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Minimization of energy functionals applied to some inverse problems

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Abstract. We consider a general class of problems of minimization of convex integral functionals subject to linear constraints. Using Fenchel duality, we prove the equality of the values of the minimization problem and its associated dual problem. This equality is a variational criterion for the existence of solution to a large class of inverse problems entering the class of generalized Fredholm integral equations. In particular, our abstract results are applied to marginal problems for stochastic processes. Such problems naturally arise from the probabilistic approaches to quantum mechanics.

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1. Introduction

We consider the energy functionals defined on the space: $M(\Omega)$, of the signed measures on the measure space $(\Omega, \mathcal{A})$ which are of the following form

$$I(Q) = \int_\Omega \gamma^* \left(\frac{dQ}{dR}\right) dR \in [0, +\infty], \quad Q \in M(\Omega)$$

if Q is absolutely continuous with respect to a given nonnegative reference measure R, and $I(Q) = +\infty$ otherwise. The function $\gamma^* : \mathbb{R} \to [0, \infty]$ is the convex conjugate of a function γ, hence it is convex and closed. With the special choice: $\gamma(x) = e^x - x - 1$, we get $\gamma^*(x) = (x+1)\log(x+1) - x$ and $I(P - R)$ is the Kullback information of P with respect to R which is sometimes called the Boltzmann-Shannon entropy.

Let $A : M(\Omega) \to \mathcal{X}$ be a linear operator on $M(\Omega)$ with its values in a vector space $\mathcal{X}$; $A(Q) = x^o$ is the expression of a linear constraint. We are concerned with the minimum energy problem

$$\inf \{I(Q)\,;\, Q \in M(\Omega), A(Q) = x^o\}\,. \tag{1.1}$$

Notice that a solution to (1.1) is absolutely continuous with respect to R. Such a minimization problem is sometimes called a maximum entropy problem: $-I$ may be seen as an entropy.

The classical moment constraint: $\int_\Omega a_k\, dQ = x^o_k$, $a_k : \Omega \to \mathbb{R}$, $1 \le k \le n$ corresponds to $\mathcal{X} = \mathbb{R}^n$ and $A(Q) = \left(\int_\Omega a_k\, dQ\right)_{1\le k\le n}$. In this paper, we consider its infinite dimensional analogue which is formally specified by

$$\left(\int_\Omega a(t,\omega)\, Q(d\omega)\right)_{t\in T} := A(Q) = (x^o_t)_{t\in T} \in \mathcal{X} = X^T \tag{1.2}$$

where T is an index set, X is a vector space and a is an X-valued function defined on $T \times \Omega$. Seen as an equation in the variable Q where a and x^o are given parameters, (1.2) is a Fredholm integral equation. The formal Fenchel dual problem associated with (1.1) is

$$\sup \left\{ \langle y, x^o\rangle - \int_\Omega \gamma(A^* y)\, dR\,;\, y \in \mathcal{Y} \right\}$$

where $\mathcal{Y}$ is a vector space in duality with $\mathcal{X}$, A^* is the formal adjoint of A and $A^* y$ is a measurable function for any $y \in \mathcal{Y}$. In this paper, it is proved that, under some assumptions,

$$\inf \{I(Q)\,;\, Q \in M(\Omega), A(Q) = x^o\} = \sup_{y\in\mathcal{Y}} \left\{ \langle y, x^o\rangle - \int_\Omega \gamma(A^* y)\, dR \right\} \tag{1.3}$$

and that if this value is finite, the infimum is attained. As a consequence, we get a variational criterion for the existence of a solution to (1.2), under the additional constraint: $I(Q) < \infty$, which implies in particular that Q is absolutely continuous with respect to R.

It appears that when the domain of the function γ is a proper subset of $\mathbb{R}$, (1.3) doesn't hold anymore. This situation will also be taken into account; in order that a dual equality holds, one must extend the energy functional I from $M(\Omega)$ to some linear forms which are not measures anymore and are singular with respect to the reference measure R. This phenomenon is already known, see for instance [5] and [21].

In [6], it is proved that (1.3) holds when γ is finite everywhere and A is a continuous operator from $L^1(\Omega, R)$ to a normed space $\mathcal{X}$. Such an equality had already been obtained by R. T. Rockafellar ([35], Theorem 23) in the case where $f \mapsto \int_\Omega \gamma(f)\, dR$ is everywhere finite and continuous on some reflexive Orlicz space built on $(\Omega, \mathcal{A}, R)$.

In the present paper, using standard Fenchel duality techniques developped in [35], (1.3) (and its extension when γ achieves the value $+\infty$) together with the primal attainment are obtained, essentially assuming that for some vector space $\mathcal{Y}$ (dually paired with $\mathcal{X}$) which separates $\mathcal{X}$, the range $A^*\mathcal{Y}$ is a function space and

$$\forall y \in \mathcal{Y}, \exists \lambda > 0 \text{ such that } \int_\Omega \gamma(\lambda A^* y)\, dR + \int_\Omega \gamma(-\lambda A^* y)\, dR < \infty.$$

Let us note that a characterization of the primal solution is stated in [27] under the stronger assumption: $\forall y \in \mathcal{Y}, \int_\Omega \gamma(A^*y)\, dR < \infty$.

The marginal problems are of type (1.2). Let us first describe the simplest one in order to give an illustration. Considering a product measure space $\Omega = \Omega_1 \times \Omega_2$, one wishes to find some measures Q on Ω which are absolutely continuous with respect to R and whose marginals Q_1 on Ω_1 and Q_2 on Ω_2 are given. The constraint is $Q_1 = \nu_1 \in M(\Omega_1)$, $Q_2 = \nu_2 \in M(\Omega_2)$ which corresponds to $A(Q) = (Q_1, Q_2) \in \mathcal{X} = M(\Omega_1) \times M(\Omega_2)$. Marginal problems of type (1.1) are considered in [10], [4] and [9]. See also [3] for a close related problem.

This problem can be extended with an infinite number of marginals. Let E be a state space and let $\Omega = E^{[0,1]}$ be the space of the E-valued paths $\omega = (\omega_t)_{0\leq t\leq 1}$. If Q is a probability measure on Ω, let $Q_t \in M(E)$ stand for its marginal at time t : the law of ω_t under Q. The problem can be stated as follows:

(1.4) "Does there exist $Q \in M_1(E^{[0,1]})$ such that $Q \ll R$ and $Q_t = \nu_t, \forall 0 \leq t \leq 1$?"

The constraint $Q_t = \nu_t, \forall 0 \leq t \leq 1$, where $(\nu_t)_{0\leq t\leq 1}$ is a given flow of probability measures on E, corresponds to $A(Q) = (Q_t)_{0\leq t\leq 1} \in \mathcal{X} = M(E)^{[0,1]}$. Such problems of reconstruction of absolutely continuous (w.r.t. R) laws with given marginal flows appear naturally in the statistical mechanics of large dynamical particle systems (see [14], [19], [7]).

If $(\nu_t)_{t\in[0,1]}$ is given by the square modulus of a solution to the Schrödinger equation and R is the Wiener measure, (1.4) is a problem which was addressed by E. Nelson (see [31]). A positive answer to this question is fundamental for the construction of stochastic mechanics. A related problem has first been solved with a positive answer by E. Carlen in [8], using partial differential equation techniques.

The intermediate problem where the only initial and final marginal laws are constrained: $Q_0 = \nu_0$, $Q_1 = \nu_1$ is related to the construction of Schrödinger bridges (see [1], [13],[17], [18], [19], [30],[37]). It is also a Fredholm equation of type (1.2).

In the above reconstruction problems motivated by physical questions, the relevant energy functional is the Kullback information. In [7], [9],[14], [15], [19] and [21], the dual equalities are by-products of large deviation principles (see [16]).

In Section 2, an abstract dual equality is derived along the usual lines of the Fenchel duality theory (see [35]). Some basic estimates which have already been used in [25] and [26] are also needed.

In Section 3, this abstract equality is applied to energy functionals. Orlicz spaces are very well designed for this purpose.

In Section 4, our main general result is stated in Theorem 4.1. It is a direct corollary of the main results of Sections 2 and 3.

Then, Theorem 4.1 is applied to typical Fredholm equations in Section 5. In particular, an example of constrained diffusion process is developed.

Finally, in Section 6, we investigate several marginal problems for stochastic processes, which have the flavour of the question (1.4).

2. An abstract dual equality

All the vector spaces are real. Let U be a vector space and $\Phi : U \to [0,\infty]$ be a real extended function which satisfies

$$\Phi \text{ is convex}, \quad \Phi(U) \subset [0,\infty] \quad \text{and} \quad \Phi(0) = 0. \tag{2.1}$$

Let U^* be the algebraic dual space of U. We define the convex conjugate of Φ for the duality (U, U^*) by:

$$\Phi^* : \ell \in U^* \mapsto \sup_{u \in U} \{\langle \ell, u\rangle - \Phi(u)\} \in [0,\infty].$$

It is easy to check that (2.1) also holds for Φ^*.

Assumption (2.1) is a normalization condition: it doesn't imply a great loss of generality. Indeed, let Φ be bounded below by an affine function and $\Phi(0) < \infty$. Its convex envelope: $\mathrm{cv}(\Phi)$, satisfies $\partial \mathrm{cv}(\Phi)(0) \neq \emptyset$. Let us define $\Phi_o(u) = \mathrm{cv}(\Phi)(u) - [\langle \ell_o, u\rangle + \mathrm{cv}(\Phi)(0)]$ with ℓ_o in $\partial \mathrm{cv}(\Phi)(0)$. Then, Φ_o satisfies (2.1) and $\Phi^*(\ell) = {\Phi^*}_o(\ell - \ell_o) - \mathrm{cv}(\Phi)(0)$.

In order to state the linear constraints on U^*, let us introduce the linear operator

$$A : \ell \in U^* \mapsto A\ell \in \mathcal{X}$$

on U^* with its values in the algebraic dual space $\mathcal{X} = \mathcal{Y}^*$ of some vector space $\mathcal{Y}$. We consider the (primal) minimization problem

$$\inf \{\Phi^*(\ell)\,;\, \ell \in U^*, A\ell = x^o\} \tag{P}$$

where $x^o \in \mathcal{X}$ specifies the linear constraint. Let us state now two regularity assumptions on Φ :

(2.2) Φ is $\sigma(U, U^*)$-lower semicontinuous,

(2.3) Φ^* has $\sigma(U^*, U)$-compact level sets.

Notice that without any assumptions, Φ^* is $\sigma(U^*, U)$-lower semicontinuous: it admits $\sigma(U^*, U)$-closed level sets. In order to state the dual problem associated with (P), let us introduce the adjoint operator of A :

$$A^* : y \in \mathcal{Y} \mapsto A^* y \in U^{**}$$

which is defined on $\mathcal{Y}$, with its values in the algebraic bidual U^{**} of U, for any $y \in \mathcal{Y}$, by: $\langle A^* y, \ell\rangle_{U^{**}, U^*} = \langle y, A\ell\rangle_{\mathcal{Y},\mathcal{X}}$. The canonical imbedding $U \subset U^{**}$ is done. The following assumption on the regularity of A :

$$A^*(\mathcal{Y}) \subset U \tag{2.4}$$

will appear to be crucial for our results. The Fenchel dual problem associated with (P):

$$\sup \{\langle y, x^o\rangle - \Phi(A^* y)\,;\, y \in \mathcal{Y}\} \tag{D}$$

is meaningful if (2.4) holds. The aim of this section is to prove in Theorem 2.3 the dual equality: $\inf(\mathrm{P}) = \sup(\mathrm{D})$, under the assumptions (2.1), (2.2), (2.3) and (2.4).

