lip}}^0} \geq \|(-\mathcal{L})^{-1}\|_{L^2(\mu) \cap \{g \in L^2(\mu); \mu(g)=0\}} = c_P(\mu).$$

But the converse is false: the symmetric exponential measure $\mu = \frac{1}{2} e^{-|x|} dx$ on \mathbb{R} satisfies the Poincaré inequality but the associated Dirichlet form $\mathcal{E}(g, g) = \int_{\mathbb{R}} g'^2 d\mu$ does not have spectral gap in C_{Lip}^0 w.r.t. the Euclidean metric.

- (ii) The concentration inequality (4.5) and its equivalent $W_1 I(\sigma C)$ are sharp, as seen for one-dimensional Ornstein-Uhlenbeck process. The estimate (4.6) for p close to 1 is sharp in the moderate deviation scale (λ very small) and extends the result of Lezaud [30] to unbounded g .
- (iii) Klein-Ma-Privault [28] developed *convex concentration inequality* (4.4) for semimartingales instead of $S_t(g)$, by means of a forward-backward martingale calculus, but their result cannot be applied directly here.
- (iv) In [27], Joulin obtained a similar result for Markov processes with jumps.

Proof of Theorem 4.1. Let $\Phi = \{(g, g); \|g\|_{\text{Lip}} \leq 1, g \text{ bounded}\}$. Then $W_1(\nu, \mu) = T_\Phi(\nu, \mu)$ by Kantorovich-Rubinstein's theorem. Let us verify that (b') of Theorem 2.4 holds.

For any $g \in C_{\text{Lip}}$ with $\|g\|_{\text{Lip}} \leq 1$, let $h \in C_{\text{Lip}} \cap \mathbb{D}_2(\mathcal{L})$ such that $-\mathcal{L}h = g$. Hence

$$M_t(h) := h(X_t) - h(X_0) + \int_0^t g(X_s) ds \quad \text{and} \quad M_t^*(h) := h(X_0) - h(X_t) + \int_0^t g(X_s) ds$$

have the same law under \mathbb{P}_μ by the reversibility of $((X_t), \mathbb{P}_\mu)$. Consequently from Lyons-Meyer-Zheng's forward-backward martingale decomposition

$$S_t(g) := \int_0^t g(X_s) ds = \frac{1}{2} (M_t(h) + M_t^*(h)), \quad (4.7)$$

it follows that for any convex function ϕ on \mathbb{R} , $\mathbb{E}_\mu \phi(S_t(g)) \leq \frac{1}{2} \mathbb{E}_\mu [\phi(M_t(h) + \phi(M_t^*(h)))] = \mathbb{E}_\mu \phi(M_t(h))$. As $M_t(h)$ is a (forward) continuous martingale, $M_t(h) = B_{\tau_t}$ where (B_t) is some Brownian motion with respect to another time-changed filtration $(\hat{\mathcal{F}}_t)$, and $\tau_t = \langle M(h) \rangle_t$ is a $(\hat{\mathcal{F}}_t)$ -stopping time (a well known result). Since by our conditions (4.2) and (4.3), $\langle M(h) \rangle_t = 2 \int_0^t \Gamma(h, h)(X_s) ds \leq 2(\sigma C)^2 t$, by Jensen's inequality we obtain for all convex

function ϕ on \mathbb{R} that $\mathbb{E}_\mu \phi(S_t(g)) \leq \mathbb{E} \phi(B_{\tau_t}) = \mathbb{E} \phi(\mathbb{E}[B_{2(\sigma C)^2 t} | \hat{\mathcal{F}}_{\tau_t}]) \leq \mathbb{E} \phi(B_{2(\sigma C)^2 t})$, which is (4.4). Applying this to $\phi(x) = e^{\lambda x}$, we get (4.5) for $\beta = \mu$. Hence Theorem 2.4-(b') holds with $\Phi = \{(g, g); \|g\|_{\text{Lip}} \leq 1, g \text{ bounded}\}$ and $\alpha(r) = r^2/(4(\sigma C)^2)$, and Theorem 2.4 also gives us (4.5) for β .

