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Convex conjugates of integral functionals

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Abstract. We consider a convex integral functional on a functional space V and compute its greatest extension to the algebraic bidual space V^{**}, among all convex functions which are lower semicontinuous with respect to the $*$-weak topology $\sigma(V^{**}, V^*)$.

Such computations are usually performed to extend these functionals to some topological closures. In the present paper, no a priori topological restrictions are imposed on the extended domain. As a consequence, this extended functional is a valuable first step for the computation of the exact shape of the minimizers of the conjugate convex integral functional subject to a convex constraint, in full generality: without constraint qualification.

These convex integral functionals are sometimes called entropies, divergences or energies. Our proofs mainly rely on basic convex duality and duality in Orlicz spaces.

1. Introduction

Let $(\Omega, \mathcal{A}, R)$ be a measure space, $\gamma : \mathbb{R} \to [0, \infty[$ a convex nonnegative function such that $\gamma(0) = 0$ and let V be a vector space of measurable functions $v : \Omega \to \mathbb{R}$. We consider the integral functional

$$I_\gamma(v) = \int_\Omega \gamma\big(v(\omega)\big)\, R(d\omega), \quad v \in V \tag{1.1}$$

defined on V with values in $[0, \infty]$. Let us denote by V^* and V^{**} the algebraic dual and bidual spaces of V. Our aim in this paper is to derive an expression for the "greatest" convex $\sigma(V^{**}, V^*)$-lower semicontinuous extension of I_γ to the bidual $V^{**} \supset V$.

The motivations. Let $V^\sharp$ be a vector space in duality with V and I_γ^* the conjugate of I_γ for the duality $(V, V^\sharp)$. A representation of some of the elements of the effective domain of I_γ^* is given by the following classical conjugacy result. If $\ell \in V^\sharp$ is such that

$$\partial I_\gamma^*(\ell) \cap V \neq \emptyset, \tag{1.2.a}$$

then

$$\ell \in \partial I_\gamma(v_\ell), \quad \text{for any } v_\ell \in \partial I_\gamma^*(\ell) \cap V \tag{1.2.b}$$

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where $\partial I_\gamma^*(\ell)$ is the algebraic subdifferential of I_γ^* at ℓ, that is the set of all the linear forms ξ on $V^\sharp$ (without any regularity restriction) such that: $I_\gamma^*(\ell') \geq I_\gamma^*(\ell) + \langle \xi, \ell' - \ell \rangle, \ \forall \ell' \in V^\sharp$.

Unfortunately, in infinite dimension, it may happen that (1.2.a) is a regularity condition which is satisfied for only a few ℓ's. Nevertheless, if these "regular" ℓ's are dense (in a sense to be made precise) in $V^\sharp$, one can extend the representation (1.2.b) as follows:

$$\text{(1.2.c)} \qquad \text{there exists a sequence } (v_\ell^n)_{n\geq 1} \text{ in } V \text{ such that } \ell \in \lim_{n\to\infty} \partial I_\gamma(v_\ell^n)$$

(with a bit of countability, otherwise one should invoke limits along filters).

Another way to improve (1.2.a) & (1.2.b) is to compute the conjugate $\widetilde{I_\gamma}$ of I_γ^* for the duality $(V^\sharp, V^{\sharp\sharp})$ for some vector space $V^{\sharp\sharp}$ with $V \subset V^{\sharp\sharp}$. Indeed, the same arguments lead us to the following representation. For any $\ell \in V^\sharp$, if

$$\text{(1.3.a)} \qquad \partial I_\gamma^*(\ell) \cap V^{\sharp\sharp} \neq \emptyset,$$

then

$$\text{(1.3.b)} \qquad \ell \in \partial \widetilde{I_\gamma}(\xi_\ell), \quad \text{for any } \xi_\ell \in \partial I_\gamma^*(\ell) \cap V^{\sharp\sharp}.$$

It appears that the largest $V^{\sharp\sharp}$ is, the better is this representation result.

Now, let us consider the general case where $V^\sharp$ is the algebraic dual space V^* of V. The less restrictive condition (1.3.a) corresponds to the choice of $V^{\sharp\sharp} = V^{**}$. Indeed, the geometric version of Hahn-Banach theorem insures that any ℓ in *ridom* I_γ^* : the relative interior of the effective domain of I_γ^*, satisfies (1.3.a) with $V^{\sharp\sharp} = V^{**}$. Therefore, $\overline{I_\gamma}$ standing for $\widetilde{I_\gamma}$ with $V^{\sharp\sharp} = V^{**}$, we have

$$\text{(1.4)} \quad \text{for any } \ell \in \textit{ridom } I_\gamma^*, \text{ one can find } \xi_\ell \in \partial I_\gamma^*(\ell); \text{ and for any such } \xi_\ell \text{ we have } \ell \in \partial \overline{I_\gamma}(\xi_\ell).$$

In addition, a representation of type (1.2.c): "there exists a sequence $(\xi_\ell^n)_{n\geq 1}$ in V^{**} such that $\ell \in \lim_{n\to\infty} \partial \overline{I_\gamma}(\xi_\ell^n)$", is still available for those ℓ's which stand on the relative boundary of the effective domain of I_γ^*.

The improvement of (1.2.c) by (1.4) makes more precise the statements of the solutions of some variational problems which are sometimes stated in terms of (1.2.c). Besides, the equality $I_\gamma^* = (\overline{I_\gamma})^*$ may be useful for the proof of the lower bound of some large deviation principles (see [DeZ] for this notion).

In [Lé2], the representation (1.4) is the first step to obtain the characterization of the solutions to the minimization of I_{γ^*} under general linear constraints (γ^* is the conjugate of γ).

Some known results. When γ is a Young function and V is an Orlicz space: $\mathcal{L}$, (built on $(\Omega, \mathcal{A}, R)$) in duality with an Orlicz space $\mathcal{L}_*$, Luxemburg and Zaanen ([LuZ]) have shown that I_γ and I_{γ^*}, respectively defined on $\mathcal{L}$ and $\mathcal{L}_*$ are conjugate to each other for the duality $(\mathcal{L}, \mathcal{L}_*)$. Later, Rockafellar ([Ro1], [Ro2], [Ro3]) obtained similar results for a larger class of convex integral functionals of the form $v \mapsto \int_\Omega \gamma[\omega, v(\omega)]\, R(d\omega)$ as well as for the duality (L_∞, L_∞'). Then, Kozec

([Ko1], [Ko2]) extended these results to the setting of general Orlicz spaces. These results are recalled below in Theorem 5.2.

If $(\mathcal{L}, \mathcal{L}_*)$ and $(\mathcal{L}_*, \mathcal{L}_{**})$ are two dual pairings of functional vector spaces, applying twice the above results, one obtains that $I_\gamma^* = I_{\gamma^*}$ and then that $(I_{\gamma^*})^* = \widetilde{I_\gamma}$ where $\widetilde{I_\gamma}$ is of the form (1.1) with $V = \mathcal{L}_{**}$. One also gets that I_{γ^*} and $\widetilde{I_\gamma}$ are conjugate to each other for the duality $(\mathcal{L}_*, \mathcal{L}_{**})$. Therefore, $\widetilde{I_\gamma}$ is the greatest convex $\sigma(\mathcal{L}_{**}, \mathcal{L}_*)$-lower semicontinuous extension of I_γ to $\mathcal{L}_{**}$ and we have a representation (1.3) with $V^{\sharp\sharp} = \mathcal{L}_{**}$. But, the spaces V^* and V^{**} not being functional spaces, one cannot apply directly these results to solve our problem.

Outline of the paper. The main idea of the paper is to obtain a priori estimates for the domains of I_γ^* and $\overline{I_\gamma}$, in order to be able to invoke Rockafellar's results. The abstract a priori estimate is stated in Lemma 2.1 at Section 2.

This estimate is valid for symmetric enough functionals I_γ : for instance when γ is an even function. If γ is not even, one can decompose I_γ into $I_\gamma(u) = I_{\gamma_+}(u_+) + I_{\gamma_-}(u_-), u \in U$, with $u = u_+ - u_-$, $\gamma_+(x) = \gamma(|x|)$ and $\gamma_-(x) = \gamma(-|x|)$ (which are even functions), provided that U is a Riesz space (in order to have $u_+ = u \vee 0 \in U$ and $u_- = (-u) \vee 0 \in U$). This decomposition in a Riesz space is proved in a general setting at Section 4. The main result of Section 4 is stated in Theorem 4.7.