Let us first establish some preliminary results in Lemmas 2.1 and 2.2. We introduce the *perturbation* $F : U^* \times \mathcal{X} \to [0, \infty]$ defined for all $\ell \in U^*, x \in \mathcal{X}$ by

$$F(\ell, x) := \Phi^*(\ell) + \delta(A\ell + x \mid x^o)$$

where $\delta(x \mid x^o) = \begin{cases} 0 & \text{if } x = x^o \\ +\infty & \text{if } x \neq x^o \end{cases}$ is the convex indicator of x^o. We also define the *optimal value function* $\theta : \mathcal{X} \to [0, \infty]$, for any $x \in \mathcal{X}$, by

$$\theta(x) := \inf_{\ell \in U^*} F(\ell, x) = \inf\{\Phi^*(\ell)\,;\, \ell \in U^*, A\ell = x^o - x\}$$

so that (P) is $\inf\{F(\ell, 0)\,;\, \ell \in U^*\}$ and its value is $\inf(\mathrm{P}) = \theta(0)$.

Lemma 2.1. *If (2.1), (2.3) and (2.4) hold, then θ is a $\sigma(\mathcal{X}, \mathcal{Y})$-closed convex function.*

<u>Proof</u>. The assumption (2.4) implies that A is $\sigma(U^*, U)$-$\sigma(\mathcal{X}, \mathcal{Y})$-continuous. Indeed, for any $y \in \mathcal{Y}$, the function $\ell \in U^* \mapsto \langle y, A\ell\rangle = \langle \ell, A^* y\rangle$ is $\sigma(U^*, U)$-continuous, since $A^* y \in U$.

But, by (2.3), Φ^* has $\sigma(U^*, U)$-compact level sets and we have just seen that $\ell \mapsto x^o - A\ell$ is $\sigma(U^*, U)$-$\sigma(\mathcal{X}, \mathcal{Y})$-continuous. Hence, the function $x \in \mathcal{X} \mapsto \theta(x) = \inf\{\Phi^*(\ell)\,;\, x^o - A\ell = x\}$ has $\sigma(\mathcal{X}, \mathcal{Y})$-compact level sets. For this standard argument, see for instance ([16], Theorem 4.2.1.(a)). In particular, θ is $\sigma(\mathcal{X}, \mathcal{Y})$-lower semicontinuous.

On the other hand, as F is convex on $U^* \times \mathcal{X}$, $x \mapsto \theta(x) = \inf_{\ell \in U^*} F(\ell, x)$ is also convex (see [35], Theorem 1).

Finally, as $\Phi^* \geq 0$ (by (2.1)), we have $\theta \geq 0 : \theta$ is bounded below by an affine function, and since $\theta(x^o) = 0 : \theta$ is not identically equal to $+\infty$. This completes the proof of the lemma. ■

Now, let us introduce the *Lagrangian* $K : U^* \times \mathcal{Y} \to]-\infty, \infty]$, which is defined for any $\ell \in U^*, y \in \mathcal{Y}$, by

$$\begin{aligned} K(\ell, y) &:= \inf_{x \in \mathcal{X}} \{\langle y, x\rangle + F(\ell, x)\} \\ &= \Phi^*(\ell) + \langle y, x^o\rangle - \langle A\ell, y\rangle \end{aligned}$$

and the *dual objective function* $g : \mathcal{Y} \to [-\infty, \infty[$, defined for any $y \in \mathcal{Y}$, by

$$g(y) := \inf_{\ell \in U^*} K(\ell, y).$$

In the general theory (see [35]), the dual problem is $\sup\{g(y)\,;\, y \in \mathcal{Y}\}$. Taking the following Lemma 2.2 into account, its expression is given by (D) above.

Lemma 2.2. *If (2.1), (2.2) and (2.4) hold, then*

$$g(y) = \langle y, x^o\rangle - \Phi(A^* y), \quad y \in \mathcal{Y}.$$

<u>Proof</u>. For any $y \in \mathcal{Y}$,

$$\begin{aligned} \inf_{\ell \in U^*} K(\ell, y) &= \langle y, x^o\rangle - \sup_{\ell \in U^*} \{\langle \ell, A^* y\rangle - \Phi^*(\ell)\} \\ &= \langle y, x^o\rangle - \Phi^{**}(A^* y) \end{aligned}$$

with $A^*y \in U$ (by (2.4)), where $\Phi^{**} : U \to [0, \infty]$ is the convex conjugate of Φ^* for the duality (U, U^*). As Φ is bounded below by an affine function ($\Phi \geq 0$), Φ^{**} is the convex $\sigma(U, U^*)$-lower semicontinuous regularized of Φ. Because of the assumptions (2.1) and (2.2), Φ is closed. Therefore, we have: $\Phi^{**} = \Phi$, which completes the proof of the lemma. ■

We are now ready to state the main result of the section.

Theorem 2.3. *We assume (2.1), (2.2), (2.3) and (2.4). The dual equality*

$$\inf\{\Phi^*(\ell)\,;\, \ell \in U^*, A\ell = x^o\} = \sup_{y \in \mathcal{Y}}\{\langle y, x^o\rangle - \Phi(A^*y)\}$$

holds. If this value is finite, then (P) admits at least a solution.

Proof. The last statement is a direct consequence of (2.3).
Let us prove inf (P) = sup (D). For any $y \in \mathcal{Y}$,

$$\begin{aligned} g(y) &= \inf\{\langle y, x\rangle + F(\ell, x)\,;\, \ell \in U^*, x \in \mathcal{X}\} \\ &= \inf_{x \in \mathcal{X}}\{\langle y, x\rangle + \theta(x)\} \\ &= (-\theta)^*(y) \qquad\qquad \text{(concave conjugate of } -\theta). \end{aligned}$$

It follows that $-g^* = \theta^{**}$, where g^* is the concave conjugate of g and θ^{**} is the convex biconjugate of θ for the duality $(\mathcal{X}, \mathcal{Y})$. But, by Lemma 2.1, θ is a $\sigma(\mathcal{X}, \mathcal{Y})$-closed convex function, so that: $\theta^{**} = \theta$. In particular, for $x = 0$ we get: $-g^*(0) = \theta(0)$. This means that

$$-\inf_{y \in \mathcal{Y}}\{\langle y, 0\rangle - g(y)\} = \inf\{\Phi^*(\ell)\,;\, A\ell = x^o - 0\}$$

which, taking Lemma 2.2 into account, is the desired dual equality. ■

3. A dual equality for energy functionals

In the present section, we are going to apply the results of Section 2 to the case where Φ^* is an energy functional which is the convex conjugate of an integral functional Φ.

Let $(\Omega, \mathcal{A})$ be a measure space and R be a σ-finite measure on $\mathcal{A}$. Let $\gamma : \mathbb{R} \to [0, \infty]$ be a function which satisfies

(3.1) γ is convex, lower semicontinuous, $\gamma(\mathbb{R}) \subset [0, +\infty]$, $\gamma(0) = 0$, γ is not identically equal to zero and $]-\alpha, +\alpha[\subset$ *dom* γ for some $\alpha > 0$.

In the sequel, R-a.e. equal functions are identified. We define the function space

(3.2) $$U := \Big\{u : \Omega \to \mathbb{R}\,;\ \text{measurable such that there exists } \lambda > 0 \text{ with } \int_\Omega \gamma(\lambda u)\, dR + \int_\Omega \gamma(-\lambda u)\, dR < \infty\Big\}$$

Let us denote

$$\gamma_o(t) = \max(\gamma(t), \gamma(-t)),\ t \in \mathbb{R}.$$

As it is a Young function, one can define the Orlicz space $L_{\gamma_o} = L_{\gamma_o}(\Omega, \mathcal{A}, R)$ by

$$L_{\gamma_o} = \{u : \Omega \to \mathbb{R};\ \text{measurable such that } \|u\|_{\gamma_o} < \infty\} \quad \text{with}$$
$$\|u\|_{\gamma_o} = \inf\left\{a > 0\,;\ \int_\Omega \gamma_o\left(\frac{|u|}{a}\right) dR \le 1\right\}$$

We have

$$U = L_{\gamma_o}. \tag{3.3}$$

It is assumed that 0 stands in the interior of *dom* γ to avoid the trivial situation: $U = \{0\}$.
In contrast with L_{γ_o}, its subspace

$$M_{\gamma_o} = \left\{u : \Omega \to \mathbb{R};\ \text{measurable such that: } \forall\lambda \ge 0, \int_\Omega \gamma_o(\lambda|u|)\, dR < \infty\right\} \tag{3.4}$$

will be considered in the proof of Theorem 3.4 below. It is a vector subspace of L_{γ_o}. The spaces $L_{\gamma_o}$and M_{γ_o} are endowed with the Orlicz norm $\|\cdot\|_{\gamma_o}$. Notice that if γ_o does not satisfy the Δ_2-condition (it is the case when γ is given by (5.5)), it may happen that M_{γ_o} is a proper subset of L_{γ_o}. Let us define

$$\Phi : u \in U \mapsto \int_\Omega \gamma(u)\, dR \in [0, \infty]. \tag{3.5}$$

It is a convex integral functional and γ is a normal integrand in the sense of Rockafellar (see [33], [34]).

Lemma 3.1. *With U and Φ defined by (3.2) and (3.5), under the assumption (3.1), the conditions (2.1), (2.2) and (2.3) hold.*

Proof. Clearly, (2.1) is true.

Let us show (2.2). Let γ_o^* be the convex conjugate of γ_o and $L_{\gamma_o^*}$ be the associated Orlicz space. By ([33], Theorem 2), the functionals $v \in L_{\gamma_o^*} \mapsto \int_\Omega \gamma^*(v)\, dR \in [0, \infty]$ and $\Phi : u \in L_{\gamma_o} \mapsto \int_\Omega \gamma(u)\, dR \in [0, \infty]$ are convex conjugates to each other. In particular, Φ is lower semicontinuous for the topology $\sigma(L_{\gamma_o}, L_{\gamma_o^*})$ and a fortiori for $\sigma(L_{\gamma_o}, U^*)$, since $L_{\gamma_o^*} \subset U^*$.