For (4.6), it is enough to show it for $\beta = \mu$ by Theorem 2.4 and for g with $\|g\|_{\text{Lip}} = 1$. By Hölder's inequality,

$$\mathbb{E}_\mu e^{\lambda S_t(g)} \leq \mathbb{E}_\mu e^{\lambda M_t} \leq \left(\mathbb{E}_\mu \exp \left(\lambda p M_t - \frac{\lambda^2 p^2}{2} \langle M \rangle_t \right) \right)^{1/p} \left(\mathbb{E}_\mu \exp \left(\frac{\lambda^2 p q}{2} \langle M \rangle_t \right) \right)^{1/q}$$

where $M_t = M_t(h)$. Since $\exp \left(\lambda p M_t - \frac{\lambda^2 p^2}{2} \langle M \rangle_t \right)$ is an exponential martingale, its expectation is 1. To estimate the last term above we use the analogue of (3.2) at Theorem 3.1, given by Theorem 2.4(b). Denoting $\delta(\Gamma(h, h)) = \|\Gamma(h, h)\|_\infty \leq \sigma C$ and $2\mu(\Gamma(h, h)) = V(g)$, this provides us with

$$\begin{aligned} \mathbb{E}_\mu \exp \left(\frac{\lambda^2 p q}{2} \langle M \rangle_t \right) &\leq \exp \left(t \left[\lambda^2 p q \mu(\Gamma(h, h)) + \frac{\lambda^4 p^2 q^2}{4} \cdot c_{\mathbb{P}} \|\Gamma(h, h)\|_\infty^2 \right] \right) \\ &= \exp \left(t \left[\frac{\lambda^2}{2} p q V(g) + \frac{\lambda^4 p^2 q^2}{4} \cdot c_{\mathbb{P}} (\sigma C)^4 \right] \right) \end{aligned}$$

where (4.6) follows. \square

Remarks. Using results from [16], one can be even more precise in the one dimensional case. Using the metric induced by the carré-du-champ operator of the diffusion, and conditions on Feller's scale and speed functions, we get sharp constant on the spectral gap for Lipschitz functions. It will be developed in [25].

5. LYAPUNOV FUNCTION CONDITIONS

5.1. We will use in this section general conditions on the generator of the process, known as Lyapunov function conditions, for deriving $W_1 I$ or more generally $T_\Phi I$ where $\Phi = \{(u, u); |u| \leq \phi\}$ with ϕ unbounded. To state properly the Lyapunov function condition, it is necessary to enlarge the domain of the generator. In this section, the Markov process $((X_t), \mathbb{P}_\mu)$ is reversible and its sample paths are \mathbb{P}_μ -càdlàg (possibly with jumps).

A continuous function h is said to be in the μ -extended domain $\mathbb{D}_e(\mathcal{L})$ of the generator of the Markov process $((X_t), \mathbb{P}_\mu)$ if there is some measurable function g such that $\int_0^t |g|(X_s) ds < +\infty$, \mathbb{P}_μ -a.s. and

$$M_t(h) := h(X_t) - h(X_0) - \int_0^t g(X_s) ds$$

is a local \mathbb{P}_μ -martingale. It is obvious that g is uniquely determined up to μ -equivalence. In such case one writes $h \in \mathbb{D}_e(\mathcal{L})$ and $\mathcal{L}h = g$.

The Lyapunov condition can now be stated:

(H) There exist a continuous function $U : \mathcal{X} \rightarrow [1, +\infty)$ in $\mathbb{D}_e(\mathcal{L})$, a nonnegative function ϕ and a constant $b > 0$ such that

$$-\frac{\mathcal{L}U}{U} \geq \phi - b, \quad \mu\text{-a.s.}$$

When the process is irreducible and the constant b is replaced by $b1_C$ for some “small set” C , then it is well-known that the existence of a positive bounded ϕ such that $\inf_{\mathcal{X} \setminus C} \phi > 0$ in (H) is equivalent to Poincaré inequality (see [2], for instance).

Lyapunov conditions are widely used to study the speed of convergence of Markov chains [36] or Markov processes [18], large or moderate deviations and essential spectral radii [47, 48]. More recently, they have been used to study functional inequalities as weak Poincaré inequality [2] or super-Poincaré inequality [12]. See Wang [43] on weak and super Poincaré inequalities.