Although a functional vector space V is not necessarily a Riesz space, it is included in a Riesz space U. This is the reason why, at Section 3, considering a convex function $\Phi : U \mapsto [0, \infty[$ on a vector space U and its restriction $\Psi : V \mapsto [0, \infty[$ to a vector subspace V of U, we investigate the relation between $\overline{\Phi}$ and $\overline{\Psi}$ which are respectively their greatest convex $\sigma(U^{**}, U^*)$ and $\sigma(V^{**}, V^*)$-lower semicontinuous extensions. The main result of Section 3 is Corollary 3.4.

As a consequence of the results of Sections 2, 3 and 4, the effective domain of $\overline{I_\gamma}$ is related to the topological bidual spaces of Orlicz spaces associated with the Young functions γ_+ and γ_-. In Section 5, known duality results for Orlicz spaces are recalled.

Finally, at Section 6, the results of Sections 2, 3, 4 and 5 allow us to compute the extension $\overline{I_\gamma}$ of I_γ. The main result of the paper is stated in Theorem 6.4.

The main motivation of this paper is to provide preliminary results for the proofs of the papers [Lé2] and [Lé3] where general problems of minimization of convex integral functionals under linear constraints are studied; [Lé2] requires our most sophisticated result: Theorem 6.4, while simpler results: Lemma 2.1 and Corollary 2.2, are invoked in [Lé3].

2. Preliminaries from convex analysis

In the first subsection, some usual results of convex analysis are recalled; this is an opportunity to state some of our notations. At the second subsection, estimates for the effective domains of the functionals are derived. This technical result has already been proved by the author in [Lé1]. Since it is important for our purpose, we reproduce its short proof in the present paper.

Basic convex analysis. Let X and Y be two vector spaces in separating duality for the bracket: $(x, y) \in X \times Y \mapsto \langle x, y \rangle \in \mathbb{R}$. In this paper, it is understood that all the convex functions are proper convex, that is: convex and $]-\infty,+\infty]$-valued with at least one finite value. We consider a proper convex function

$$f : x \in X \mapsto f(x) \in]-\infty,+\infty].$$

Its conjugate f^* is defined by

$$f^* : y \in Y \mapsto \sup_{x\in X}\{\langle x,y\rangle - f(x)\} \in]-\infty,+\infty]$$

and its biconjugate $\bar{f}$ is defined by:

$$\bar{f} : x \in X \mapsto \sup_{y\in Y}\{\langle x,y\rangle - f^*(y)\} \in]-\infty,+\infty].$$

If $f(0)=0$, then $f^*(Y)\subset[0,+\infty]$. As supremums of affine functions, f^* and $\bar{f}$ are convex functions.

The topology $\sigma(X,Y)$ is the weakest topology on X such that every linear form $\langle\cdot,y\rangle$, $y\in Y$ is continuous and $\sigma(Y,X)$ is the weakest topology on Y such that every linear form $\langle x,\cdot\rangle$, $x\in X$ is continuous.
As supremums of continuous functions, f^* and $\bar{f}$ are respectively $\sigma(Y,X)$ and $\sigma(X,Y)$-lower semicontinuous.

As a definition, the convex $\sigma(X,Y)$-lower semicontinuous regularized of f is the largest convex $\sigma(X,Y)$-lower semicontinuous function below f.

(2.1) Result. (See [EkT], Ch. 1, Proposition 3.3). If f is convex and bounded below by an affine $\sigma(X,Y)$-continuous function, then $\bar{f}$ is the convex $\sigma(X,Y)$-lower semicontinuous regularized of f. In particular, if f is nonnegative, convex and $\sigma(X,Y)$-lower semicontinuous, then $f=\bar{f}$; in this case, we shall say that f and f^* are proper convex functions conjugate to each other. Of course, $\bar{f}$ and f^* are in this situation.

The geometric interior of a subset A of X is the set of those $a\in A$ such that for any $x\in X$, there exists $\lambda>0$ satisfying $[a,\lambda x[\subset A$. The affine hull of A : *aff* A, is the smallest affine space containing A. The relative interior of A : *ri* A, is the geometric interior of A considered as a subset of its affine hull: $ri\ A=\{a\in aff\ A\,;\,\forall x\in aff\ A, \exists\lambda>0, [a,\lambda x[\subset A\}$.
The effective domain of f is *dom* $f := \{x\in X\,;\,f(x)<+\infty\}$ and the relative interior of *dom* f is denoted by *ridom* f.

(2.2) Result. (See [EkT], Ch.1, Proposition 2.5). Suppose that f is a convex function on the normed vector space $(X,\|\cdot\|)$. Then, f is $\|\cdot\|$-continuous on *ridom* f if and only if there exists a non-empty open $\|\cdot\|$-ball on which f is bounded above. In this situation, *ridom* f is equal to the $\|\cdot\|$-interior of *dom* f.

(2.3) Result. (Goldstine's lemma). See ([Bo1], Ch. 4, §2, Proposition 5). Let E be a normed vector space. Then, the unit ball of E is $\sigma(E'',E')$-dense in the unit ball of its topological bidual space E''.

A priori estimates for the effective domains. Let V be a vector space, V^* is its algebraic dual space and V^{**} its algebraic bidual space; V is canonically imbedded in V^{**} : $V\subset V^{**}$.

We consider a nonnegative convex function $\Psi : V \to [0, \infty]$ such that $\Psi(0) = 0$ and its conjugate Ψ^* for the duality (V, V^*) :

$$v^* \in V^* \mapsto \Psi^*(v^*) = \sup_{v \in V}\{\langle v, v^*\rangle - \Psi(v)\} \in [0, \infty].$$

Let $\overline{\Psi}$ stand for the conjugate of Ψ^* for the duality (V^{**}, V^*) :

$$\overline{\Psi} : \zeta \in V^{**} \mapsto \overline{\Psi}(\zeta) = \sup_{v^* \in V^*}\{\langle \zeta, v^*\rangle - \Psi^*(v^*)\} \in [0, \infty].$$

Thanks to (2.1), $\overline{\Psi}$ is the convex $\sigma(V^{**}, V^*)$-lower semicontinuous regularized of the convex function on V^{**} which matches with Ψ on $V \subset V^{**}$ and is equal to $+\infty$ on $V^{**} \setminus V$. The aim of the present paper is to compute $\overline{\Psi}$ when Ψ is an integral functional of the form I_γ (see (1.1)). If one knows that *dom* $\overline{\Psi}$ is included in some function space, then it becomes possible to take advantage of Theorem 5.2 below to get an explicit expression of $\overline{\Psi}$. These a priori estimate is stated in the following lemma.

Lemma 2.1. *Suppose that there exists a norm* $\|\cdot\|$ *on* V *such that:*

$$\text{(2.4)} \qquad \textit{there exists } r > 0 \textit{ such that } \sup\{\Psi(v)\,;\, \|v\| \le r\} \le 1.$$

Then, Ψ *is* $\|\cdot\|$*-continuous on ridom* Ψ *and*

$$\textit{dom}\ \Psi^* \subset V',$$

where V' *is the topological dual space of* $(V, \|\cdot\|)$ *endowed with the dual norm* $\|\cdot\|^*$.
If in addition,

$$\text{(2.5)} \qquad \textit{there exists } t_o > 0 \textit{ such that } 0 < \inf\{\Psi(v)\,;\, \|v\| = t_o\} < \infty,$$

then, Ψ^* *is* $\|\cdot\|^*$*-continuous on ridom* Ψ^* *and*

$$\textit{dom}\ \overline{\Psi} \subset V'',$$

where V'' *is the topological bidual space of* $(V, \|\cdot\|)$.

Proof. Thanks to (2.2), the condition (2.4) implies that Ψ is $\|\cdot\|$-continuous on *ridom* Ψ.
Let $\|\cdot\|$ satisfying (2.4) be given. Let ℓ be an element of V^*, then for all $v \in V$ and all $a > 0$

$$\langle \ell, v/a\rangle \le \Psi(v/a) + \Psi^*(\ell)$$

so that choosing $a = \|v\|/r$ and $a = -\|v\|/r$ when $v \neq 0$, one gets

$$\text{(2.6)} \qquad |\langle \ell, v\rangle| \le \frac{1 + \Psi^*(\ell)}{r}\|v\|, \ \forall v \in V.$$

It follows that, if $\Psi^*(\ell) < \infty$, then $\ell \in V'$.