Let us show (2.3). As $\sup\{\Phi(u)\,;\ \|u\|_{\gamma_o} \le \frac{1}{2}\} \le 1$, one can apply ([26], Lemma 2.1 and Corollary 2.2) and (3.3) to deduce that

$$dom\ \Phi^* \subset L'_{\gamma_o} \tag{3.6}$$

and that Φ^* has $\sigma(U^*, U)$-compact level sets. ■

Now, we wish to give an explicit expression for Φ^*. In order to do this, the inclusion (3.6) suggests us to state a representation of L'_{γ_o}. We recall this description which has been derived in ([28], Theorem

4.7), see also ([24], Theorem 2.2) in case γ is finite. Any continuous linear form ℓ on L_{γ_o} is relatively bounded. Hence, it admits a decomposition in nonnegative and nonpositive parts: $\ell = \ell_+ - \ell_-$ with $\ell_+ = \ell \vee 0$ and $\ell_- = (-\ell) \vee 0$ (for the natural order relation on the dual of an ordered space). We define $|\ell| = \ell_+ + \ell_-$. With the usual integral duality bracket, Hölder's inequality in Orlicz spaces, yields: $L_{\gamma_o^*} \subset L'_{\gamma_o}$. As a definition, a form $\ell \in L'_{\gamma_o}$ is said to be singular with respect to $L_{\gamma_o^*}$, if: $|\ell| \wedge v = 0, \forall v \in L_{\gamma_o^*}, v \geq 0$. Let us denote $L^{\mathrm{s}}_{\gamma_o}$ the space of all these singular forms. Then, any $\ell \in L'_{\gamma_o}$ is uniquely decomposed as

$$\ell = \ell^{\mathrm{ac}} + \ell^{\mathrm{s}} \tag{3.7}$$

with $\ell^{\mathrm{ac}} \in L_{\gamma_o^*}$ and $\ell^{\mathrm{s}} \in L^{\mathrm{s}}_{\gamma_o}$. In other words, L'_{γ_o} is the direct sum

$$L'_{\gamma_o} = L_{\gamma_o^*} \oplus L^{\mathrm{s}}_{\gamma_o}.$$

In ([28], Propositon 4.6), it is proved that any singular form with respect to $L_{\gamma_o^*}$ is singular with respect to R, according to the following definition.

A relatively bounded linear form: ℓ, on L_{γ_o} is said to be singular with respect to R if there exists an R-localizing sequence $(\Omega_p)_{p\geq 1}$ and a nonincreasing sequence $(A_k)_{k\geq 1}$ in $\mathcal{A}$ such that:

$$\lim_{k\to\infty} R(A_k) = 0 \quad \text{and} \quad \langle |\ell|, \mathbb{1}_{(\Omega_p \setminus A_k)} \rangle = 0,\ \forall p, k \geq 1.$$

In the above definition, a sequence $(\Omega_p)_{p\geq 1}$ in $\mathcal{A}$ was said to be R-localizing if it is nondecreasing, $\bigcup_{p\geq 1} \Omega_p = \Omega$ and $R(\Omega_p) < \infty,\ \forall p \geq 1$. As a definition, since R is σ-finite, such a sequence exists. If R is bounded, ℓ is singular with respect to R if and only if the above conditions are satisfied with $\Omega_p = \Omega,\ \forall p \geq 1$.

We adopt the excessive integral notation: $\langle \ell, u \rangle_{L'_{\gamma_o}, L_{\gamma_o}} = \int_\Omega u\, d\ell$. Incidentally, one can make this definition meaningful: ℓ can be represented as an additive set function on $\mathcal{A}$ (not necessarily σ-additive) (see [32]). We also denote $\ell^{\mathrm{ac}} = \frac{d\ell^{\mathrm{ac}}}{dR} \cdot R$ with $\frac{d\ell^{\mathrm{ac}}}{dR} \in L_{\gamma_o^*}$.

Proposition 3.2. *We assume (3.1). The convex conjugate Φ^* of Φ for the duality (U, U^*), when U and Φ are specified by (3.2) and (3.5), is given for any $\ell \in U^*$, by*

$$\Phi^*(\ell) = \begin{cases} \int_\Omega \gamma^* \left(\frac{d\ell^{\mathrm{ac}}}{dR}\right) dR + \sup\{\langle \ell^{\mathrm{s}}, u\rangle\, ;\, u \in U, \Phi(u) < \infty\} & \text{if } \ell \in L'_{\gamma_o} \\ +\infty & \text{otherwise} \end{cases}$$

where $\ell \in L'_{\gamma_o}$ is decomposed as in (3.7).

Proof. See ([28], Proposition 5.1). See also ([24], Theorem 2.6) in case γ is finite. ■

Let us describe now, the linear constraints. We take a function $\varphi : \Omega \to \mathcal{X}$ on Ω with its values in the algebraic dual space $\mathcal{X}$ of a vector space $\mathcal{Y}$. We also suppose that

(3.8) for all $y \in \mathcal{Y}$, $\omega \in \Omega \mapsto \langle y, \varphi(\omega)\rangle \in \mathbb{R}$ is measurable and there exists $\lambda > 0$ such that
$$\int_\Omega \gamma\Big(\lambda \langle y, \varphi(\omega)\rangle\Big)\, R(d\omega) + \int_\Omega \gamma\Big(-\lambda\langle y, \varphi(\omega)\rangle\Big)\, R(d\omega) < \infty.$$

In other words: $\forall y \in \mathcal{Y}, \langle y, \varphi(\cdot)\rangle \in L_{\gamma_o} = U$. This allows us to define the linear operator $A : U^* \to \mathcal{X}$, for all $\ell \in U^*$, by

$$\text{(3.9.a)} \qquad \langle y, A\ell\rangle_{\mathcal{Y},\mathcal{X}} := \Big\langle \ell, \langle y, \varphi(\cdot)\rangle \Big\rangle_{U^*,U}, \forall y \in \mathcal{Y}$$

that is: $A^*y = \langle y, \varphi(\cdot)\rangle \in U, \forall y \in \mathcal{Y}$, which is (2.4). To signify (3.9.a), we denote

$$\text{(3.9.b)} \qquad A\ell = \int_\Omega \varphi\, d\ell$$

(3.10) Remark. Any linear operator $A : U^* \mapsto \mathcal{X}$ such that the range $A^*\mathcal{Y}$ is included in the space $\mathcal{M}$ of the measurable functions on Ω (without R-a.e. equality) can be written in the form (3.9). Indeed, in this situation, the Dirac measures $\delta_\omega, \omega \in \Omega$, act on $\mathcal{M}$ and for any $y \in \mathcal{Y}$, one gets: $A^*y(\omega) = \langle A^*y, \delta_\omega\rangle_{\mathcal{M},\mathcal{M}^*} = \langle y, B(\delta_\omega)\rangle_{\mathcal{Y},\mathcal{X}}$ where $B : \mathcal{M}^* \to \mathcal{X}$ is the adjoint of $A^* : \mathcal{Y} \to \mathcal{M}$ and it is sufficient to take $\varphi(\omega) = B(\delta_\omega), \omega \in \Omega$.

We have just checked that with U and Φ given by (3.2) and (3.5), the assumptions of Theorem 2.3 are satisfied. Taking Proposition 3.2 into account, we have proved the following result.

Theorem 3.3. *We assume that γ satisfies (3.1) and that φ satisfies (3.8). Then, for any $x^o \in \mathcal{X}$, we have*

$$\begin{aligned}
&\inf\Big\{\int_\Omega \gamma^*\Big(\frac{d\ell^{\mathrm{ac}}}{dR}\Big)\, dR + \sup\Big\{\langle \ell^{\mathrm{s}}, u\rangle\, ;\, u \in L_{\gamma_o}, \int_\Omega \gamma(u)\, dR < \infty\Big\}\, ;\, \ell^{\mathrm{ac}}, \ell^{\mathrm{s}} \textit{ such that} \\
&\qquad\qquad\qquad \ell^{\mathrm{ac}} \ll R, \frac{d\ell^{\mathrm{ac}}}{dR} \in L_{\gamma_o^*}, \ell^{\mathrm{s}} \in L^{\mathrm{s}}_{\gamma_o}, \int_\Omega \varphi\, d(\ell^{\mathrm{ac}} + \ell^{\mathrm{s}}) = x^o\Big\} \\
&= \sup_{y\in\mathcal{Y}}\Big\{\langle y, x^o\rangle - \int_\Omega \gamma\Big(\langle y, \varphi(\omega)\rangle\Big)\, R(d\omega)\Big\}.
\end{aligned}$$

If the value is finite, the infimum is attained.

Remark. In the case where *dom* γ is a proper subset of $\mathbb{R}$ and R is a bounded measure, we have: $L_{\gamma_o} = L^\infty$ and $L_{\gamma_o^*} = L^1$.

Let us consider the following strengthening of (3.8):

$$\text{(3.11)} \qquad \forall y \in \mathcal{Y}, \int_\Omega \gamma_o(\langle y, \varphi(\omega)\rangle)\, R(d\omega) < \infty.$$

Notice that under (3.11), (3.1) becomes:

(3.12) $\qquad \gamma$ is convex finite nonnegative, $\gamma(0) = 0$ and γ is not identically equal to zero.

Theorem 3.4. *We assume that γ satisfies (3.12) and that φ satisfies (3.11). Then, for any $x^o \in \mathcal{X}$, we have*

$$\begin{aligned}
&\inf\Big\{\int_\Omega \gamma^*(v)\, dR\, ;\, v \in L_{\gamma_o^*} \textit{such that} \int_\Omega \varphi v\, dR = x^o\Big\} \\
&= \sup_{y\in\mathcal{Y}}\Big\{\langle y, x^o\rangle - \int_\Omega \gamma\Big(\langle y, \varphi(\omega)\rangle\Big)\, R(d\omega)\Big\}.
\end{aligned}$$

If the value is finite, the infimum is attained.

Proof. Following the proof of Theorem 3.3 with $U = M_{\gamma_o}$ (see (3.4)) instead of L_{γ_o} (assumption (3.11) yields (2.4)), we obtain the dual equality of Theorem 3.3 with $\sup\Big\{\langle \ell^{\mathrm{s}}, u\rangle\,;\, u \in L_{\gamma_o}, \int_\Omega \gamma(u)\,dR < \infty\Big\}$ replaced by $\sup\Big\{\langle \ell^{\mathrm{s}}, u\rangle\,;\, u \in M_{\gamma_o}, \int_\Omega \gamma(u)\,dR < \infty\Big\} = 0$, since M_{γ_o} is in the kernel of any singular form. ■

Remark. If R is bounded or if R is unbounded and $(\gamma(t) = 0 \Leftrightarrow t = 0)$, we have $M'_{\gamma_o} = L_{\gamma_o^*}$ so that: $\ell \in M'_{\gamma_o} \Rightarrow \ell^{\mathrm{s}} = 0$. However, this theorem still holds when R is unbounded and $\gamma(t_o) = 0$ for some $t_o \neq 0$.

About generalized projections. It can be shown that the absolutely continuous part of the minimizer of Theorem 3.3 is the generalized D-projection, in the sense of Csiszár ([11], [12]), of R on the linear set $\{Q \in M(\Omega)\,;\, \int_\Omega \varphi\,dQ = x^o\}$. Also using Orlicz spaces, it is proved in ([12], Theorem 3) that (3.11) is a sufficient condition for the D-projection to exist. This also follows directly from Theorem 3.4.