Theorem 5.1. *Assume that μ satisfies a Poincaré inequality with best constant $c_P < \infty$ and that the Lyapunov condition (H) holds. Suppose moreover that $\phi \in L^2(\mu)$, that is $\|\phi\|_2 := (\int \phi^2 d\mu)^{1/2} < \infty$. Then for any $p \in [1, +\infty)$, we have*

$$\alpha_{p,R} (\|\phi^{1/p}(\nu - \mu)\|_{TV}) \leq I(\nu|\mu), \quad \forall \nu \quad (5.1)$$

where $R = 2^{1/q} \left(\frac{3}{c_P} + 6b + 2\sqrt{2}\|\phi\|_2 \right)^{1/p}$, $1/p + 1/q = 1$ and

$$\alpha_{p,R}(r) = \begin{cases} r^2/(R^2 c_P), & \text{if } 0 \leq r \leq R; \\ r^p/(R^p c_P), & \text{if } r > R. \end{cases} \quad (5.2)$$

In particular, if moreover $d(x, x_0) \leq C\phi^{1/p}$, $\forall x \in \mathcal{X}$ for some $x_0 \in \mathcal{X}$, then

$$\alpha_{p,R}(W_1(\nu, \mu)/C) \leq I(\nu|\mu). \quad (5.3)$$

Remarks 5.2. Since $\|\phi(\nu - \mu)\|_{TV} = \sup_{u: |u| \leq \phi} \int u d(\nu - \mu)$, the inequality (5.1) in this theorem may be regarded as $T_\Phi I$ in Theorem 2.4 with $\Phi = \{(u, u); u \in b\mathcal{B}, |u| \leq \phi^{1/p}\}$. Since

$$W_1(\nu, \mu) = \sup_{f: \|f\|_{\text{Lip}} \leq 1} \int f d[\nu - \mu] \leq \inf_{x_0 \in \mathcal{X}} \|d(\cdot, x_0)(\nu - \mu)\|_{TV}, \quad (5.4)$$

one sees that the $W_1 I$ inequality (5.3) is a direct consequence of (5.1).

An important feature of this result is: $(1/t) \int_0^t u(X_s) ds$ has different concentration behaviors according to different p so that $|u| \leq C\phi^{1/p}$, by (5.1).

The explicit constants in the inequalities of this theorem, produced by the Lyapunov function condition (H), are in general far from being optimal, but are sharp in order, as will be seen for the Ornstein-Uhlenbeck process at Example 5.5.

Example 5.3. ($M/M/\infty$ queue). In this example $\mathcal{X} = \mathbb{N}$, μ is the Poisson measure with mean $\lambda > 0$ and the Dirichlet form is

$$\mathcal{E}(h, h) = \sum_{n \in \mathbb{N}} (h(n+1) - h(n))^2 \mu(n)$$

The associated generator is

$$\mathcal{L}h(n) = \lambda(h(n+1) - h(n)) + n(h(n-1) - h(n)), \quad \forall n \geq 0$$

(with the convention that $h(-1) = h(0)$). Let $U(n) = e^{cn}$ where $c > 0$. We have

$$-\frac{\mathcal{L}U}{U}(n) = n(1 - e^{-c}) - (e^c - 1).$$

Thus condition (H) is satisfied with $\phi(n) := (n+1)(1-e^{-c})$ and $b = e^c + e^{-c} - 2$, and it is well known that $c_P(\mu) = \lambda$. Noting $\|\phi\|_2 = (1-e^{-c})\sqrt{\lambda^2 + 3\lambda + 1}$, we have by Theorem 5.1 that for the distance $d(m, n) := |\sqrt{m} - \sqrt{n}|$,

$$W_1(\nu, \mu)^2 \leq K(c)I(\nu|\mu), \quad \forall \nu \in M_1(\mathbb{N}), c > 0$$

where $K(c) = \frac{2}{1-e^{-c}} [3(1+2(e^c + e^{-c} - 2)\lambda) + 2\sqrt{2}(1-e^{-c})\sqrt{\lambda^2 + 3\lambda + 1}]$. By Theorem 2.4, this gives the Gaussian deviation inequality for any observable u so that $|u(n+1) - u(n)| \leq C|\sqrt{n+2} - \sqrt{n+1}|$ or $|u(n)| \leq C\sqrt{1+n}$. Furthermore if $|u(n)| \leq C[(1-e^{-c})(1+n)]^{1/p}$, we have by (5.2) and Theorem 2.4

$$\mathbb{P}_\beta \left(\frac{1}{t} \int_0^t u(X_s) ds > \mu(u) + r \right) \leq \left\| \frac{d\beta}{d\mu} \right\|_2 \exp(-t\alpha_{p,R}(Cr)), \quad \forall t, r > 0.$$

See Joulin [26] and Liu-Ma [31] for previous studies on deviation inequalities of this model. Note that they only obtain Poisson tail for observable u so that $|u(n+1) - u(n)| \leq C|\sqrt{n+2} - \sqrt{n+1}|$.