Suppose now that (2.5) is satisfied. Let ℓ stand in $dom\,\Psi^* \subset V'$. Let us denote by $\theta(t) = \inf\{\Psi(v)\,;\,\|v\| = t\}, t \geq 0$ and θ^* its conjugate. Then, $\Psi^*(\ell) = \sup_{v\in V}\{\langle \ell, v\rangle - \Psi(v)\} \leq \sup_{v\in V}\{\|\ell\|^*\|v\| - \Psi(v)\} = \sup_{t\geq 0}\{t\|\ell\|^* - \theta(t)\} = \theta^*(\|\ell\|^*)$.
But (2.5) implies that

$$\theta^*(\theta(t_o)/t_o) \leq \theta(t_o) < \infty.$$

Let us prove this estimate. We have $\theta(0) = 0$, $\theta(t) \geq 0$ for all $t \geq 0$. Let us first show that $\theta(t)/t$ is non-decreasing on $[0, \infty[$. Let $0 < s < t$. For all $\varepsilon > 0$, there exists v_ε such that $\|v_\varepsilon\| = t$ and $\theta(t) \leq \Psi(v_\varepsilon) \leq \theta(t) + \varepsilon$. The application $f(x) = \Psi(xv_\varepsilon)$ is convex, non-negative with $f(0) = 0$. Hence, $f(x)/x$ is non-decreasing. With $x = s/t < 1$, we have: $\Psi(\frac{s}{t}v_\varepsilon)/s \leq \Psi(v_\varepsilon)/t \leq (\theta(t) + \varepsilon)/t$. But, $\|\frac{s}{t}v_\varepsilon\| = s$ so that $\theta(s) \leq \Psi(\frac{s}{t}v_\varepsilon)$. Therefore, for all $\varepsilon > 0$, $\theta(s)/s \leq (\theta(t) + \varepsilon)/t$.
Now, let $t_o > 0$ be such that $0 < \theta(t_o) < \infty$. As $\theta(t)/t$ is non-decreasing, we have: $\theta(t) \geq \alpha(t)$ where $\alpha(t) = \frac{\theta(t_o)}{t_o}(t - t_o)$ if $t \geq t_o$ and $\alpha(t) = 0$ if $t < t_o$. Hence, $\theta^* \leq \alpha^*$ and $\theta^*(\theta(t_o)/t_o) \leq \alpha^*(\theta(t_o)/t_o) = \theta(t_o) < \infty$, which is the desired result.
Therefore, with $\beta := \theta^*(\theta(t_o)/t_o)$ and $\mathcal{T} := \{\ell \in V'\,;\, \|\ell\|^* < \theta(t_o)/t_o\}$, one obtains

$$\sup_{\ell\in\mathcal{T}} \Psi^*(\ell) \leq \beta < \infty. \tag{2.7}$$

Thanks to (2.2), a consequence of (2.7) is that the convex function Ψ^* is $\|\cdot\|^*$-continuous on *ridom* Ψ^*.
Denote $\delta(\ell \mid \mathcal{T}) = \begin{cases} 0 & \text{if } \ell \in \mathcal{T} \\ +\infty & \text{otherwise} \end{cases}$, (2.7) leads us to: $\forall \ell \in V^*$, $\Psi^*(\ell) \leq \beta + \delta(\ell \mid \mathcal{T})$. It follows that for all $\zeta \in V^{**}$,

$$\overline{\Psi}(\zeta) \geq \sup_{\ell\in V^*}\{\langle \zeta, \ell\rangle - \delta(\ell \mid \mathcal{T}) - \beta\} = \sup_{\ell\in\mathcal{T}}\langle \zeta, \ell\rangle - \beta, \tag{2.8}$$

which completes the proof of the lemma. ■

An immediate consequence of (2.6) and (2.8) is the following result.

Corollary 2.2 . *If (2.4) is satisfied, then Ψ^* has $\sigma(V^*, V)$-compact sublevel sets included in V'. If (2.4) and (2.5) are satisfied, then $\overline{\Psi}$ has $\sigma(V^{**}, V')$-compact sublevel sets which are $\|\cdot\|$-bounded subsets of V''.*

<u>Remark</u>. The bidual V'' is endowed with its natural norm $\|\cdot\|$ whose restriction to $V \subset V''$ is the initial $\|\cdot\|$.

3. The convex biconjugate of a restricted function

Let U be a vector space and V a vector subspace of U. The algebraic dual and bidual spaces of U and V are denoted by: U^*, U^{**}, V^* and V^{**}. We consider a nonnegative convex function

$$\Phi : u \in U \mapsto \Phi(u) \in [0, \infty]$$

such that $\Phi(0) = 0$ and its restriction Ψ to V :

$$\Psi : u \in V \mapsto \Phi(u) \in [0, \infty].$$

Their conjugates are

$$\begin{array}{lcllll} \Phi^* & : & \ell \in U^* & \mapsto & \sup_{u\in U}\{\langle \ell, u\rangle - \Phi(u)\} & \in [0,\infty], \\ \overline{\Phi} & : & \xi \in U^{**} & \mapsto & \sup_{\ell\in U^*}\{\langle \xi, \ell\rangle - \Phi^*(\ell)\} & \in [0,\infty], \\ \Psi^* & : & v^* \in V^* & \mapsto & \sup_{u\in V}\{\langle v^*, u\rangle - \Phi(u)\} & \in [0,\infty], \\ \overline{\Psi} & : & \zeta \in V^{**} & \mapsto & \sup_{v^*\in V^*}\{\langle \zeta, v^*\rangle - \Psi^*(v^*)\} & \in [0,\infty]. \end{array}$$

In this section, we investigate the relations between $\overline{\Psi}$ and $\overline{\Phi}$. Let us begin with the relations between the vector spaces.

Let us define the equivalence relation on $U^* : \ell \sim \ell'$ for any $\ell, \ell' \in U^*$ if and only if $\ell(u) = \ell'(u), \forall u \in V$. In other words: $\ell \sim \ell' \iff \ell_{|V} = \ell'_{|V}$. We identify V^* with the factor space:

$$V^* = U^*/\sim$$

and $\dot{\ell} \in V^*$ stands for the equivalence class of $\ell \in U^*$. Therefore, one can identify V^{**} with a vector subspace of $U^{**} : V^{**} \subset U^{**}$ as follows. For any $\xi \in U^{**}$,

$$\xi \in V^{**} \iff \Big(\forall \ell, \ell' \in U^*, \ell \sim \ell' \implies \langle \xi, \ell - \ell'\rangle = 0\Big).$$

Notice that the topolologies $\sigma(V^{**}, V^*)$ and $\sigma(V^{**}, U^*)$ are equal.

Let $\overline{V}$ stand for the $\sigma(U^{**}, U^*)$-closure of V.

Lemma 3.1. *The inclusion $\overline{V} \subset V^{**}$ holds. In particular, $\overline{V}$ is also the $\sigma(V^{**}, V^*)$-closure of V.*

Proof. Let $\xi \in \overline{V}$. For every $\ell, \ell' \in U^*$ such that $\ell_{|V} = \ell'_{|V}$, one gets $\langle \xi, \ell - \ell'\rangle = \lim_\alpha \langle v_\alpha, \ell - \ell'\rangle = \lim_\alpha 0 = 0$, where (v_α) is a generalized sequence in V which $\sigma(U^{**}, U^*)$-converges to ξ. Hence, ξ belongs to V^{**}.

The second statement is a direct consequence of the previous inclusion and of the identity: $\sigma(V^{**}, U^*) = \sigma(V^{**}, V^*)$. ■

For any subset B of U^{**} and any $\xi \in U^{**}$, let us define $\delta(\xi \mid B) = \begin{cases} 0 & \text{if } \xi \in B \\ +\infty & \text{otherwise} \end{cases}$. We denote by

$$\begin{array}{lcllll} \Phi_V & : & u \in U & \mapsto & \Phi(u) + \delta(u \mid V) & \in [0,\infty], \\ \Phi_V^* & : & \ell \in U^* & \mapsto & \sup_{u\in U}\{\langle \ell, u\rangle - \Phi_V(u)\} & \in [0,\infty], \\ \overline{\Phi_V} & : & \xi \in U^{**} & \mapsto & \sup_{\ell\in U^*}\{\langle \xi, \ell\rangle - \Phi_V^*(\ell)\} & \in [0,\infty]. \end{array}$$

It appears that $\overline{\Phi_V}$ is the biconjugate with respect to the duality (U^{**}, U^*) of the "restricted" function $\xi \in U^{**} \mapsto \begin{cases} \Phi(\xi) & \text{if } \xi \in V \\ +\infty & \text{otherwise} \end{cases}$. This justifies the title of the present section.

Proposition 3.2. *If Φ is $\sigma(U, U^*)$-lower semicontinuous and Φ^* is $\sigma(U^*, U)$-inf-compact, then*

$$\overline{\Phi_V} = \overline{\Phi} + \delta(\cdot \mid \overline{V}).$$

where $\overline{V}$ stands for the $\sigma(U^{**},U^*)$-closure of V in U^{**}.