It is of interest to notice that the infimum of Theorem 3.3 may not be attained at a measure (if the singular part of the minimizer is not equal to zero). Therefore, the statements of Theorems 3.3 and 3.4 shed light on the discovery by I. Csiszár that the generalized D-projection may not share the constraint, see for instance ([11], Example 3.2): this may happen for some x^o when φ satisfies (3.8) but not (3.11). These considerations, together with the form of the minimizers, will be developed in a forthcoming paper.

4. Some abstract inverse problems

In this section, Theorems 3.3 and 3.4 are applied to a general class of inverse problems. We shall specialize our investigations in Sections 5 and 6, where Fredholm integral equations and marginal problems for stochastic processes will be considered.

An abstract result. In addition to $(\Omega, \mathcal{A}, R)$ and the function γ, we consider a measure space $(T, \mathcal{T}, \rho)$ where ρ is a nonnegative measure on $\mathcal{T}$ and a pair (X, Y) of vector spaces in duality, Y separating X. We are going to describe the linear constraint with the functions $a : T \times \Omega \to X$ and $x^o : T \to X$. Its formal expression is

$$\ell \in L'_{\gamma_o} \text{ is subject to } \int_\Omega a(t,\omega)\,\ell(d\omega) = x^o_t, \text{ for } \rho\text{-a.e. } t \in T. \tag{4.1}$$

To do this, let us introduce the following spaces $\mathcal{Y}$ and $\mathcal{X}^o$. We consider a vector space $\mathcal{Y}$ of functions $y : T \to Y$ and a vector space $\mathcal{X}^o$ of functions $x : T \to X$. We denote $y = (y_t)_{t\in T} \in \mathcal{Y} \subset Y^T$ and $x = (x_t)_{t\in T} \in \mathcal{X}^o \subset X^T$. We assume that

$$\text{for any } x \in \mathcal{X}^o \text{ and } y \in \mathcal{Y},\, t \in T \mapsto \langle y_t, x_t\rangle \in \mathbb{R} \text{ is measurable and } \int_T |\langle y_t, x_t\rangle|\,\rho(dt) < \infty. \tag{4.2}$$

In other words: $\forall x \in \mathcal{X}^o, y \in \mathcal{Y},\ \langle y_\cdot, x_\cdot\rangle \in L^1(T, \rho)$.

This property allows us to define a dual bracket between $\mathcal{Y}$ and $\mathcal{X}^o$, by

$$\langle y, x\rangle_{\mathcal{Y},\mathcal{X}^o} = \int_T \langle y_t, x_t\rangle\,\rho(dt). \tag{4.3}$$

One identifies $\mathcal{X}^o$ as a subset of $\mathcal{X} = \mathcal{Y}^*$. We want $\mathcal{Y}$ to separate $\mathcal{X}^o$. Therefore, in $\mathcal{X}^o$, x and x' are identified if: $\langle y, x - x'\rangle_{\mathcal{Y},\mathcal{X}^o} = 0, \forall y \in \mathcal{Y}$. We assume that the function $a : T \times \Omega \to X$ appearing in (4.1) satisfies the following conditions:

$$\left|\begin{array}{l}\text{for any } \omega \in \Omega, \text{ the function } a(\cdot,\omega) : t \mapsto a(t,\omega) \text{ belongs to } \mathcal{X}^o \text{ and}\\ \text{for any } y \in \mathcal{Y}, \text{ the function } \langle y, a(\cdot)\rangle_{\mathcal{Y},\mathcal{X}^o} : \omega \mapsto \int_T \langle y_t, a(t,\omega)\rangle\,\rho(dt) \text{ is measurable}\end{array}\right.$$

and

$$\text{(4.4.}\exists\text{)}\quad \left|\ \text{there exists } \lambda > 0 \text{ such that: } \int_\Omega \gamma_o\Big(\lambda\langle y, a(\cdot)\rangle_{\mathcal{Y},\mathcal{X}^o}\Big)\,dR < \infty\right.$$

or

$$\text{(4.4.}\forall\text{)}\quad \left|\ \text{for any } \lambda > 0,\ \int_\Omega \gamma_o\Big(\lambda\langle y, a(\cdot)\rangle_{\mathcal{Y},\mathcal{X}^o}\Big)\,dR < \infty\right.$$

In other words, under (4.4.$\exists$) (resp. (4.4.$\forall$)): $\forall y \in \mathcal{Y},\ \langle y, a(\cdot)\rangle_{\mathcal{Y},\mathcal{X}^o} \in U = L_{\gamma_o}$ (resp. $\in U = M_{\gamma_o}$). We are now ready to reformulate correctly (4.1) by:

$$\forall y \in \mathcal{Y}, \int_\Omega \left[\int_T \langle y_t, a(t,\omega)\rangle\,\rho(dt)\right]\,\ell(d\omega) = \int_T \langle y_t, x^o_t\rangle\,\rho(dt) \tag{4.5}$$

where $x^o \in \mathcal{X}^o$ and the left hand side means: $\Big\langle \ell, \langle y, a(\cdot)\rangle_{\mathcal{Y},\mathcal{X}^o}\Big\rangle_{L'_{\gamma_o}, L_{\gamma_o}}$ when ℓ isn't a measure. As a corollary of Theorems 3.3 and 3.4, we get the following result.

Theorem 4.1. *Let us suppose that γ satisfies the condition (3.1) and that (X, Y), $(T, \mathcal{T}, \rho)$, $\mathcal{X}^o$, $\mathcal{Y}$ and a satisfy the previous assumptions (in particular (4.2) and (4.4)).*

<u>*Case 1*</u>*. Suppose that (4.4.$\forall$) holds. Then, for any $x^o \in \mathcal{X}^o$, we have:*

$$\begin{aligned}&\inf\Big\{\int_\Omega \gamma^*(v)\,dR\,;\ v \in L_{\gamma_o^*} \text{ such that} : \forall y \in \mathcal{Y}, \int_{T\times\Omega} \langle y_t, a(t,\omega)\rangle v(\omega)\,R(d\omega)\rho(dt)\\ &\qquad\qquad = \int_T \langle y_t, x^o_t\rangle\,\rho(dt)\Big\}\\ &= \sup_{y\in\mathcal{Y}}\Big\{\int_T \langle y_t, x^o_t\rangle\,\rho(dt) - \int_\Omega \gamma\Big(\int_T \langle y_t, a(t,\omega)\rangle\,\rho(dt)\Big)\,R(d\omega)\Big\}.\end{aligned}$$

<u>*Case 2*</u>*. Suppose that (4.4.$\exists$) holds. Then, for any $x^o \in \mathcal{X}^o$, we have:*

$$\begin{aligned}&\inf\Big\{\int_\Omega \gamma^*\left(\frac{d\ell^{\mathrm{ac}}}{dR}\right)dR + \sup\Big\{\langle \ell^{\mathrm{s}}, u\rangle\,;\ u \in L_{\gamma_o}, \textstyle\int_\Omega \gamma(u)\,dR < \infty\Big\},\ \ell^{\mathrm{ac}}, \ell^{\mathrm{s}} \text{ such that} :\\ &\qquad\qquad \ell^{\mathrm{ac}} \ll R,\ \tfrac{d\ell^{\mathrm{ac}}}{dR} \in L_{\gamma_o^*}, \ell^{\mathrm{s}} \in L^{\mathrm{s}}_{\gamma_o} \text{ and } \ell = \ell^{\mathrm{ac}} + \ell^{\mathrm{s}} \text{ satisfies (4.5)}\Big\}\\ &= \sup_{y\in\mathcal{Y}}\Big\{\int_T \langle y_t, x^o_t\rangle\,\rho(dt) - \int_\Omega \gamma\Big(\int_T \langle y_t, a(t,\omega)\rangle\,\rho(dt)\Big)\,R(d\omega)\Big\}.\end{aligned}$$

In these two cases, if the value is finite, then the infimum is attained.

Proof. Apply Theorems 3.3 and 3.4 with $\mathcal{Y}$ and $\mathcal{X}^o$ dualy paired by (4.3) and

$$\varphi : \omega \in \Omega \mapsto a(\cdot,\omega) \in \mathcal{X}.$$

Notice that, by (4.4), we have: $a(\cdot,\omega) \in \mathcal{X}^o$. In the present situation, the condition (4.4.∃) is (3.8) and (4.4.∀) is (3.11). ■

An important specific case. It is the case where $\rho = \sum_{t\in T} \delta_t$ is the counting measure on T endowed with the σ-field all its subsets,

$$\begin{aligned}\mathcal{Y} &= \{(y_t)_{t\in T}\,;\, y_t = 0 \text{ for all but finitely many } t \in T\}\\ &\simeq \Big\{\Big((t_1, y_{t_1}), \dots, (t_d, y_{t_d})\Big)\,;\, d \geq 1, t_1, \dots, t_d \in T \text{ distinct}\Big\} \quad \text{and}\\ \mathcal{X}^o &= X^T.\end{aligned}$$

Then, (4.2) is clearly satisfied, $\langle y, x\rangle_{\mathcal{Y},\mathcal{X}^o} = \sum_{i=1}^d \langle y_{t_i}, x_{t_i}\rangle$, the constraint (4.5) is equivalent to

$$\forall t \in T, \theta \in Y,\ \int_\Omega \langle \theta, a(t,\omega)\rangle\, \ell(d\omega) = \langle \theta, x^o_t\rangle \tag{4.6}$$

and, thanks to the convexity of γ, the condition (4.4) is equivalent to

| $\forall t \in T, \theta \in Y,\ \omega \mapsto \langle \theta, a(t,\omega)\rangle$ is measurable

and

(4.7.∃) | there exists $\lambda > 0$ such that: $\int_\Omega \gamma_o\Big(\lambda\langle \theta, a(t,\omega)\rangle\Big)\, R(d\omega) < \infty$

or

(4.7.∀) | for any $\lambda > 0$, $\int_\Omega \gamma_o\Big(\lambda\langle \theta, a(t,\omega)\rangle\Big)\, dR < \infty$.

We have just shown that in the present situation, Theorem 4.1 yields the following result.

Proposition 4.2. *Let us suppose that γ satisfies the condition (3.1). Let us take a dual pairing (X, Y) of vector spaces such that Y separates X and a function $a : T \times \Omega \to X$ which satisfies the condition (4.7).*

Case 1. Suppose that (4.7.∀) holds. Then, for any $x^o \in X^T$, we have:

$$\begin{aligned}&\inf\Big\{\int_\Omega \gamma^*(v)\, dR\,;\, v \in L_{\gamma_o^*} \text{ such that}: \forall t \in T, \theta \in Y, \int_\Omega \langle \theta, a(t,\omega)\rangle v(\omega)\, R(d\omega) = \langle \theta, x^o_t\rangle\Big\}\\ =&\sup\left\{\sum_{i=1}^d \langle \theta_i, x^o_{t_i}\rangle - \int_\Omega \gamma\left(\sum_{i=1}^d \langle \theta_i, a(t_i,\omega)\rangle\right) R(d\omega)\,;\, d \geq 1, t_1, \dots, t_d \in T, \theta_i \dots, \theta_d \in Y\right\}.\end{aligned}$$

Case 2. Suppose that (4.7.∃) holds. Then, for any $x^o \in X^T$, we have:

$$\begin{aligned}&\inf\Big\{\int_\Omega \gamma^*\left(\tfrac{d\ell^{ac}}{dR}\right) dR + \sup\{\langle \ell^s, u\rangle\,;\, u \in L_{\gamma_o}, \textstyle\int_\Omega \gamma(u)\, dR < \infty\},\ \ell^{ac}, \ell^s \text{ such that}:\\ &\qquad \ell^{ac} \ll R, \tfrac{d\ell^{ac}}{dR} \in L_{\gamma_o^*}, \ell^s \in L^s_{\gamma_o} \text{ and } \ell = \ell^{ac} + \ell^s \text{ satisfies (4.6)}\Big\}\\ =&\sup\left\{\sum_{i=1}^d \langle \theta_i, x^o_{t_i}\rangle - \int_\Omega \gamma\left(\sum_{i=1}^d \langle \theta_i, a(t_i,\omega)\rangle\right) R(d\omega)\,;\, d \geq 1, t_1, \dots, t_d \in T, \theta_i \dots, \theta_d \in Y\right\}.\end{aligned}$$

In these two cases, if the value is finite, then the infimum is attained.