Corollary 5.4. *Let $\mu = e^{-V} dx/Z$ be a probability measure where $V \in C^\infty(\mathcal{X})$ is bounded from below and $|\nabla V|^2 \in L^2(\mu)$. Let $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ be the generator of the diffusion (X_t) on the non-compact connected complete Riemannian manifold \mathcal{X} . Assume that*

$$\gamma := \limsup_{d(x, x_0) \rightarrow \infty} \frac{\Delta V(x)}{|\nabla V|^2(x)} < 1 \quad (5.5)$$

and for some $p \geq 1$, $d(x, x_0) \leq C(1 + |\nabla V|^2(x))^{1/p}$, $\forall x \in \mathcal{X}$. Then for every $\delta \in (0, (1-\gamma)^2/4)$, the Lyapunov function condition (H) is satisfied with $\phi = \delta(1 + |\nabla V|^2(x))$ and some $b = b(\delta) > 0$, and $c_P = c_P(\mu) < +\infty$. If moreover ϕ is in $L^2(\mu)$, then w.r.t. the Riemannian metric d ,

$$\alpha_{p,R}(W_1(\nu, \mu)/C) \leq I(\nu|\mu), \quad \forall \nu \in M_1(\mathcal{X}). \quad (5.6)$$

where $\alpha_{p,R}$ is given in (5.2). In particular for every Lipschitz function u with $\|u\|_{\text{Lip}} \leq 1$ and any initial law β with $d\beta/d\mu \in L^2(\mu)$,

$$\mathbb{P}_\beta \left(\frac{1}{t} \int_0^t u(X_s) ds > \mu(u) + r \right) \leq \left\| \frac{d\beta}{d\mu} \right\|_2 \exp(-t\alpha_{p,R}(Cr)), \quad \forall t, r > 0. \quad (5.7)$$

Proof. For any $0 < \delta < (1-\gamma)^2/4$ let $\varepsilon \in (0, 1-\gamma)$ so that $\delta = (1-\gamma-\varepsilon)^2/4$ (or $\gamma + \varepsilon = 1 - 2\sqrt{\delta}$). Choose $\lambda = \sqrt{\delta}$ we have $\lambda - \lambda^2 = (\gamma + \varepsilon)\lambda + \delta$. For $U = e^{\lambda V}$, we have

$$-\frac{\mathcal{L}U}{U} = -\lambda\mathcal{L}V - \lambda^2|\nabla V|^2 = (\lambda - \lambda^2)|\nabla V|^2 - \lambda\Delta V \geq \delta(1 + |\nabla V|^2) - b$$

where $b := \delta + \sqrt{\delta} \sup_{\mathcal{X}} (\Delta V - (1 - 2\sqrt{\delta})|\nabla V|^2)$ is finite under our assumption (5.5). Thus (H) is satisfied with $\phi = \delta(1 + |\nabla V|^2)$ which is in $L^2(\mu)$. On the other hand our assumptions imply that ϕ tends to infinity at infinity. Hence $(1-\mathcal{L})^{-1}$ is compact on $L^2(\mu)$ and $c_P(\mu) < \infty$. The statement now follows directly from Theorem 5.1. \square

In the example below the constant C may change from one place to other.

Example 5.5. Let $\mathcal{X} = \mathbb{R}^n$, $V(x) = a|x|^\beta$ where $\beta > 1, a > 0$. Then (5.5) is verified with $\gamma = 0$ and then (H) is satisfied with $\phi = \delta(1 + |\nabla V|^2) \sim C|x|^{2(\beta-1)}$ when $|x|$ large (though V is not C^2 at $x = 0$, one can choose $U = e^{\lambda \tilde{V}}$ where $\tilde{V} \in C^2(\mathbb{R}^n)$ and $\tilde{V}(x) = V(x)$ for $|x| > 1$ in the proof of Corollary 5.4), where $0 < \delta < 1/4$.