Proposition 3.3. *Under the assumption of Proposition 3.2, we have*

$$\overline{\Psi} = \overline{\Phi}_{|V^{**}} + \delta(\cdot \mid \overline{V}).$$

where $\overline{V}$ stands for the $\sigma(V^{**},V^*)$-closure of V in V^{**}.

Because of Lemma 3.1, $\overline{V} \subset V^{**} \subset U^{**}$ in Proposition 3.3 is identified with $\overline{V} \subset U^{**}$ appearing in Proposition 3.2.

Proof of Proposition 3.3. Let $\dot{\ell} \in V^*$. For any $\ell' \in U^*$ such that $\ell' \in \dot{\ell}$, we have: $\Psi^*(\dot{\ell}) = \sup_{v\in V}\{\langle \dot{\ell}, v\rangle - \Psi(v)\} = \sup_{v\in V}\{\langle \ell', v\rangle - \Phi(v)\} = \Phi_V^*(\ell')$.
As a consequence, for any $\xi \in V^{**}$: $\overline{\Psi}(\xi) = \sup_{\dot{\ell}\in V^*}\{\langle \xi, \dot{\ell}\rangle - \Psi^*(\dot{\ell})\} = \sup_{\ell\in U^*}\{\langle \xi, \ell\rangle - \Phi_V^*(\ell)\} = \overline{\Phi_V}(\xi)$, which proves that $\overline{\Psi}$ and $\overline{\Phi_V}$ match on V^{**}. Now, the result follows from Lemma 3.1 and Proposition 3.2. ■

Proof of Proposition 3.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be two vector spaces in separating duality. We endow U^* and $\mathcal{X}$ with the weak topologies $\sigma(U^*,U)$ and $\sigma(\mathcal{X},\mathcal{Y})$. Let $A : U^* \to \mathcal{X}$ be a continuous linear operator whose adjoint operator $A^* : \mathcal{Y} \to U^{**}$ satisfies $A^*(\mathcal{Y}) \subset U$. Under the assumptions of Proposition 3.2, ([Lé3], Thm. 2.3) states the following dual equality : for all $x \in \mathcal{X}$, $\inf\{\Phi^*(\ell)\,;\, \ell \in U^*, A\ell = x\}$ $= \sup_{y\in\mathcal{Y}}\{\langle y, x\rangle - \Phi(A^*y)\}$. Taking $\mathcal{Y} = V \subset U$, $\mathcal{X} = V^*$ and $A : \ell \in U^* \mapsto \ell_{|V} \in V^*$, we have $A^* : v \in V \mapsto v \in U$, $A^*(\mathcal{Y}) \subset U$ and A is continuous. Therefore, we obtain for all $v^* \in V^*$

$$\inf\{\Phi^*(\ell)\,;\, \ell \in U^*, \ell_{|V} = v^*\} = \sup_{v\in V}\{\langle v^*, v\rangle - \Phi(v)\}. \tag{3.1}$$

Clearly, $\Phi \le \Phi_V \Rightarrow \overline{\Phi} \le \overline{\Phi_V} \Rightarrow$ *dom* $\overline{\Phi_V} \subset$ *dom* $\overline{\Phi}$. We also have $\Phi_V^*(\ell) = \Psi^*(\ell_{|V})$ for all $\ell \in U^*$.
Let us define $\widetilde{V} \subset U^{**}$ by: for all $z \in U^{**}$, $z \in \widetilde{V} \iff \forall \ell^1, \ell^2 \in U^*$, $\ell^1_{|V} = \ell^2_{|V} \Rightarrow \langle z, \ell^1\rangle = \langle z, \ell^2\rangle$. For all $z \in \widetilde{V}$, we have:

$$\begin{aligned}
\overline{\Phi_V}(z) &= \sup_{\ell\in U^*}\{\langle \ell, z\rangle - \Psi^*(\ell_{|V})\}\\
&\overset{(a)}{=} \sup_{\ell\in U^*}\Big\{\langle \ell, z\rangle - \inf\{\Phi^*(\ell')\,;\, \ell'_{|V} = \ell_{|V}\}\Big\}\\
&= \sup_{\ell\in U^*}\ \sup_{\ell'\in U^*\,;\,\ell'_{|V}=\ell_{|V}}\{\langle \ell, z\rangle - \Phi^*(\ell')\}\\
&\overset{(b)}{=} \sup_{\ell'\in U^*}\{\langle \ell', z\rangle - \Phi^*(\ell')\}\\
&= \overline{\Phi}(z)
\end{aligned}$$

where equality (a) is due to (3.1) and equality (b) follows from $z \in \widetilde{V}$. Consequently, $\overline{\Phi_V}$ and $\overline{\Phi}$ match on $\widetilde{V}$.
We show as in Lemma 3.1 that $\overline{V} \subset \widetilde{V}$. Moreover, *dom* $\Phi_V \subset V$ implies that *dom* $\overline{\Phi_V} \subset \overline{V}$. All these arguments allow us to conclude. ■

4. Convex conjugates on a Riesz space

Let U be a Riesz vector space for the order relation $\le$ and Φ be a $[0,\infty]$-valued function on U. Since U is a Riesz space, any $u \in U$ admits a nonnegative part: $u_+ := u \vee 0$, and a nonpositive part: $u_- := (-u) \vee 0$. Of course, $u = u_+ - u_-$ and as usual, we state: $|u| = u_+ + u_-$. Throughout this section, it is assumed that Φ satisfies the following conditions:

$$\forall u \in U, \Phi(u) = \Phi(u_+ - u_-) = \Phi(u_+) + \Phi(-u_-) \tag{4.1}$$

$$\forall u, v \in U, \begin{cases} 0 \le u \le v & \Longrightarrow \quad \Phi(u) \le \Phi(v) \\ u \le v \le 0 & \Longrightarrow \quad \Phi(u) \ge \Phi(v) \end{cases} \tag{4.2}$$

$$\forall u \in U, \exists \lambda > 0, \Phi(\lambda u) < +\infty. \tag{4.3}$$

Let Φ^* be the conjugate of Φ on the algebraic dual space U^* of U, for the duality (U, U^*).

It is clear that (4.1) implies $\Phi(0) = 0$, (4.1) and (4.2) together imply that for any $u \in U$, $\Phi(u) = \Phi(u_+) + \Phi(-u_-) \ge \Phi(0) + \Phi(0) = 0$. Therefore, Φ^* is $[0, \infty]$-valued and $\Phi^*(0) = 0$.

Let us recall that there is a natural order on U^* which is defined by: $u^* \le v^*$ if and only if $\langle u^*, u \rangle \le \langle v^*, u \rangle$ for any $u \in U$ with $u \ge 0$. A linear form $u^* \in U^*$ is said to be relatively bounded if for any $v \in U$, $v \ge 0$, we have $\sup_{u\,;\,|u| \le v} |\langle u^*, u \rangle| < +\infty$. Although U^* may not be a Riesz space in general, the vector space: U^{b}, of all relatively bounded linear forms on U is always a Riesz space (see [Bo2], Ch. 2, Th. 1, p. 28). In particular, the elements of U^{b} admit a decomposition in nonnegative and nonpositive parts $u^* = u^*_+ - u^*_-$ where (see ([Bo2], Ch. 2, §2, Prop. 4)), for any $u \in U$, $u \ge 0$:

$$\langle u^*_+, u \rangle = \sup\{\langle u^*, v \rangle\,;\, 0 \le v \le u\},\ \langle u^*_-, u \rangle = \sup\{\langle -u^*, v \rangle\,;\, 0 \le v \le u\} \text{ and } u^*_+ \wedge u^*_- = 0.$$

Lemma 4.1. *Under the assumptions (4.1), (4.2) and (4.3), we have:*

$$\mathit{dom}\ \Phi^* \subset U^{\mathrm{b}}.$$

Proof. Let $v \in U$ be nonnegative. By (4.2), for any $u \in U$: $|u| = u_+ + u_- \le v \Longrightarrow \begin{cases} 0 \le u_+ \le v \\ -v \le -u_- \le 0 \end{cases} \Longrightarrow \begin{cases} \Phi(u_+) \le \Phi(v) \\ \Phi(-u_-) \le \Phi(-v) \end{cases}$. Together with (4.1), this yields $\sup_{u\,;\,|u| \le v} \Phi(u) \le \Phi(v) + \Phi(-v)$. Since, $\langle u^*, u \rangle \le \Phi^*(u^*) + \Phi(u), \forall u \in U$, we obtain $\sup_{u\,;\,|u| \le v} |\langle u^*, u \rangle| = \sup_{u\,;\,|u| \le v} \langle u^*, u \rangle \le \Phi^*(u^*) + \sup_{u\,;\,|u| \le v} \Phi(u) \le \Phi^*(u^*) + \Phi(v) + \Phi(-v)$.