Remark. In the special case where T is finite, and $X = Y = \mathbb{R}^d$ which corresponds to finitely many constrained moments, these dual equalities have been obtained, when Ω is compact, in [21] via large deviations techniques.

Some criteria of existence. It is clear that Theorem 4.1 provides variational criteria of existence of solutions to the considered inverse problems. The following situation, is particularly tractable. It has been introduced in [15] and then exploited in [21], [22], [29] among others.

Let R be a probability measure and $\mathcal{C} := \{Q \in M(\Omega)\,;\, Q \ll R \text{ and } \kappa_- \leq \frac{dQ}{dR} \leq \kappa_+, R\text{-a.e.}\}$ with $-\infty < \kappa_- \leq 0 \leq \kappa_+ < \infty$. We consider the inverse problem:

$$\text{"Does there exists } Q \in \mathcal{C} \text{ such that } \int_\Omega a_t\, dQ = x^o_t, \text{ for } \rho\text{-a.e. } t \in T?\text{"}$$

where the correct statement of the above constraint is (4.5). In order to give a full answer to this question, let us chose γ with two asymptots with slopes κ_- and κ_+ : $\lim_{s\to-\infty}(\kappa_- s - \gamma(s)) = b_- < \infty$ and $\lim_{s\to+\infty}(\kappa_+ s - \gamma(s)) = b_+ < \infty$. Then, $b_- = \gamma^*(\kappa_-)$ and $b_+ = \gamma^*(\kappa_+)$ and the effective domain of γ^* is $[\kappa_-, \kappa_+]$. As a direct consequence of Theorem 4.1, we obtain the following result (it slightly extends previous similar results).

Corollary 4.3. *Let γ be as above and satisfy (3.1).*
Let us assume (4.2) and $\int_\Omega \gamma\Big| \int_T \langle y_t, a(t,\omega)\rangle\, \rho(dt)\Big|\, R(d\omega) < \infty$, for all $y \in \mathcal{Y}$.
Then, there exists $Q \in \mathcal{C}$ such that $\int_\Omega a_t\, dQ = x^o_t$, for ρ-a.e. $t \in T$, if and only if x^o satisfies

$$\forall y \in \mathcal{Y}, \int_T \langle y_t, x^o_t\rangle\, \rho(dt) - \int_\Omega \gamma\Big(\int_T \langle y_t, a(t,\omega)\rangle\, \rho(dt)\Big)\, R(d\omega) \leq \max(b_-, b_+)$$

5. Fredholm equations

In the whole section, it is assumed that (4.4.$\forall$) holds. In particular, this implies that *dom* $\gamma = \mathbb{R}$. We are going to apply Theorem 4.1 in different specific situations.

Let us first consider the simple case where $X = Y = \mathbb{R}$ and $T = [0,1]$. We assume that a satisfies the following condition:

$$(5.1)\qquad \left|\begin{array}{l} a \text{ is jointly measurable in } (t,\omega) \text{ with respect to } \mathcal{B}([0,1]) \otimes \mathcal{A}, \\ \text{there exists a function } \overline{a} \in M_{\gamma_o} \text{ such that: } \forall t \in [0,1], \omega \in \Omega, |a(t,\omega)| \leq \overline{a}(\omega) \\ \text{and } \begin{cases} \text{for any } \omega \in \Omega,\ a(\cdot,\omega) \text{ is right continuous on } [0,1[\text{ and left continuous at } t = 1 \\ \text{or} \\ \text{for any } \omega \in \Omega,\ a(\cdot,\omega) \text{ is left continuous on }]0,1] \text{ and right continuous at } t = 0. \end{cases} \end{array}\right.$$

Because of Proposition 4.2 ((5.1) implies (4.7.$\forall$)), the finite energy constraints are the functions $x^o : [0,1] \to \mathbb{R}$ such that there exists $v \in L_{\gamma_o^*}$ with $\int_\Omega a(t,\omega)v(\omega)\, R(d\omega) = x^o_t, \forall 0 \leq t \leq 1$. Consequently, thanks to the dominated convergence theorem and Hölder's inequality for $(L_{\gamma_o}, L_{\gamma_o^*})$, the condition (5.1) implies that x^o is bounded and is $\begin{cases} \text{right continuous on } [0,1[\text{ and left continuous at } t = 1 \\ \text{or} \\ \text{left continuous on }]0,1] \text{ and right continuous at } t = 0 \end{cases}$.

Therefore, x^o is perfectly described by the knowledge of x^o_t, for dt-a.e. $t \in [0,1]$ and one can identify x^o as an element of $L^\infty([0,1], dt)$ without any loss of information. In other words, the regular path x^o is determined by $\langle y, x^o\rangle = \int_{[0,1]} y_t x^o_t \, dt,\ \forall y \in \mathcal{Y}$, where $\mathcal{Y}$ is any subspace of $L^1([0,1], dt)$ (this insures (4.2)) which separates $L^\infty([0,1], dt)$. For instance: $\mathcal{Y} = L^1([0,1], dt)$ or $\mathcal{Y} = C^\infty_c(]0,1[)$ or $\mathcal{Y} = \mathcal{S}_{[0,1]}$ with

$$\mathcal{S}_{[0,1]} := \{\text{simple functions on } [0,1]\}.$$

Choosing for $\mathcal{X}^o : L^\infty([0,1], dt)$, and for ρ : the Lebesgue measure on $[0,1]$, we get (4.2). In addition, (4.4.$\forall$) follows from (5.1). We have just shown that in the present situation, Theorem 4.1 yields the following result.

Proposition 5.1. *Let us assume (3.12) and suppose that a satisfies the condition (5.1). Then, for any $x^o \in L^\infty([0,1], dt)$, we have:*

$$\begin{aligned}
&\inf\left\{ \int_\Omega \gamma^*(v)\, dR\,;\ v \in L_{\gamma^*_o} \text{ such that} : \int_\Omega a(t,\omega)v(\omega)\, R(d\omega) = x^o_t,\ \text{for a.e. } t \in [0,1]\right\}\\
&= \sup_{y\in\mathcal{Y}} \left\{ \int_{[0,1]} y_t x^o_t \, dt - \int_\Omega \gamma\left(\int_{[0,1]} y_t a(t,\omega)\, dt\right) R(d\omega)\right\}
\end{aligned}$$

where $\mathcal{Y} = L^1([0,1], dt)$ or $\mathcal{Y} = C^\infty_c(]0,1[)$ or $\mathcal{Y} = \mathcal{S}_{[0,1]}$.

If this value is finite, then the infimum is attained.

One proves similarly the following result.

Proposition 5.2. *Let us assume (3.12). Let us consider a sequence $(a_n)_{n\geq 1}$ of measurable functions on Ω such that there exists a function $\overline{a} \in M_{\gamma_o}$ with $\sup_{n\geq 1} |a_n(\omega)| \leq \overline{a}(\omega)$, $\forall \omega \in \Omega$. Then, for any bounded sequence $(x^o_n)_{n\geq 1}$, we have:*

$$\begin{aligned}
&\inf\left\{ \int_\Omega \gamma^*(v)\, dR\,;\ v \in L_{\gamma^*_o} \text{ such that} : \forall n \geq 1, \int_\Omega a_n v\, dR = x^o_n\right\}\\
&= \sup_{y\in\mathcal{Y}} \left\{ \sum_{n\geq 1} y_n x^o_n - \int_\Omega \gamma\Big(\sum_{n\geq 1} y_n a_n(\omega)\Big) R(d\omega)\right\}.
\end{aligned}$$

where $\mathcal{Y} = \ell^1$: the space of the summable sequences, or $\mathcal{Y}$ is the space of finite sequences.

If this value is finite, then the infimum is attained.

Proof. Apply Theorem 4.1 with $X = Y = \mathbb{R}$, $T = \{1, 2, \ldots\}$, $\rho = \sum_{n\geq 1} \delta_n$ and $a(n,\omega) = a_n(\omega)$, noticing that the conditions (4.2) and (4.4.$\forall$) are satisfied. ■

Let us extend the previous Propositions 4.2, 5.1 and 5.2. We consider a topological space T endowed with its Borel σ-field $\mathcal{T} = \mathcal{B}(T)$ and a nonnegative measure ρ on $\mathcal{B}(T)$ such that:

(5.2) | any $t \in T$ admits an open neighbourhood with a finite ρ-mass and any ρ-negligible subset has an empty interior.

In any case, one may choose for ρ the counting measure on T. With $T=[0,1]$, the Lebesgue measure is also convenient.

Let us take a dual pairing (X,Y) of vector spaces where Y separates X and a function $a:T\times\Omega\to X$ which satisfies the condition

$$
(5.3)\qquad \left|\begin{array}{l}
\text{for any } \theta\in Y,\omega\in\Omega, \text{ the function } t\in T\mapsto\langle\theta,a(t,\omega)\rangle\in\mathbb{R} \text{ is continuous,}\\
\text{for any } \theta\in Y,t\in T, \text{ the function } \omega\in\Omega\mapsto\langle\theta,a(t,\omega)\rangle\in\mathbb{R} \text{ is measurable and}\\
\text{for any } \theta\in Y, \text{ there exists } \overline{a}_\theta\in M_{\gamma_o} \text{ such that: } \sup_{t\in T}|\langle\theta,a(t,\omega)\rangle|\leq\overline{a}_\theta(\omega),\forall\omega\in\Omega.
\end{array}\right.
$$

Notice that under (5.3), for any $\theta\in Y$, $(t,\omega)\mapsto\langle\theta,a(t,\omega)\rangle$ is jointly measurable.