- (i) If $\beta \geq 3/2$, then the condition in Corollary 5.4 is verified with $p = 2(\beta - 1) \geq 1$, so we have (5.7) for Lipschitz observable u with $p = 2(\beta - 1)$ (we can prove that (5.7) is false once $p > 2(\beta - 1)$). Then we have Gaussian behavior for small r , and even a super-Gaussian tail for large r whenever $\beta > 2$.
- (ii) Let $\beta \in (1, 3/2)$. Then for $\psi = (1 + |x|)^{\beta-1}$, we have by Theorem 5.1(5.1), $\|\psi(\nu - \mu)\|_{TV}^2 \leq CI(\nu|\mu)$. Then the Gaussian deviation inequality holds true for the observable u satisfying $|u| \leq C(1 + |x|)^{\beta-1}$.
- (iii) Let $\beta = 2$ (Ornstein-Uhlenbeck process). (5.1) holds with $\phi(x) = \delta(1 + |x|^2)$ and so does (5.3) with $p = 2$. They are both correct in order. Indeed for any $p \in [1, +\infty)$ fixed, if $\psi(x) \gg |x|^{2/p} \sim C\phi^{1/p}$ at infinity with $\mu(\psi) < +\infty$, one cannot hope that

$$\alpha_{p,R}(\|\psi(\nu - \mu)\|_{TV}) \leq I(\nu|\mu), \quad \forall \nu$$

for some $R > 0$, since by Theorem 2.4, this would imply that

$$\mathbb{P}_\mu \left(\int_0^1 \psi(X_s) ds > \mu(\psi) + r \right) \leq e^{-\alpha_{p,R}(r)}, \quad \forall r > 0$$

which is impossible for large r .

For this example and for the empirical mean with unbounded u , only Gaussian concentration inequality is known and that is only in the case $\beta = 2$ (cf. Djellout and al. [15]).

5.2. Proof of Theorem 5.1. The starting point is the following large deviation result.

Lemma 5.6. *For every continuous function $U \geq 1$ in $\mathbb{D}_e(\mathcal{L})$ such that $-\mathcal{L}U/U$ is μ -a.e. lower bounded,*

$$\int -\frac{\mathcal{L}U}{U} g^2 d\mu \leq \mathcal{E}(g, g), \quad \forall g \in \mathbb{D}(\mathcal{E}). \quad (5.8)$$

When U is bounded, this is contained in Deuschel-Stroock [14, Lemme 4.2.35].

Proof. For any initial law β ,

$$N_t = U(X_t) \exp \left(- \int_0^t \frac{\mathcal{L}U}{U}(X_s) ds \right)$$

is a local \mathbb{P}_β -martingale. Indeed, denoting $A_t := \exp \left(- \int_0^t \frac{\mathcal{L}U}{U}(X_s) ds \right)$, Itô's formula is $dN_t = A_t [dM_t(U) + \mathcal{L}U(X_t) dt] - \frac{\mathcal{L}U}{U}(X_t) A_t U(X_t) dt = A_t dM_t(U)$ where $M(U)$ is a local \mathbb{P}_β -martingale. As (N_t) is nonnegative, it is also a \mathbb{P}_β -supermartingale. Choosing $\beta := U^{-1} \mu/Z$ with $0 < Z = \mu(U^{-1}) \leq 1$, one sees that for all $t \geq 0$

$$\mathbb{E}_\beta \exp \left(- \int_0^t \frac{\mathcal{L}U}{U}(X_s) ds \right) \leq \mathbb{E}_\beta N_t \leq \beta(U) = 1/Z < +\infty.$$

Let $u_n := \min\{-\mathcal{L}U/U, n\}$. The previous estimation implies that

$$F(u_n) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_\beta \exp \left(\int_0^t u_n(X_s) ds \right) \leq 0.$$

On the other hand by the lower bound of large deviation in [46, Theorem B.1, Corollary B.11] and Laplace-Varadhan principle, as in the proof of (c') \Rightarrow (a) in Theorem 2.4, $F(u_n) \geq \sup\{\nu(u_n) - I(\nu|\mu); \nu \in M_1(E)\}$. Thus $\int u_n d\nu \leq I(\nu|\mu)$, which yields (by letting $n \rightarrow \infty$ and monotone convergence)

$$\int -\frac{\mathcal{L}U}{U} d\nu \leq I(\nu|\mu), \quad \forall \nu \in M_1(E). \quad (5.9)$$

This is equivalent to (5.8) by the fact that $\mathcal{E}(|h|, |h|) \leq \mathcal{E}(h, h)$ for all $h \in \mathbb{D}(\mathcal{E})$.