But, for any $\lambda > 0$, we have: $\sup_{u\,;\,|u| \le v} |\langle u^*, u \rangle| = \dfrac{1}{\lambda} \sup_{u\,;\,|u| \le \lambda v} |\langle u^*, u \rangle|$.

Therefore, we get $\sup_{u\,;\,|u| \le v} |\langle u^*, u \rangle| \le \dfrac{1}{\lambda} [\Phi^*(u^*) + \Phi(\lambda v) + \Phi(-\lambda v)]$, from which the desired result follows, choosing $\lambda > 0$ small enough and taking (4.3) into account. ■

Let us define

$$\begin{aligned} \Phi_+(u) &= \Phi(u_+) + \Phi(u_-), & u \in U \\ \Phi_-(u) &= \Phi(-u_+) + \Phi(-u_-), & u \in U \end{aligned}$$

so that: $\Phi_+(u) = \Phi_+(-u)$, $\Phi_-(u) = \Phi_-(-u)$ and (4.1) is: $\Phi(u) = \Phi_+(u_+) + \Phi_-(u_-)$.

Lemma 4.2. *Under the assumptions (4.1), (4.2) and (4.3), for any $u^* \in U^{\mathrm{b}}$, we have:*

$$\Phi^*(u^*) \le \Phi^*_+(u^*_+) + \Phi^*_-(u^*_-).$$

Proof. For any $u \in U$, we have

$$\begin{aligned}
\langle u^*, u\rangle - \Phi(u) &= \langle u^*_+ - u^*_-, u_+ - u_-\rangle - [\Phi_+(u_+) + \Phi_-(u_-)] \\
&= [\langle u^*_+, u_+\rangle - \Phi_+(u_+)] + [\langle u^*_-, u_-\rangle - \Phi_-(u_-)] - [\langle u^*_+, u_-\rangle + \langle u^*_-, u_+\rangle] \\
&\le [\langle u^*_+, u_+\rangle - \Phi_+(u_+)] + [\langle u^*_-, u_-\rangle - \Phi_-(u_-)] \\
&\le \sup_{u\in U}\{\langle u^*_+, u_+\rangle - \Phi_+(u_+)\} + \sup_{u\in U}\{\langle u^*_-, u_-\rangle - \Phi_-(u_-)\} \\
&= \Phi^*_+(u^*_+) + \Phi^*_-(u^*_-),
\end{aligned}$$

from which the desired result follows. ■

Now, let us prove the converse inequality.

Lemma 4.3. *Under the assumptions (4.1), (4.2) and (4.3), for any $u^* \in U^{\mathrm{b}}$, we have:*

$$\Phi^*(u^*) \ge \Phi^*_+(u^*_+) + \Phi^*_-(u^*_-). \tag{4.4}$$

For any $u^* \in U^*$, if $u^* \ge 0$, then:

$$\Phi^*_+(u^*) = \sup_{u\in U\,;\,u\ge 0}\{\langle u^*, u\rangle - \Phi_+(u)\}, \tag{4.5}$$

$$\Phi^*_-(u^*) = \sup_{u\in U\,;\,u\ge 0}\{\langle u^*, u\rangle - \Phi_-(u)\}, \tag{4.6}$$

$$\Phi^*(u^*) = \sup_{u\in U\,;\,u\ge 0}\{\langle u^*, u\rangle - \Phi(u)\}. \tag{4.7}$$

Proof. Let us begin with the second part of the lemma. Let $u^* \in U^*, u^* \ge 0$. For every $u \in U$,

$$\begin{aligned}
\langle u^*, u\rangle - \Phi(u) &= \langle u^*, u_+ - u_-\rangle - \Phi(u_+) - \Phi(-u_-) \qquad \text{(with (4.1))} \\
&= \langle u^*, u_+\rangle - \Phi(u_+) - [\langle u^*, u_-\rangle + \Phi(-u_-)] \\
&\le \langle u^*, u_+\rangle - \Phi(u_+).
\end{aligned}$$

This gives (4.7). One shows the equalities (4.5) and (4.6) similarly.

Now, let us show (4.4). Let $u^* \in U^{\mathrm{b}}$, $0 = u^*_+ \wedge u^*_-$ is equivalent to: $\forall w \ge 0, \inf\{\langle u^*_+, w_1\rangle + \langle u^*_-, w_2\rangle\,;\, w_1, w_2 \ge 0, w_1 + w_2 = w\} = 0$ (see [Bo2], Ch. 2). Hence, for any $\varepsilon > 0, u, v \ge 0$, with $w = u + v$, one obtains the existence of $w_1, w_2 \ge 0$ such that $u + v = w_1 + w_2$ and $\langle u^*_+, w_1\rangle + \langle u^*_-, w_2\rangle \le \varepsilon$.

Stating $h := w_2 - v = u - w_1 \in U$, we get:

$$\begin{aligned}
\langle u^*, h\rangle &= \langle u^*_+, u - w_1\rangle - \langle u^*_-, w_2 - v\rangle \\
&= \langle u^*_+, u\rangle + \langle u^*_-, v\rangle - [\langle u^*_+, w_1\rangle + \langle u^*_-, w_2\rangle] \\
&\ge \langle u^*_+, u\rangle + \langle u^*_-, v\rangle - \varepsilon.
\end{aligned}$$

Moreover, as $0 \le h_+ \le u$ and $0 \le h_- \le v$, (4.1) and (4.2) provide us with $\Phi(h) = \Phi_+(h_+) + \Phi_-(h_-) \le \Phi_+(u) + \Phi_-(v)$. Hence, $\langle u^*, h\rangle - \Phi(h) \ge [\langle u^*_+, u\rangle - \Phi_+(u)] + [\langle u^*_-, v\rangle - \Phi_-(v)] - \varepsilon$. It comes out that for any $u, v \ge 0, \varepsilon > 0$:

$$\Phi^*(u^*) \ge [\langle u^*_+, u\rangle - \Phi_+(u)] + [\langle u^*_-, v\rangle - \Phi_-(v)] - \varepsilon$$

which gives

$$
\begin{aligned}
\Phi^*(u^*) &\geq \sup_{u\in U, u\geq 0}\{\langle u^*_+, u\rangle - \Phi_+(u)\} + \sup_{u\in U, u\geq 0}\{\langle u^*_-, v\rangle - \Phi_-(v)\} \\
&= \Phi^*_+(u^*_+) + \Phi^*_-(u^*_-)
\end{aligned}
$$

where the last equality is a consequence of (4.5) and (4.6). ■

Let U^{**} be the algebraic bidual of U. We consider the following functions.

$$
\begin{aligned}
\overline{\Phi}(\xi) &= \sup_{u^*\in U^*}\{\langle \xi, u^*\rangle - \Phi^*(u^*)\}, \qquad \xi\in U^{**} \\
\overline{\Phi_+}(\xi) &= \sup_{u^*\in U^*}\{\langle \xi, u^*\rangle - \Phi^*_+(u^*)\}, \qquad \xi\in U^{**} \\
\overline{\Phi_-}(\xi) &= \sup_{u^*\in U^*}\{\langle \xi, u^*\rangle - \Phi^*_-(u^*)\}, \qquad \xi\in U^{**}.
\end{aligned}
$$

Proposition 4.4. *We assume that Φ shares the properties (4.1), (4.2) and (4.3). Then,*

$$
\Phi^*(u^*) = \begin{cases} \Phi^*_+(u^*_+) + \Phi^*_-(u^*_-) & \text{if } u^*\in U^{\mathrm{b}} \\ +\infty & \text{otherwise.} \end{cases} \tag{4.8}
$$

If in addition,

$$
\text{span } (\text{dom } \Phi^*) \text{ is a Riesz space,} \tag{4.9}
$$

then

$$
\overline{\Phi}(\xi) = \begin{cases} \overline{\Phi_+}(\xi_+) + \overline{\Phi_-}(\xi_-) & \text{if } \xi\in U^{\mathrm{bb}} \\ +\infty & \text{otherwise} \end{cases}
$$

where $U^{\mathrm{bb}} \subset U^{**}$ stands for the subspace of linear forms on U^* whose restrictions to U^{b} are relatively bounded.