In the part of $\mathcal{X}^o$, let us choose the space of the functions $x:T\to X$ such that for all $\theta\in Y$, $\langle\theta,x.\rangle$ belongs to $L^\infty(T,\rho)$. Let us choose $\mathcal{Y}=\{y=\sum_{i=1}^d\alpha_i\theta_i\,;\,d\geq1,\alpha_i\in\mathcal{H},\theta_i\in Y,1\leq i\leq d\}$ where $\mathcal{H}$ is any subspace of $L^1(T,\rho)$ separating $L^\infty(T,\rho)$.
The condition (4.2) is satisfied and the duality bracket (4.3) is:
$\langle\sum_i\alpha_i\theta_i,x\rangle_{\mathcal{Y},\mathcal{X}^o}=\int_T\sum_i\alpha_i(t)\langle\theta_i,x_t\rangle\,\rho(dt)$. Thanks to (5.3) and Proposition 4.2, for any finite energy constraint x^o, $\langle\theta,x^o(\cdot)\rangle$ is bounded and continuous, for any $\theta\in Y$. Thanks to (5.2), for any $t\in T$, there exists a decreasing net (V_α) of open neighbourhoods of t such that $\lim_\alpha\dfrac{\mathbb{1}_{V_\alpha}}{\rho(V_\alpha)}\cdot\rho=\delta_t$ for the weak topology $\sigma(M_1(T),C_b(T))$ of the probability measures on T. Therefore, the measures $\langle\theta,x^o_t\rangle\rho(dt)$ completely determine x^o when θ describes Y and one doesn't lose any information, identifying $\langle\theta,x^o(\cdot)\rangle$ with an element of $L^\infty(T,\rho)$. On the other hand, (5.3) implies (4.4.$\forall$). We have just checked that the assumptions of Theorem 4.1 are satisfied in the present situation. As a corollary, we obtain the following statement.

Proposition 5.3. *Let us assume (3.12). We take (X,Y), $(T,\mathcal{B}(T),\rho)$ and a as above; in particular (5.2) and (5.3) are satisfied. Then, for any $x^o\in X^T$ such that for any $\theta\in Y$, $\sup_{t\in T}|\langle\theta,x^o_t\rangle|<\infty$, we have:*

$$
\begin{aligned}
&\inf\Big\{\int_\Omega\gamma^*(v)\,dR\,;\,v\in L_{\gamma_o^*}\text{ such that}:\forall t\in T,\theta\in Y,\int_\Omega\langle\theta,a(t,\omega)\rangle v(\omega)\,R(d\omega)=\langle\theta,x^o_t\rangle\Big\}\\
&=\sup\Big\{\int_T\sum_{i=1}^d\alpha_i(t)\langle\theta_i,x^o_t\rangle\,\rho(dt)-\int_\Omega\gamma\left(\int_T\sum_{i=1}^d\alpha_i(t)\langle\theta_i,a(t,\omega)\rangle\,\rho(dt)\right)R(d\omega)\,;\\
&\qquad\qquad d\geq1,\theta_i\in Y,\alpha_i\in\mathcal{H},1\leq i\leq d\Big\}
\end{aligned}
$$

where $\mathcal{H}$ is any subspace of $L^1(T,\rho)$ separating $L^\infty(T,\rho)$. For instance, $\mathcal{H}=L^1(T,\rho)$ or $\mathcal{H}=\{$simple functions on $T\}$ or, if ρ is a Radon measure on the Polish space T : $\mathcal{H}=C_c(T)$.

If this value is finite, then the infimum is attained.

Remark. Similar results hold with relaxed assumptions based on (4.4.$\exists$) rather than on (4.4.$\forall$). The dominating functions $\overline{a}$ of the assumptions of Propositions 5.1, 5.2 and 5.3 belonging to L_{γ_o} instead of M_{γ_o}, we get dual equalities similar to the Case 2 of Theorem 4.1, involving singular components.

Example 1. Let γ be a Young function with $dom\,\gamma = \mathbb{R}$ and x^o a bounded measurable function on $[0,1]$. We consider the following minimization problem

$$(\mathrm{P}_1) \qquad \inf\left\{\int_{[0,1]} \gamma^*(v(\omega))\,d\omega\,;\, v \in L_{\gamma^*}([0,1]) \text{ such that : } \int_{[0,t]} v(\omega)\,d\omega = x^o_t, \forall 0 \le t \le 1\right\}.$$

Applying Proposition 5.1 with $\Omega = [0,1[$, $R(d\omega) = d\omega$ and $a(t,\omega) = \mathbb{1}_{[0,t]}(\omega) = \mathbb{1}_{[\omega,1]}(t)$ ((5.1) holds with $\overline{a} = \mathbb{1}$), one obtains the dual equality

$$\inf(\mathrm{P}_1) = \sup_{y \in C_c^\infty(]0,1[)} \left\{\int_{[0,1]} y_t x^o_t\,dt - \int_{[0,1]} \gamma\left(\int_\omega^1 y_t\,dt\right)\,d\omega\right\}.$$

Denoting $Y(\omega) = \int_\omega^1 y_t\,dt$ and $\dot{x}^o$ the derivative of x^o in the distribution sense, the right hand side of the previous equality becomes $\sup_{Y \in \mathcal{U}} \left\{Y_0 x^o_0 + \int Y\dot{x}^o - \int_{[0,1]} \gamma(Y_t)\,dt\right\}$, where $\mathcal{U} := \{Y : \omega \in [0,1] \mapsto Y(\omega) = \int_\omega^1 y_t\,dt\,;\, y \in C_c^\infty(]0,1[)\}$. If this quantity is finite, then $x^o_0 = 0$ (consider $|y_t|$ arbitrarily large on an arbitrarily small neighbourhood of $t = 0$). And if $x^o_0 = 0$, $\sup_{Y\in\mathcal{U}}\{\int Y\dot{x}^o - \int_{[0,1]}\gamma(Y_t)\,dt\}$ is finite if and only if $\dot{x}^o$ is a $\|\cdot\|_\gamma$-continuous linear form on $\mathcal{U}$ (see [26], Lemma 2.1). But $\mathcal{U}$ is included in $M_\gamma([0,1])$ (see (3.4)), hence: $\dot{x}^o \in M'_\gamma = L_{\gamma^*}$ (see [32], Theorem 4.1.7). In addition, since $C_c^\infty \subset \mathcal{U}$, $\mathcal{U}$ is $\|\cdot\|_\gamma$-dense in M_γ. Therefore, $\sup_{Y\in\mathcal{U}} = \sup_{Y\in M_\gamma}$ and from the conjugacy of convex integral functionals (see [33], Theorem 2), it follows that $\sup_{Y\in M_\gamma}\{\int_{[0,1]} Y_t\dot{x}^o_t\,dt - \int_{[0,1]}\gamma(Y_t)\,dt\} = \int_{[0,1]}\gamma^*(\dot{x}^o_t)\,dt$. Finally, we have recovered that the value of (P_1) is finite if and only if $x_0 = 0$ and $\int_{[0,1]}\gamma^*(\dot{x}^o_t)\,dt < \infty$. If this holds, this value is precisely $\int_{[0,1]}\gamma^*(\dot{x}^o_t)\,dt$, hence $v = \dot{x}^o$ is a solution of (P_1). It is unique if γ^* is strictly convex (or equivalently if γ is differentiable).

Example 2. (Suggested by D. Dacunha-Castelle). Let Ω be the set of the left-continuous and right-limited (càdlàg) paths on $[0,1]$ which are left continuous at $t = 1$. It is endowed with its usual σ-field: $\mathcal{A}$, generated by the t-cylinders. The set of probability measures on Ω is called $M_1(\Omega)$. Let us pick R in $M_1(\Omega)$. The Kullback information of $P \in M_1(\Omega)$ with respect to R is defined by,

$$I(P \mid R) = \begin{cases} \int_\Omega \log\left(\frac{dP}{dR}\right)\,dP & \text{if } P \ll R \\ +\infty & \text{otherwise.}\end{cases}$$

Let us take $\xi^o : [0,1] \to \mathbb{R}$ and a measurable function $f : \mathbb{R} \to \mathbb{R}$ which satisfies

$$(5.4) \qquad \forall 0 \le t \le 1, \lambda > 0, \int_\Omega \exp(\lambda|f(\omega_t)|)\,R(d\omega) < \infty.$$

We consider the minimization problem

$$(\mathrm{P}_2) \qquad \inf\left\{I(P \mid R)\,;\, P \in M_1(\Omega), \int_\Omega f(\omega_t)\,P(d\omega) = \xi^o_t, \forall 0 \le t \le 1\right\}.$$

Let us take

$$(5.5) \qquad \gamma(x) = e^x - x - 1, x \in \mathbb{R}.$$

Its convex conjugate is

$$\gamma^*(x) = \begin{cases} (x+1)\log(x+1) - x & \text{if } x \in]-1,\infty[\\ +1 & \text{if } x = -1 \\ +\infty & \text{if } x \in]-\infty,-1[\end{cases}. \tag{5.6}$$

The Young function γ_o associated with it is $\tau(x) = e^{|x|} - |x| - 1$ and its convex conjugate is $\tau^*(x) = (|x|+1)\log(|x|+1) - |x|$. We denote L_τ and L_{τ^*} the corresponding Orlicz spaces on $(\Omega, \mathcal{A}, R)$. Considering the new variables $x_t^o = \xi_t^o - \int_\Omega f(\omega_t)\, R(d\omega)$ and $v(\omega) = \dfrac{d(P-R)}{dR}(\omega)$, one sees that (P_2) is equivalent to

$$\inf \left\{ \int_\Omega \gamma^*(v)\, dR\,;\, v \in L_{\tau^*} \text{ such that} : \int_\Omega v\, dR = 0 \text{ and } \int_\Omega f(\omega_t) v(\omega)\, R(d\omega) = x_t^o, \forall 0 \le t \le 1 \right\}.$$

We apply Proposition 4.2 with $X = Y = \mathrm{I\!R}$ and $a(t,\omega) = f(\omega_t)$, remarking that (5.4) implies (4.7.$\forall$) in the present situation. We obtain

$$\begin{aligned} &\inf(\mathrm{P}_2) \qquad (5.7)\\ &= \sup\Big\{ \textstyle\sum_{i=1}^d \theta_i \xi_{t_i}^o - \displaystyle\int_\Omega \exp\Big(c + \sum_{i=1}^d \theta_i f(\omega_{t_i})\Big)\, R(d\omega)\,; \\ &\qquad\qquad c \in \mathrm{I\!R}, d \ge 1, \theta_1, \dots, \theta_d \in \mathrm{I\!R}, 0 \le t_1 < \dots < t_d \le 1 \Big\} + 1 \\ &= \sup\Big\{ \theta_0 \xi_0^o + \textstyle\sum_{i=1}^d \theta_i (\xi_{t_i}^o - \xi_{t_{i-1}}^o) \\ &\qquad - \displaystyle\int_\Omega \exp\Big(c + \theta_0 f(\omega_0) + \sum_{i=1}^d \theta_i [f(\omega_{t_i}) - f(\omega_{t_{i-1}})]\Big)\, R(d\omega)\,; \\ &\qquad\qquad c \in \mathrm{I\!R}, d \ge 1, \theta_0, \dots, \theta_d \in \mathrm{I\!R}, 0 = t_0 < t_1 < \dots < t_d = 1 \Big\} + 1 \end{aligned}$$