Note that one was allowed to apply the large deviation lower bound [46, Theorem B.1] under \mathbb{P}_β since β is absolutely continuous with respect to μ . In addition, in the symmetric case, [46, Corollary B.11] states that the large deviation rate function is $I(\cdot|\mu)$; it doesn't depend on β under the underlying assumption that \mathbb{P}_μ is ergodic. As this lower bound holds for the topology of probability measures weakened by all bounded measurable test functions (sometimes called τ -topology), one can apply the Laplace-Varadhan principle to the continuous bounded function $\nu \mapsto \nu(u_n)$. \square

Lemma 5.7. *In the framework of Theorem 5.1, for all $a \geq 2$ and $\nu \in M_1(\mathcal{X})$,*

$$\|\phi(\nu - \mu)\|_{TV} \leq (1 + 2bc_P) \frac{a+1}{a-1} I(\nu|\mu) + a\sqrt{2}\|\phi\|_2 \sqrt{c_P I(\nu|\mu)}. \quad (5.10)$$

Proof. We may assume that $\nu = f\mu$ with $\sqrt{f} \in \mathbb{D}(\mathcal{E})$ (trivial otherwise). For any $a \geq 2$, define $h : \mathbb{R} \rightarrow \mathbb{R}^+$ by

$$h(t) = \begin{cases} 0 & \text{if } t \leq 1; \\ \sqrt{\frac{a+1}{a-1}}(t-1) & \text{if } t \in [1, a]; \\ \sqrt{t^2-1} & \text{if } t \geq a. \end{cases}$$

It is easy to see that $\|h\|_{\text{Lip}} \leq \sqrt{\frac{a+1}{a-1}}$. Decompose

$$\|\phi(\nu - \mu)\|_{TV} = \int \phi|f-1|d\mu = \int \phi h^2(\sqrt{f})d\mu + \int \phi[|f-1| - h^2(\sqrt{f})]d\mu.$$

First consider the last term. Since $t^2-1-h^2(t) \leq a(t-1)$ for $t \in [1, a]$, and $= 0$ for $t \geq a \geq 2$, $\int \phi[|f-1| - h^2(\sqrt{f})]d\mu = \int \phi[1_{\{f \leq 1\}}(1-f) + 1_{\{1 \leq f \leq a^2\}}a(\sqrt{f}-1)]d\mu \leq a \int \phi|1-\sqrt{f}|d\mu$ which is not greater than $a\|\phi\|_2\|1-\sqrt{f}\|_2 = a\|\phi\|_2\sqrt{2}\sqrt{1-\mu(\sqrt{f})} \leq a\|\phi\|_2\sqrt{2\text{Var}_\mu(\sqrt{f})} \leq a\sqrt{2c_P}\|\phi\|_2\sqrt{I(\nu|\mu)}$.

We turn now to bound the crucial first term by means of (5.8):

$$\begin{aligned} \int \phi h^2(\sqrt{f})d\mu &\leq \int \left(-\frac{\mathcal{L}U}{U} + b\right) h^2(\sqrt{f})d\mu \\ &\leq \mathcal{E}(h(\sqrt{f}), h(\sqrt{f})) + b\|h\|_{\text{Lip}}^2 \int (\sqrt{f}-1)^2 d\mu \\ &\leq \|h\|_{\text{Lip}}^2 \mathcal{E}(\sqrt{f}, \sqrt{f}) + 2b\|h\|_{\text{Lip}}^2 \text{Var}_\mu(\sqrt{f}) \\ &\leq (1 + 2bc_P) \frac{a+1}{a-1} I(\nu|\mu). \end{aligned}$$

Substituting these two estimates into our previous decomposition, we obtain (5.10). \square

Proof of Theorem 5.1. As noticed in Remarks 5.2, (5.3) follows directly from (5.1). Note that if $p = 1$, (5.1) is a direct consequence of Lemma 5.7 (5.10) with $a = 2$. It remains to

show (5.1) in the case $p > 1$. Noting that $\int |f - 1| d\mu \leq 2 \min\{1, \sqrt{c_P I(\nu|\mu)}\}$ by Theorem 3.1, we have by Hölder's inequality and (5.10) with $a = 2$,

$$\begin{aligned} \|\phi^{1/p}(\nu - \mu)\|_{TV} &\leq \left(\int |f - 1| d\mu \right)^{1/q} \left(\int \phi |f - 1| d\mu \right)^{1/p} \\ &\leq 2^{1/q} \min\{1, \sqrt{c_P I(\nu|\mu)}\}^{1/q} \left(3(1 + 2bc_P) I(\nu|\mu) + 2\sqrt{2} \|\phi\|_2 \sqrt{c_P I(\nu|\mu)} \right)^{1/p} \end{aligned}$$

which entails immediately the desired (5.1). \square

6. PROOF OF THEOREM 2.4

The proof of this result is similar to [24, Theorems 2 and 15]'s ones. It takes advantage of large deviation results previously obtained by L. Wu. Namely,

- the identification of the rate function in the symmetric case and the large deviation lower bound are taken from [46] and
- the non-asymptotic Cramér's upper bounds which are used in [24] are replaced by the following result.