Remarks. Since the domains of Φ^* and $\overline{\Phi}$ consist of relatively bounded forms, one can consider their decompositions into nonnegative and nonpositive parts.
By Lemma 4.6.b below, $\overline{\Phi}(\xi)$ only depends on the restriction of $\xi\in U^{**}$ to $U^{\mathrm{b}}\subset U^*$. Hence, the expression of $\overline{\Phi}(\xi)$ is unambiguous.

Proof. The equality (4.8) is a direct consequence of Lemmas 4.1, 4.2 and 4.3.

In order to prove the equality concerning $\overline{\Phi}$, as *dom* $\Phi^*\subset U^{\mathrm{b}}$ by Lemma 4.1, it is enough to check that the conditions (4.1), (4.2) and (4.3) are satisfied for (Φ^*, U^{b}) instead of (Φ, U). Then, one is allowed to apply (4.8) to (Φ^*, U^{b}), which yields the desired result.

For every $u^*\in U^{\mathrm{b}}$, we have:

$$
\begin{aligned}
(\Phi^*)_+(u^*) &:= \Phi^*(u^*_+) + \Phi^*(u^*_-) \\
&= \Phi^*_+(u^*_+) + \Phi^*_+(u^*_-) \qquad \text{(thanks to (4.8))} \\
&= \Phi^*_+(u^*)
\end{aligned}
$$

where the last equality is (4.8) applied to the function Φ_+ which satisfies conditions similar to the conditions satisfied by Φ. Therefore, $(\Phi^*)_+ = \Phi^*_+$ and analogously $(\Phi^*)_- = \Phi^*_-$, so that (4.8) is the condition (4.1) for Φ^*.

Let us show the analogue of (4.2) for Φ^*. Let $0 \le u^* \le v^*$ be in U^*. For any $u \ge 0$, $\langle u^*, u\rangle - \Phi(u) \le \langle v^*, u\rangle - \Phi(u)$. Therefore, $\sup_{u\in U, u\ge 0}\{\langle u^*, u\rangle - \Phi(u)\} \le \sup_{u\in U, u\ge 0}\{\langle v^*, u\rangle - \Phi(u)\}$. Together with (4.7), this gives: $\Phi^*(u^*) \le \Phi^*(v^*)$.

Finally, with (4.9), the analogue of (4.3) is satisfied on a Riesz space. Taking into account the remarks before the proof, this completes the proof of the proposition. ■

Together with Lemma 4.3, the above proof immediately provides us with the following result.

Corollary 4.5. *Let us assume (4.1), (4.2), (4.3) and (4.9). For any $\xi \ge 0$ in U^{**}, we have:*

$$\begin{aligned}
\overline{\Phi}(\xi) &= \sup_{u^*\in U^*, u^*\ge 0}\{\langle \xi, u^*\rangle - \Phi^*(u^*)\},\\
\overline{\Phi_+}(\xi) &= \sup_{u^*\in U^*, u^*\ge 0}\{\langle \xi, u^*\rangle - \Phi_+^*(u^*)\},\\
\overline{\Phi_-}(\xi) &= \sup_{u^*\in U^*, u^*\ge 0}\{\langle \xi, u^*\rangle - \Phi_-^*(u^*)\}.
\end{aligned}$$

We still need one more lemma for the proof of Theorem 4.7.

Lemma 4.6.

(a) *Let $\|\cdot\|$ be a norm on U and denote by U' the topological dual of $(U, \|\cdot\|)$. If $\|\cdot\|$ satisfies: $(\forall u, v, 0 \le u \le v \Longrightarrow \|u\| \le \|v\|)$, then*

$$U' \subset U^{\mathrm{b}} \quad \text{and} \quad U'' \subset (U')^{\mathrm{b}}.$$

(b) *Let E be a subset of U^{b}. Suppose that dom $\Phi^* \subset E$. Then,*

$$\forall \xi, \zeta \in U^{**}, \Big(\langle \xi, u^*\rangle = \langle \zeta, u^*\rangle, \forall u^* \in E\Big) \Longrightarrow \overline{\Phi}(\xi) = \overline{\Phi}(\zeta).$$

In particular, this holds with $E = U'$ or $E = U^{\mathrm{b}}$ if dom $\Phi^* \subset U' \subset U^{\mathrm{b}}$.

(4.10) Conventions. We denote $\overline{\Phi}(\xi) = \overline{\Phi}(\xi_{|E}), \xi \in U^{**}$ to signify the implication above in (b). Also, by $\xi \in U''$, it is understood that $\xi_{|U'}$ belongs to U''.

Proof. Let us show (a). We have $U' \subset U^{\mathrm{b}}$, that is: the elements of U^* which are $\|\cdot\|$-bounded are relatively bounded. Indeed, let $u^* \in U'$ and $v \in U, v \ge 0$. Then,
$\sup\{|\langle u^*, u\rangle|\,;\, u \in U, |u| \le v\} \le \sup\{|\langle u^*, u\rangle|\,;\, u \in U, \|u\| \le 2\|v\|\} \le 2\|v\|\|u^*\|^*$,
since $|u| \le v \Longrightarrow \begin{cases} 0 \le u_+ \le v \\ 0 \le u_- \le v \end{cases} \Longrightarrow \|u\| \le \|u_+\| + \|u_-\| \le 2\|v\|$, using the assumption of (a).
On the other hand, the analogue of the assumption of (a) for the norm $\|\cdot\|^*$ of U' is satisfied. Indeed, $\forall 0 \le u^* \le v^* \in U'$, $\|u^*\|^* = \sup_{u\ge 0, \|u\|\le 1}\langle u^*, u\rangle \le \sup_{u \ge 0, \|u\|\le 1}\langle v^*, u\rangle = \|v^*\|^*$. It comes out as before that: $U'' \subset (U')^{\mathrm{b}}$.

Let us show (b). Let $\xi, \zeta \in U^{**}$ be such that $\xi_{|E} = \zeta_{|E}$. Then,

$$\begin{aligned}
\overline{\Phi}(\xi) &= \sup_{u^*\in U^*}\{\langle \xi, u^*\rangle - \Phi^*(u^*)\}\\
&= \sup_{u^*\in E}\{\langle \xi, u^*\rangle - \Phi^*(u^*)\} && (\textit{dom } \Phi^* \subset E)\\
&= \sup_{u^*\in E}\{\langle \zeta, u^*\rangle - \Phi^*(u^*)\} && (\xi_{|E} = \zeta_{|E})\\
&= \overline{\Phi}(\zeta) && (\textit{dom } \Phi^* \subset E)
\end{aligned}$$

This completes the proof of the lemma. $\blacksquare$

We are now ready to state and establish the main result of the section.

Theorem 4.7. *Let us suppose that $\Phi : U \to [0,\infty]$ is a convex function which satisfies (4.1) and (4.2) and that there exist two norms $\|\cdot\|_+$ and $\|\cdot\|_-$ on U such that*

$$\forall 0 \le u \le v \in U, \|u\|_+ \le \|v\|_+ \text{ and } \|u\|_- \le \|v\|_- \tag{4.11}$$

and

$$\exists r_+, r_- > 0 \text{ such that } \sup\{\Phi_+(u)\,;\, \|u\|_+ \le r_+\} \le 1 \text{ and } \sup\{\Phi_-(u)\,;\, \|u\|_- \le r_-\} \le 1 \tag{4.12}$$

and

$$\exists t_+, t_- > 0 \text{ such that } \begin{cases} 0 < \inf\{\Phi_+(u)\,;\, \|u\|_+ = t_+\} < \infty \\ 0 < \inf\{\Phi_-(u)\,;\, \|u\|_- = t_-\} < \infty \end{cases}. \tag{4.13}$$

We denote by U'_+ and U'_- the topological dual spaces of $U_+ := (U, \|\cdot\|_+)$ and $U_- := (U, \|\cdot\|_-)$, U''_+ and U''_- are their topological bidual spaces. Then,

$$U'_+ \subset U^{\mathrm{b}}, \quad U'_- \subset U^{\mathrm{b}}$$

and for any $u^* \in U^*$,

$$\Phi^*(u^*) = \begin{cases} \Phi^*_+(u^*_+) + \Phi^*_-(u^*_-) & \text{if } u^* \in U^{\mathrm{b}} \text{ and } (u^*_+, u^*_-) \in U'_+ \times U'_- \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover,

$$U''_+ \subset U^{\mathrm{bb}},\ U''_- \subset U^{\mathrm{bb}},\ \mathit{dom}\ \overline{\Phi} \subset U^{\mathrm{bb}}$$

and for any $\xi \in U^{**}$,

$$\overline{\Phi}(\xi) = \begin{cases} \overline{\Phi_+}(\xi_{+|U'_+}) + \overline{\Phi_-}(\xi_{-|U'_-}) & \text{if } \xi \in U^{\mathrm{bb}} \text{ and } (\xi_{+|U'_+}, \xi_{-|U'_-}) \in U''_+ \times U''_- \\ +\infty & \text{otherwise} \end{cases}$$

with the conventions (4.10).