If instead of (5.4), we have:

$$f \text{ is continuous and for all } \lambda > 0, \int_\Omega \exp\Big(\lambda \sup_{0 \le t \le 1} |f(\omega_t)|\Big)\, R(d\omega) < \infty, \tag{5.8}$$

then, one can apply Proposition 5.1, noticing that (5.8) yields (5.1) with $\overline{a}(\omega) = \sup_{0 \le t \le 1} |f(\omega_t)|$ (one uses the left continuity of ω at $t = 1$). This leads us to

$$\inf(\mathrm{P}_2) = \sup\left\{ \int_{[0,1]} y_t \xi_t^o\, dt - \int_\Omega \exp\Big(c + \int_{[0,1]} y_t f(\omega_t)\, dt\Big)\, R(d\omega)\,;\, c \in \mathrm{I\!R}, y \in \mathcal{H} \right\} + 1$$

where $\mathcal{H} = L^1([0,1], dt)$ or $\mathcal{H} = C_c^\infty(]0,1[)$ or $\mathcal{H} = \mathcal{S}_{[0,1]}$. In particular, under the assumption (5.8), we have

$$\begin{aligned} &\sup\Big\{ \theta_0 \xi_0^o + \textstyle\sum_{i=1}^d \theta_i (\xi_{t_i}^o - \xi_{t_{i-1}}^o) \\ &\qquad - \displaystyle\int_\Omega \exp\Big(c + \theta_0 f(\omega_0) + \sum_{i=1}^d \theta_i [f(\omega_{t_i}) - f(\omega_{t_{i-1}})]\Big)\, R(d\omega)\,; \\ &\qquad\qquad c \in \mathrm{I\!R}, d \ge 1, \theta_0, \dots, \theta_d \in \mathrm{I\!R}, 0 = t_0 < t_1 < \dots < t_d = 1 \Big\} \\ &= \sup\left\{ \int_{[0,1]} y_t \xi_t^o\, dt - \int_\Omega \exp\left(c + \int_{[0,1]} y_t f(\omega_t)\, dt\right)\, R(d\omega)\,;\, c \in \mathrm{I\!R}, y \in \mathcal{H} \right\} \end{aligned}$$

and if this quantity is finite, then (P_2) admits a unique solution.

Let us specialize the example choosing the Wiener measure $\mathcal{W}$ for R and $f(x) = x, x \in \mathbb{R}$. The problem ($\mathrm{P}_2$) becomes

$$(\mathrm{P}_2') \qquad \inf \left\{ I(P \mid \mathcal{W})\,;\, P \in M_1(\Omega_o), \int_{\Omega_o} \omega_t \, P(d\omega) = \xi_t^o, \forall 0 \le t \le 1 \right\}$$

where Ω_o is the space of all continuous paths ω on $[0,1]$ with $\omega_0 = 0$. We impose $\xi_0^o = 0$ (without losing generality). By (5.7), the value $\inf\,(\mathrm{P}_2')$ is

$$\begin{aligned}
&\sup \Bigg\{ \sum_{i=1}^{d} \theta_i(\xi_{t_i}^o - \xi_{t_{i-1}}^o) - \int_{\Omega_o} \exp\Big(c + \sum_{i=1}^{d} \theta_i [\omega_{t_i} - \omega_{t_{i-1}}]\Big)\, \mathcal{W}(d\omega)\,; \\
&\qquad\qquad c \in \mathbb{R}, d \ge 1, \theta_0, \dots, \theta_d \in \mathbb{R}, 0 = t_0 < t_1 < \dots < t_d = 1 \Bigg\} + 1 \\
&= \sup \left\{ \int_{[0,1]} z_t d\xi_t^o - \exp\left(c + \frac{1}{2} \int_{[0,1]} z_t^2 \, dt \right) \,;\, z \in \mathcal{S}_{[0,1]}, c \in \mathbb{R} \right\} + 1.
\end{aligned}$$

If it is finite, taking $c = 0$, for any $a \in \mathbb{R}$ and $z \in \mathcal{S}_{[0,1]}$, we get $a \int_{[0,1]} z \, d\xi^o - \exp\left(\frac{a^2}{2}\|z\|_2^2\right) \le \inf\,(\mathrm{P}_2') < \infty$ and choosing $|a| = 1/\|z\|_2$, leads us to: $\forall z \in \mathcal{S}_{[0,1]}, |\int_{[0,1]} z \, d\xi^o| \le [\inf\,(\mathrm{P}_2') + e^{\frac{1}{2}}]\|z\|_2$. Hence, $z \mapsto \int_{[0,1]} z \, d\xi^o$ is continuous on $\mathcal{S}_{[0,1]} \subset L^2([0,1])$, that is: $\int_{[0,1]} z \, d\xi^o = \int_{[0,1]} z\dot{\xi}^o \, dt$ with $\dot{\xi}^o \in L^2([0,1])$ ($\xi^o \in H^1$).

Conversely, if $\xi^o \in H^1$, then, $\sup\left\{ \int_{[0,1]} z_t d\xi_t^o - \exp\left(c + \frac{1}{2}\int_{[0,1]} z_t^2 \, dt\right) \,;\, z \in \mathcal{S}_{[0,1]}, c \in \mathbb{R} \right\} < \infty$. Therefore, ($\mathrm{P}_2'$) admits a solution if and only if $\xi^o \in H^1$. In addition, this solution P_* is unique since γ^* is strictly convex (see (5.6)).

By means of ([27], Theorem 4.5), one can identify P_* as the (unique) solution P^{ξ^o} to the stochastic differential equation $dX_t(\omega) = \dot{\xi}_t^o \, dt + d\omega_t,\ X_0 = 0$ under the law $\mathcal{W}$ (ω is a $\mathcal{W}$-Brownian motion). Indeed, Girsanov's theorem provides us with

$$\frac{dP^{\xi^o}}{d\mathcal{W}}(\omega) = \exp\left(\int_{[0,1]} \dot{\xi}_t^o \, d\omega_t - \frac{1}{2} \int_{[0,1]} (\dot{\xi}_t^o)^2 \, dt \right)$$

where $\int_{[0,1]} \dot{\xi}_t^o \, d\omega_t$ is a stochastic integral which is well-defined since ξ^o belongs to H^1. Applying ([27], Theorem 4.5), we have to check that $\int_{[0,1]} \dot{\xi}_t^o \, d\omega_t - \frac{1}{2}\int_{[0,1]} (\dot{\xi}_t^o)^2 \, dt$ is an admissible force field (see [27], (4.9)) as a function of $(1, a(\cdot, \omega)) = (1, (\omega_t)_{0 \le t \le 1}) = (1, \omega)$. The linearity is clear, while the approximation condition in ([27], (4.9.a)) follows from the stronger property:

$$(5.9) \qquad \forall \varepsilon > 0, \exists z \in \mathcal{S}_{[0,1]} \text{ such that : } \left\| \omega \mapsto \int_{[0,1]} (\dot{\xi}_t^o - z_t) \, d\omega_t \right\|_\tau \le \varepsilon$$

($\|\cdot\|_\tau$ is the Orlicz norm of L_τ). Let us prove (5.9). For any $\lambda > 0$, we have:

$$\begin{aligned}
&\int_{\Omega_o} \exp(\lambda | \int_{[0,1]} (\dot{\xi}_t^o - z_t) \, d\omega_t |) \, \mathcal{W}(d\omega) \\
\le &\int_{\Omega_o} \exp(\lambda \int_{[0,1]} (\dot{\xi}_t^o - z_t) \, d\omega_t) \, \mathcal{W}(d\omega) + \int_{\Omega_o} \exp(-\lambda \int_{[0,1]} (\dot{\xi}_t^o - z_t) \, d\omega_t) \, \mathcal{W}(d\omega) \\
= &2 \exp\left(\frac{\lambda^2}{2} \|\dot{\xi}^o - z\|_2^2 \right).
\end{aligned}$$

This leads us to the estimate: $\forall\lambda>0, \exists z\in\mathcal{S}_{[0,1]}$ such that : $\int_{\Omega_o}\exp(\lambda|\int_{[0,1]}(\dot\xi^o_t-z_t)\,d\omega_t|)\,\mathcal{W}(d\omega)\le 3$, from which (5.9) follows.

6. Marginal problems

In this section, we apply Proposition 5.3 to a class of Fredholm equations: the marginal problems for stochastic processes. Let us describe them.

Let E be a state space endowed with a σ-field $\mathcal{E}$, T is an index set. The space: $\Omega=E^T$, of the paths from T to E is endowed with the product σ-field $\mathcal{A}=\mathcal{E}^{\otimes T}$ and we denote $M_1(E)$ and $M_1(\Omega)$ the sets of the probability measures on E and Ω. For any $P\in M_1(\Omega)$ and any $t\in T$, the t-marginal of P is defined by

$$P_t(A)=P(\{\omega\in\Omega\,;\,\omega_t\in A\}),\ \forall A\in\mathcal{E}.$$

Let us fix a reference law on $\Omega: R\in M_1(\Omega)$. A typical marginal problem is:

$$\text{“Does there exist } P\in M_1(E^T) \text{ such that } P\ll R \text{ and } P_t=\nu^o_t,\forall t\in T\ \text{?”}\ , \tag{6.1}$$

where $\nu^o: t\in T\mapsto \nu^o_t\in M_1(E)$ is a given flow of marginals. Notice that $\bigotimes\limits_{t\in T}\nu^o_t$ has the desired marginals, but it may not be absolutely continuous with respect to R.

One way to answer (6.1) is to consider the following energy minimization problem:

$$\inf\left\{\int_{E^T}\gamma^*\Big(\frac{dP}{dR}+\kappa_-\Big)\,dR\,;\,P\in M_1(E^T),P\ll R\text{ and }P_t=\nu^o_t,\forall t\in T\right\} \tag{6.2.a}$$

with $\kappa_-:=\lim\limits_{x\to-\infty}\gamma(x)/x>-\infty$. Let us assume that $dom\,\gamma=\mathrm{I\!R}$, then $\Phi^*(P+\kappa_-R)=\begin{cases}\int_{E^T}\gamma^*(\frac{dP}{dR}+\kappa_-)\,dR & \text{if } P\ll R\\ +\infty & \text{otherwise}\end{cases}$. Clearly, if $\Phi^*(P+\kappa_-R)<\infty$, then P is a nonnegative measure and since ν^o_t has a unit mass, (6.2.a) is equivalent to

$$\inf\{\Phi^*(P+\kappa_-R)\,;\,P\text{ signed measure on }E^T,P_t=\nu^o_t,\forall t\in T\}. \tag{6.2.b}$$

When $\gamma(x)=e^x-x-1$ (see (5.5)), in restriction to $M_1(E^T)$, $\Phi^*(P+\kappa_-R)$ is the Kullback information of $P\in M_1(E^T)$ with respect to $R: I(P\mid R)=\begin{cases}\int_{E^T}\log\left(\frac{dP}{dR}\right)\,dP & \text{if } P\ll R\\ +\infty & \text{otherwise}\end{cases}$.

We introduce the function (when $\kappa_->-\infty$)

$$\widetilde\gamma(x)=\gamma(x)-\kappa_-x,\ x\in\mathrm{I\!R}. \tag{6.3}$$

With (5.5), we have: $\widetilde\gamma(x)=e^x-1$. The next proposition states a variational criterion for the existence of a solution to (6.2).