Lemma 6.1 (L. Wu [45]). *For any $u \in b\mathcal{B}$ with $\mu(|u|) < +\infty$ and any $t > 0$, the following statements hold true.*

(1) Denoting

$$\Lambda(u) := \sup \left\{ \int u g^2 d\mu - \mathcal{E}(g, g); g \in \mathbb{D}(\mathcal{E}), \mu(g^2) = 1, \mu(g^2|u|) < +\infty \right\}, \quad (6.1)$$

one has

$$\|P_t^u\|_{L^2(\mu)} \leq e^{t\Lambda(u)} \quad (6.2)$$

and the equality holds in the symmetric case;

(2) For all $r > 0$,

$$\mathbb{P}_\beta \left(\frac{1}{t} \int_0^t u(X_s) ds - \mu(u) \geq r \right) \leq \left\| \frac{d\beta}{d\mu} \right\|_2 \exp \left(-t \lim_{\delta \downarrow 0} I_u(\mu(u) + r - \delta) \right) \quad (6.3)$$

where $I_u(r) := \inf \{I(\nu|\mu); \nu \in M_1(\mathcal{X}), \nu(u) = r\}$, $r \in \mathbb{R}$.

It is proved in [45, 46] that in the symmetric case, $I_u(r)$ is exactly the rate function governing the large deviation principle of $\frac{1}{t} \int_0^t u(X_s) ds$ for bounded u . In these papers no mixing assumptions are required, this is in contrast with the usual assumptions for the large deviation principle as discovered by Donsker and Varadhan [17] and reconsidered by Deuschel and Stroock [14]. This relaxation of the usual assumptions is allowed by the assumed restriction that the initial law is absolutely continuous with respect to the ergodic measure μ .

Proof of Theorem 2.4. Part (1). As $\nu \rightarrow I(\nu|\mu)$ is convex on $M_1(\mathcal{X})$, so is $I_u : \mathbb{R} \rightarrow [0, +\infty]$. Since $I_u(\mu(u)) = 0$, I_u is increasing on $[\mu(u), +\infty)$. For all $(u, v) \in \Phi$ and all $\lambda \geq 0$, we have

$$\Lambda(\lambda u) = I_u^*(\lambda) \quad (6.4)$$

where I_u^* is the convex conjugate of I_u . Indeed for $\lambda \geq 0$, by (6.1)

$$\begin{aligned} \Lambda(\lambda u) &= \sup\{\lambda \int u g^2 d\mu - \mathcal{E}(g, g); g \in \mathbb{D}(\mathcal{E}), \mu(g^2) = 1\} \\ &= \sup\{\lambda \int u g^2 d\mu - \mathcal{E}(g, g); 0 \leq g \in \mathbb{D}(\mathcal{E}), \mu(g^2) = 1\} \\ &= \sup\{\lambda \int u d\nu - I(\nu|\mu); \nu \in M_1(\mathcal{X})\} = \sup_{a \in \mathbb{R}}\{\lambda a - I_u(a)\} \end{aligned}$$

where the second equality follows from the fact that $\mathcal{E}(|g|, |g|) \leq \mathcal{E}(g, g)$ for all $g \in \mathbb{D}(\mathcal{E})$.