Proof. The condition (4.11) appears in Lemma 4.6, while the conditions (4.12) and (4.13) are those of Lemma 2.1. Hence, we get $U'_+ \subset U^{\mathrm{b}}$, $U'_- \subset U^{\mathrm{b}}$, *dom* $\Phi^*_+ \subset U'_+$ and *dom* $\Phi^*_- \subset U'_-$.
Let us show now that the assumptions of Proposition 4.4 are satisfied. The conditions (4.1) and (4.2) are assumed to be satisfied and the condition (4.3) is implied by (4.1) and (4.12). It remains to check the condition (4.9). Because of (4.8), it is enough to check (4.9) for Φ^*_+ and Φ^*_-.

Let us check (4.9) for Φ^*_+ (the demonstration for Φ^*_- is completely similar). Since *span*(*dom* Φ^*_+) $\subset U'_+$, it is enough to check (4.9) on U'_+. Let us denote by $\|\cdot\|^*_+$ the uniform norm on U'_+ and

$\alpha_+ := \inf\{\Phi_+(u)\,;\, \|u\|_+ = t_+\}$. Because of (4.13), we have: $\Phi_+(u) \geq \mathbb{1}_{(\|u\|_+ \geq t_+)} \frac{\alpha_+}{t_+} \|u\|_+, \forall u \in U$ and for every $u^* \in U'_+$, choosing $\lambda > 0$ in such a way that $\lambda \|u^*\|^*_+ \leq \frac{\alpha_+}{t_+}$, we obtain

$$\begin{aligned}\Phi^*_+(\lambda u^*) &= \sup_{u \in U} \{\lambda \langle u^*, u\rangle - \Phi_+(u)\} \\ &\leq \sup_{u \in U} \{\lambda \|u^*\|^*_+ \|u\|_+ - \mathbb{1}_{(\|u\|_+ \geq t_+)} \frac{\alpha_+}{t_+} \|u\|_+\} \\ &= \sup_{x \geq 0} \{\lambda \|u^*\|^*_+ \, x - \mathbb{1}_{x \geq t_+} \frac{\alpha_+}{t_+} x\} \\ &= \lambda \|u^*\|^*_+ \, t_+ \leq \alpha_+ < +\infty,\end{aligned}$$

that is (4.9) for Φ^*_+. One completes the proof of the theorem thanks to Proposition 4.4, Lemma 4.6 and Lemma 2.1. ■

5. Duality in Orlicz spaces

In Section 6, the results of Sections 3 and 4 will be applied to compute the extension $\overline{\Psi} = \overline{I_\gamma}$ of $\psi = I_\gamma$ given by (1.1). In this context, it appears that the relevant topological spaces U_+ and U_- are Orlicz spaces. In the present section we recall for future use known results of A. Kozek about Orlicz spaces (see [Ko2]).

Dual space of an Orlicz space. The function $\theta : \mathbb{R} \to [0, +\infty]$ is called a Young function if it is convex, even and satisfies $\theta(0) = 0$ and $\lim_{x\to\infty} \theta(x) = +\infty$ and there exists $x_o > 0$ such that $0 \leq \theta(x_o) < \infty$.
Let Ω be an arbitrary set, $\mathcal{A}$ be a σ-field of subsets of Ω and let R be a nonnegative σ-finite measure on $\mathcal{A}$. One identifies functions which are R-almost everywhere equal.
The Orlicz space associated with θ is defined by :

$$L_\theta(\mathcal{A}, R) = \{f : \Omega \to \mathbb{R}\ ;\ f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_\theta < +\infty\}$$

with

$$\|f\|_\theta = \inf\left\{\beta > 0\ ;\ \int_\Omega \theta\left(\frac{|f(\omega)|}{\beta}\right) R(d\omega) \leq 1\right\}.$$

The function $\|\cdot\|_\theta$ is a norm and $(L_\theta, \|\cdot\|_\theta)$ is a Banach space.
The duality in Orlicz spaces is intimately linked with the convex conjugation. The convex conjugate θ^* of θ is also a Young function so that one may consider the Orlicz space L_{θ^*}.
Hölder's inequality in Orlicz spaces is as follows. For any $f \in L_\theta$ and $g \in L_{\theta^*}$, fg belongs to L_1 and $\int_\Omega |fg|\, dR \leq 2\|f\|_\theta \|g\|_{\theta^*}$. Therefore, any g in L_{θ^*} defines a continuous linear form on L_θ for the duality bracket:

$$\langle f, g\rangle = \int_\Omega fg\, dR, \quad \forall f \in L_\theta. \tag{5.1}$$

We have: $L_{\theta^*} \subset L'_\theta$. But in the general case, the dual space of $(L_\theta, \|\cdot\|_\theta)$ may be larger than L_{θ^*}.

Definitions. *A sequence $(\Omega_p)_{p\geq 1}$ in $\mathcal{A}$ is said to be R-localizing if it is nondecreasing, $\bigcup_{p\geq 1} \Omega_p = \Omega$ and $R(\Omega_p) < \infty,\ \forall p \geq 1$.*

A continuous linear form: ℓ, on L_θ is said to be <u>*absolutely continuous with respect to R*</u> *if there exists $g \in L_{\theta^*}$ such that $\langle \ell, f\rangle = \int_\Omega fg\, dR$, for all $f \in L_\theta$.*

A continuous linear form: ℓ, on L_θ is said to be <u>*singular with respect to R*</u> *if there exists an R-localizing sequence $(\Omega_p)_{p\geq 1}$ and a nonincreasing sequence $(A_k)_{k\geq 1}$ in $\mathcal{A}$ such that:*

$$\lim_{k\to\infty} R(A_k) = 0 \quad \text{and} \quad \langle |\ell|, \mathbb{1}_{(\Omega_p \setminus A_k)}\rangle = 0,\ \forall p, k \geq 1.$$

Let L_θ^{s} stand for the space of all the forms in L'_θ which are singular with respect to R.

<u>Remarks</u>. Since R is σ-finite, there exists an R-localizing sequence.
Since ℓ belongs to $L'_\theta \subset L_\theta^{\mathrm{b}}$ (see Lemma 4.6), $|l| = \ell_+ + \ell_-$ is well defined.
It is easy to check that, if R is bounded, ℓ is singular with respect to R if and only if the above conditions are satisfied with $\Omega_p = \Omega,\ \forall p \geq 1$.

Theorem 5.1. *Let θ be any finite Young function. Any continuous linear form on $L_\theta : \ell \in L'_\theta$, is uniquely decomposed as*

$$\ell = \ell^{\mathrm{ac}} + \ell^{\mathrm{s}} \tag{5.2}$$

with ℓ^{ac} in L_{θ^*} and ℓ^{s} in L_θ^{s}. This means that L'_θ is isomorphic to the direct sum

$$L'_\theta \simeq L_{\theta^*} \oplus L_\theta^{\mathrm{s}}.$$

<u>Proof</u>. See ([Ko2], Theorem 2.2). ■

Integrals which are convex functionals (see [Ro1], [Ro2]). In this subsection, we present results on the conjugacy of convex integral functionals. Theorem 5.2 below is particular case of a theorem proved by A. Kozek. It is an extension of a theorem by Rockafellar ([Ro2], Theorem 1) which was stated in the case where $L_\theta = L_\infty$.

It is assumed that $\mathcal{A}$ is R-complete. Let $\gamma : \mathbb{R} \to [0, +\infty]$ be a nonnegative lower semicontinuous convex function such that $\gamma(0) = 0$. Its conjugate γ^* is $[0, +\infty]$-valued, convex lower semicontinuous and also satisfies $\gamma^*(0) = 0$. We are interested in the functionals

$$I_\gamma(u) = \int_\Omega \gamma(u)\, dR \in [0, \infty], u \in L_\theta \tag{5.3}$$

and

$$I_{\gamma^*}(v) = \int_\Omega \gamma^*(v)\, dR \in [0, \infty], v \in L_{\theta^*}.$$

Theorem 5.2. *Let θ be a finite Young function. We consider the convex conjugate I_γ^* of I_γ for the duality between L_θ and L'_θ. For any $\ell \in L'_\theta$, I_γ^* is given by*

$$\begin{aligned} I_\gamma^*(\ell) &:= \sup_{u\in L_\theta} \{\langle \ell, u\rangle - I_\gamma(u)\} \\ &= I_{\gamma^*}(\ell^{\mathrm{ac}}) + \sup\{\langle \ell^{\mathrm{s}}, u\rangle\,;\, u \in dom\ I_\gamma\} \in [0, \infty] \end{aligned}$$

where $\ell = \ell^{\mathrm{ac}} + \ell^{\mathrm{s}}$ is the decomposition (5.2).