Proposition 6.1. *We assume that γ satisfies dom $\gamma=\mathrm{I\!R}$, (3.1) and $\kappa_-:=\lim\limits_{x\to-\infty}\gamma(x)/x>-\infty$. Then, for any flow $(\nu^o_t)_{t\in T}\in M_1(E)^T$, we have:*

$$\begin{aligned}&\inf\left\{\int_{E^T}\gamma^*\left(\frac{dP}{dR}+\kappa_-\right)\,dR\,;\,P\in M_1(E^T),P\ll R\text{ and }P_t=\nu^o_t,\forall t\in T\right\}\\ &=\sup\left\{\sum_{i=1}^d\int_E\theta_i\,d\nu^o_{t_i}-\int_{E^T}\widetilde\gamma\Big(\sum_{i=1}^d\theta_i(\omega_{t_i})\Big)\,R(d\omega)\,;\,d\ge1,\theta_1,\dots,\theta_d\in B(E),t_1,\dots,t_d\in T\right\}\end{aligned}$$

where $\widetilde{\gamma}$ is given at (6.3) and $B(E)$ stands for the space of the bounded measurable functions on E.

If this value is finite, then the infimum is attained.

Remark. With $T = \{1, 2\}$, one recovers the usual marginal problem with an absolute continuity constraint (see [10], [4], [9], [27]).
Large deviations may be useful to obtain such dual equalities. In [14], this equality is proved when γ is given by (5.5) and R is the law of a diffusion process. The infimum comes from a contraction principle, while the supremum is obtained via a projective limit of finite dimensional large deviation principles. In [9], γ being the log-Laplace transform of a probability measure, the 2-marginal problem ($T = \{1, 2\}$) is investigated with another large deviation technique: the *maximum entropy method on the mean* (MEM).

Proof. The above minimization problem appears in the form (6.2.a), but under its alternate formulation (6.2.b), it enters the framework of the present paper. We are going to check that it is a Fredholm problem which satisfies the assumptions of Proposition 5.3. To do this, we give T its discrete topology and the associated Borel σ-field, we choose ρ as the counting measure on T, $X = M_b(E)$: the space of the bounded measures on $(E, \mathcal{E})$, $Y = B(E)$ and

$$a : (t, \omega) \in T \times E^T \mapsto \delta_{\omega_t} \in M_b(E).$$

We note that (X, Y) is a separating dual pairing and ρ satisfies (5.2). In addition, (5.3) is satisfied since for any $\theta \in B(E)$, $\langle \theta, a(t, \omega)\rangle = \theta(\omega_t)$ is continuous in t (T is endowed with the discrete topology), measurable in ω ($\omega \mapsto \omega_{t_o}$ is measurable by the very definition of $\mathcal{A} = \mathcal{E}^{\otimes T}$) and $\sup_{t \in T} |\theta(\omega_t)| \leq \overline{a}_\theta(\omega) := \|\theta\|_\infty$ (hence: $\overline{a}_\theta \in L^\infty(R) \subset M_{\gamma_o}(R)$). Therefore, one can apply Proposition 5.3. This completes the proof of the proposition. ∎

One can specialize the previous result when the paths ω are continuous. Let us consider a metric space E endowed with its Borel σ-field $\mathcal{E} = \mathcal{B}(E)$ and a topological index set T with its Borel σ-field $\mathcal{T} = \mathcal{B}(T)$. The set of paths is $\Omega = C(T, E)$: the space of the continuous functions from T to E. It is endowed with the relative σ-field associated with $\mathcal{B}(E)^{\otimes T}$.

Proposition 6.2. *We assume that γ satisfies dom $\gamma = \mathrm{I\!R}$, (3.1) and $\kappa_- := \lim_{x \to -\infty} \gamma(x)/x > -\infty$. We also choose a measure ρ on $(T, \mathcal{B}(T))$ which satisfies (5.2). Then, for any continuous flow $(\nu_t^o)_{t \in T} \in C(T, M_1(E))$, we have:*

$$\begin{aligned}
&\inf \left\{ \int_{C(T,E)} \gamma^* \left(\frac{dP}{dR} + \kappa_- \right) dR \,;\, P \in M_1(C(T, E)), P \ll R \text{ and } P_t = \nu_t^o, \forall t \in T \right\} \\
&= \sup \left\{ \int_{T \times E} \sum_{i=1}^d \alpha_i(t)\theta_i(z)\, \nu_t^o(dz)\rho(dt) - \int_{C(T,E)} \widetilde{\gamma} \left(\int_T \sum_{i=1}^d \alpha_i(t)\theta_i(\omega_t)\, \rho(dt) \right) R(d\omega)\,; \right. \\
&\qquad\qquad\qquad\qquad \left. d \geq 1, \theta_i \in C_b(E), \alpha_i \in \mathcal{H}, 1 \leq i \leq d \right\}
\end{aligned}$$

where $\widetilde{\gamma}$ is given at (6.3), $C_b(E)$ stands for the space of the bounded continuous functions on E and $\mathcal{H}$ is any subspace of $L^1(T,\rho)$ separating $L^\infty(T,\rho)$. For instance, $\mathcal{H} = L^1(T,\rho)$ or $\mathcal{H} = \{$simple functions on $T\}$ or, if ρ is a Radon measure on the Polish space T : $\mathcal{H} = C_c(T)$.

If this value is finite, then the infimum is attained and $t \in T \mapsto \nu_t^o \in M_1(E)$ is continuous, $M_1(E)$ being endowed with its usual weak topology $\sigma(M_1(E), C_b(E))$.

Proof. Again, it is a corollary of Proposition 5.3. Its proof is very similar to Proposition 6.1's one. Notice that, as E is a metric space, $Y = C_b(E)$ separates $X = M_b(E)$. ■

When $T = [0,1[$, there are many relevant stochastic processes whose laws are supported by the set $\Omega = D([0,1[, E)$ of the right-continuous and left-limited (càdlàg) paths from $[0,1[$ to the topological space E. A slight modification of the proofs of Proposition 6.1 and 6.2, in the spirit of Proposition 5.1 with its condition (5.1), leads us to the following result.

Proposition 6.3. *We assume that γ satisfies dom $\gamma = \mathbb{R}$, (3.1) and $\kappa_- := \lim_{x\to-\infty} \gamma(x)/x > -\infty$. Then, for any flow $(\nu_t^o)_{t\in[0,1[} \in M_1(E)^{[0,1[}$, we have:*

$$\begin{aligned}
&\inf\left\{\int_{D([0,1[,E)} \gamma^*\left(\frac{dP}{dR} + \kappa_-\right) dR\,;\, P \in M_1(D([0,1[,E)), P \ll R \text{ and } P_t = \nu_t^o, \forall t \in T\right\}\\
&= \sup\Bigg\{\int_{[0,1[\times E} \sum_{i=1}^d \alpha_i(t)\theta_i(z)\,\nu_t^o(dz)dt - \int_{D([0,1[,E)} \widetilde{\gamma}\left(\int_T \sum_{i=1}^d \alpha_i(t)\theta_i(\omega_t)\,dt\right) R(d\omega)\,;\\
&\qquad\qquad d \geq 1, \theta_i \in B(E), \alpha_i \in C_c^\infty(]0,1[), 1 \leq i \leq d\Bigg\}
\end{aligned}$$

where $\widetilde{\gamma}$ is given at (6.3).

If this value is finite, then the infimum is attained.

Remarks. This result still holds when $[0,1[$ is replaced by any interval T of $\mathbb{R}$ which is open at its right bound, provided that $C_c^\infty(]0,1[)$ is replaced by $C_c^\infty(\mathrm{int}(T))$.
If R is a probability measure on $D([0,1], E)$ (including $t = 1$) and if $t = 1$ isn't a fixed discontinuity time of R, considering R as a probability measure on $D([0,1[, E)$ (without $t = 1$), one doesn't lose any information.
In the above statement, $B(E)$ may be replaced by any of its subspace which separates $M_1(E)$; for instance by the simple functions on E, or by $C_b(E)$ when E is a metric space, or by $C_c^\infty(\mathbb{R}^d)$ if $E = \mathbb{R}^d$.
In connection with stochastic mechanics (see [31], [8]), such a dual equality has been obtained in [7] with large deviation techniques.

Finally, we state a result connected to the problem of the existence of Schrödinger bridges (see [36], [2], [20], [3], [23], [37],[19], [30], [1], [18], [9], [13]).

Proposition 6.4. *Let $S \subset T$ be a nonempty subset of T. In the situation of Proposition 6.1 and* *under its assumptions, for any subflow* $(\nu_s^o)_{s\in S}\in M_1(E)^S$*, we have:*

$$\begin{aligned}&\inf\left\{\int_{E^T}\gamma^*\left(\frac{dP}{dR}+\kappa_-\right)dR\,;\,P\in M_1(E^T),P\ll R\text{ and }P_s=\nu_s^o,\forall s\in S\right\}\\ =&\sup\left\{\sum_{i=1}^d\int_E\theta_i\,d\nu_{s_i}^o-\int_{E^T}\widetilde{\gamma}\Big(\sum_{i=1}^d\theta_i(\omega_{s_i})\Big)\,R(d\omega)\,;\,d\geq 1,\theta_1,\dots,\theta_d\in B(E),s_1,\dots,s_d\in S\right\}\end{aligned}$$

where $\widetilde{\gamma}$ *is given at (6.3).*

If this value is finite, then the infimum is attained.

Proof. Similar to the proof of Proposition 6.1. ■

A typical application of this result is obtained with $T=[0,1]$ and $S=\{0,1\}$: the initial and final times. It is the problem of the Schrödinger bridges. As a consequence of Propositions 6.1 and 6.4 in this context, for any $\nu_0^o,\nu_1^o\in M_1(E)$, we have:

$$\begin{aligned}&\inf\left\{\int_{E^{[0,1]}}\gamma^*\left(\frac{dP}{dR}+\kappa_-\right)dR\,;\,P\in M_1(E^{[0,1]}),P\ll R,P_0=\nu_0^o\text{ and }P_1=\nu_1^o\right\}\\ =&\sup\left\{\int_E\theta_0\,d\nu_0^o+\int_E\theta_1\,d\nu_1^o-\int_{E^2}\widetilde{\gamma}\Big(\theta_0(\omega_0)+\theta_1(\omega_1)\Big)\,R_{01}(d\omega_0d\omega_1)\,;\,\theta_0,\theta_1\in B(E)\right\}\\ =&\inf\left\{\int_{E^2}\gamma^*\left(\frac{d\pi}{dR_{01}}+\kappa_-\right)dR_{01}\,;\,\pi\in M_1(E^2),\pi\ll R_{01},\pi_0=\nu_0^o\text{ and }\pi_1=\nu_1^o\right\}\end{aligned}$$

where $R_{01}\in M_1(E^2)$ is the image law of R by the application $(\omega_t)_{0\leq t\leq 1}\in E^{[0,1]}\mapsto(\omega_0,\omega_1)\in E^2$ and π_0,π_1 are the first and second marginals of π. If this value is finite, then the infima are attained.

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