Note also that $T_\Phi I$ implies that for any $(u, v) \in \Phi$,

$$I_u(\mu(v) + r) \geq \tilde{\alpha}(r), \quad \forall r \in \mathbb{R} \quad (6.5)$$

where $\tilde{\alpha}(r) = \alpha(r)$ for $r \geq 0$ and $= 0$ for $r \leq 0$. Indeed it is trivial for $r \leq 0$ and for any $r \geq 0$ and $\nu \in M_1(\mathcal{X})$ such that $\nu(u) = \mu(v) + r$, $T_\Phi I$ implies that

$$I(\nu|\mu) \geq \alpha(T_\Phi(\nu, \mu)) \geq \alpha(\nu(u) - \mu(v)) = \alpha(r).$$

- (a) \Rightarrow (b): Putting together (6.4) and (6.5) leads us to

$$\Lambda(\lambda u) = \sup_{a \in \mathbb{R}} [\lambda a - I_u(a)] \leq \sup_{r \in \mathbb{R}} [\lambda(\mu(v) + r) - \tilde{\alpha}(r)] = \lambda\mu(v) + \alpha^{\otimes}(\lambda)$$

for all $\lambda \geq 0$. Statement (b) now follows from inequality (6.2).

- (a) \Rightarrow (c): This follows from (6.3) and (6.5), noting that by (A1), $\mu(u) \leq \mu(v)$ for all $(u, v) \in \Phi$.
- (b) \Rightarrow (b') and (c) \Rightarrow (c'): These implications are trivial.

Part (2). (b) \Rightarrow (a) in the case where α is convex. By (2.6), we have for $(u, v) \in \Phi$ fixed and for any $g \in \mathbb{D}_2(\mathcal{L})$, $\langle P_t^{\lambda u} g, P_t^{\lambda v} g \rangle_\mu \leq e^{2t(\lambda\mu(v) + \alpha^{\otimes}(\lambda))} \langle g, g \rangle_\mu$. Differentiating at time zero we obtain $2\langle g, \mathcal{L}g + \lambda u g \rangle_\mu = 2(\lambda\mu(g^2 u) - \mathcal{E}(g, g)) \leq 2(\lambda\mu(v) + \alpha^{\otimes}(\lambda))\mu(g^2)$. Then for all $g \in \mathbb{D}_2(\mathcal{L})$, $\lambda[\mu(g^2 u) - \mu(v)\mu(g^2)] - \alpha^{\otimes}(\lambda)\mu(g^2) \leq \mathcal{E}(g, g)$. It can be extended to $g \in \mathcal{D}(\mathcal{E})$. Now for any $\nu \in M_1(\mathcal{X})$ such that $I(\nu|\mu) < +\infty$, applying the above inequality to $g = \sqrt{\frac{d\nu}{d\mu}}$, we get $\lambda[\nu(u) - \mu(v)] - \alpha^{\otimes}(\lambda) \leq I(\nu|\mu)$. Taking the supremum over all $\lambda \in \mathbb{R}$, as α assumed to be convex and $\alpha^{\otimes} = \tilde{\alpha}^*$ on $[0, \infty)$ (see the remark below (1.3)), we get

$$\tilde{\alpha}(\nu(u) - \mu(v)) \leq I(\nu|\mu)$$

and taking the supremum over all $(u, v) \in \Phi$ leads to the desired result.

Part (3). Let us assume from now on that the semigroup (P_t) is symmetric in $L^2(\mu)$.

- (c') \Rightarrow (a) : By the large deviation lower bound in [46, Theorem B.1] and the identification of the rate function in the symmetric case in [46, Corollary B.11], we have for any initial probability measure $\beta \ll \mu$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}_\beta \left(\frac{1}{t} \int_0^t u(X_s) ds \geq \mu(v) + r \right) \geq -\inf\{I(\nu|\mu); \nu(u) > \mu(v) + r\}.$$

This together with (c') implies that for any $r \geq 0$, $\inf\{I(\nu|\mu); \nu(u) > \mu(v) + r\} \geq \alpha(r)$. Fix now ν such that $r_0 = T_\Phi(\nu, \mu) > 0$ (otherwise $T_\Phi I$ is obviously true.) Choosing a sequence

$(u_n, v_n) \in \Phi$ so that $\nu(u_n) - \mu(v_n) > r_0 - 1/n$, for all large enough n ,

$$\alpha(r_0 - 1/n) \leq I(\nu|\mu)$$

where $T_\Phi I$ follows by letting $n \rightarrow \infty$ and by the left-continuity of α .

• α is convex and (P_t) is symmetric. $(b') \Rightarrow (c')$ with $\beta = \mu$: The proof is standard and consists in optimizing exponential upper bounds. So doing, one obtains by means of (b') the asymptotic upper bound (c') with the convex envelope of $\tilde{\alpha}$ instead of $\tilde{\alpha}$. As α is assumed to be convex, (c') is proved. This completes the proof of the theorem. \square

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