Proof. See ([Ko2], Theorem 2.6). ■

6. The biconjugate of a convex integral functional

In this section, we compute $\overline{\Psi}$ when Ψ is an integral functional of the type (5.3). The ingredients of this computation are Proposition 3.3 and Theorems 4.7 and 5.2.

Let Ω be an arbitrary set, $\mathcal{A}$ a σ-field of subsets of Ω and R a nonnegative measure on $\mathcal{A}$.
Our assumptions are

$$(6.1)\qquad \left|\begin{array}{l} \mathcal{A} \text{ is } R\text{-complete,} \\ R \text{ is } \sigma\text{-finite,} \\ \gamma : \mathbb{R} \to [0, +\infty[\text{ is a nonnegative convex function such that } \gamma(0) = 0 \text{ and} \\ dom\ \gamma = \mathbb{R}. \end{array}\right.$$

We consider the integral functional

$$\Phi : u \in U \mapsto \int_\Omega \gamma(u)\, dR \in [0, \infty[$$

defined on

$$\mathbf{U} := \{u : \Omega \to \mathbb{R}, \text{measurable such that } \int_\Omega \gamma(\lambda u)\, dR < \infty, \forall \lambda \in \mathbb{R}\}$$

where R-almost equal functions are identified. Notice that since γ is nonnegative, the integral $\int_\Omega \gamma(u)\, dR \in [0, \infty]$ is defined for any measurable u and that, as γ is convex, U is the largest vector space included in the effective domain of the functional $\Gamma : u \in M \mapsto \Gamma(u) := \int_\Omega \gamma(u)\, dR \in [0, \infty]$ where M is the space of all the real valued measurable functions.

For any $u \in M$, denote $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$. As $\gamma(0) = 0$, we have $\Gamma(u) = \Gamma(u_+ - u_-) = \Gamma(u_+) + \Gamma(-u_-)$ and as γ is nonincreasing on $]-\infty, 0]$ and nondecreasing on $[0, +\infty[$, it comes out that for any $u, v \in M$, $0 \le u \le v \Longrightarrow \Gamma(u) \le \Gamma(v)$ and $u \le v \le 0 \Longrightarrow \Gamma(u) \ge \Gamma(v)$. These are properties (4.1) and (4.2) for the function Γ defined on the Riesz space M. On the other hand, property (4.3) cannot be satisfied for Γ on the whole space M. This is the reason why we have introduced the above subspace U and the restriction Φ of Γ to U.

Let us check that U is a Riesz space. For any $u, v \in U$ and any $\lambda \ge 0$, we have

$$\begin{aligned} \Gamma\big(\lambda(u \vee v)\big) &= \Gamma\big(\lambda(u \vee v)_+\big) + \Gamma\big(-\lambda(u \vee v)_-\big) && \text{(property (4.1))} \\ &\le \Gamma\big(\lambda(u_+ + v_+)\big) + \Gamma\big(-\lambda(u_- + v_-)\big) && \text{(property (4.2))} \\ &\le 1/2\big[\Gamma(2\lambda u_+) + \Gamma(2\lambda v_+) + \Gamma(-2\lambda u_-) + \Gamma(-2\lambda v_-)\big] && \text{(convexity)} \\ &= 1/2\big[\Gamma(2\lambda u) + \Gamma(2\lambda v)\big] && \text{(property (4.1))} \\ &< +\infty \end{aligned}$$

and similarly for any $\lambda \le 0$, replacing $u, v, \lambda, \vee$ by $-u, -v, -\lambda, \wedge$. This proves that for any $u, v \in U$ and any $\lambda \in \mathbb{R}$, $\lambda(u \vee v)$ belongs to U, i.e.: U is a Riesz space. Hence, Φ and U satisfy (4.1) and (4.2), as well as (4.3) which is a direct consequence of the definition of U.

In this setting, the functions Φ_+ and Φ_- are given, for any $u \in U$, by

$$
(6.2) \qquad \begin{array}{rclcll}
\Phi_+(u) &=& \int_\Omega \gamma_+(u)\, dR & \text{with} & \gamma_+(x) := \gamma(|x|), & x \in \mathbb{R}, \\
\Phi_-(u) &=& \int_\Omega \gamma_-(u)\, dR & \text{with} & \gamma_-(x) := \gamma(-|x|), & x \in \mathbb{R}.
\end{array}
$$

Notice that γ_+ and γ_- are even functions.

Now, let us build the norms $\|\cdot\|_+$ and $\|\cdot\|_-$ which will satisfy the assumptions (4.11), (4.12) and (4.13) of Theorem 4.7. Let us suppose for a while that γ is an even function. Let us define the Orlicz gauge norm

$$
\|u\|_\gamma := \inf \left\{ \beta > 0\, ; \int_\Omega \gamma\left(\frac{|u|}{\beta}\right) dR \le 1 \right\}, \quad u \in U.
$$

In order to avoid the pathological and trivial case where $\gamma \equiv 0$, we suppose that there exists $x_o > 0$ such that $\gamma(x_o) > 0$, so that $\|\cdot\|_\gamma$ is a norm on U. It clearly satisfies (4.11).

It also satisfies (4.12). Indeed, with $r = 1/2$, for any $u \in U$,

$$
\begin{aligned}
\|u\|_\gamma \le 1/2 &\Longrightarrow \forall \beta > 1/2, \int_\Omega \gamma\left(\frac{|u|}{\beta}\right) dR \le 1 \\
&\Longrightarrow \Phi(u) = \int_\Omega \gamma(|u|)\, dR \le 1
\end{aligned}
$$

where we choosed $\beta = 1$ and used the evenness of γ.

It also satisfies (4.13). Indeed, with $t = 2$, for any $u \in U$,

$$
\begin{aligned}
\|u\|_\gamma = 2 &\Longrightarrow \forall 0 < \beta < 2, \int_\Omega \gamma\left(\frac{|u|}{\beta}\right) dR \ge 1 \\
&\Longrightarrow 1 \le \int_\Omega \gamma(|u|)\, dR = \Phi(u) < \infty
\end{aligned}
$$

where we choosed $\beta = 1$ and used the evenness of γ.

Since γ_+ and γ_- are even and share the other properties of γ, we choose $\|\cdot\|_+ = \|\cdot\|_{\gamma_+}$ and $\|\cdot\|_- = \|\cdot\|_{\gamma_-}$, that is, for any $u \in U$,

$$
(6.3) \qquad \begin{array}{rcl}
\|u\|_+ &:=& \inf\{\beta > 0\, ; \int_\Omega \gamma_+(\frac{|u|}{\beta})\, dR \le 1\} \\
\|u\|_- &:=& \inf\{\beta > 0\, ; \int_\Omega \gamma_-(\frac{|u|}{\beta})\, dR \le 1\}.
\end{array}
$$

Remark. The evenness of γ is not necessary to establish property (4.12): the gauge norm built upon $\max(\gamma_+, \gamma_-)$ satisfies (4.12). On the other hand, this evenness is essential for the property (4.13) to be satisfied: if the norms $\|\cdot\|_+$ and $\|\cdot\|_-$ are not equivalent, then (4.13) may fail for the gauge norm built upon $\max(\gamma_+, \gamma_-)$. This is precisely the case when $\gamma(x) = e^x - x - 1, x \in \mathbb{R}$; this function γ is attached to the usual entropy (the opposite of the Kullback-Leibler information with respect to R). This is the reason why the decomposition $\Phi(u) = \Phi_+(u_+) + \Phi_-(u_-)$ has been introduced and Theorem 4.7 has been proved.

Since all the assumptions of Theorem 4.7 are satisfied, we have just proved the following lemma.

Lemma 6.1. *With Φ_+ and Φ_- given by (6.2) and with $\|\cdot\|_+$ and $\|\cdot\|_-$ given by (6.3), all the conclusions of Theorem 4.7 hold.*

Hence, the computation of $\overline{\Phi}$ reduces to the separate computations of $\overline{\Phi_+}$ and $\overline{\Phi_-}$.

Proposition 6.2. *Under the assumptions (6.1), for any $\ell \in U^*$,*

$$\Phi^*(\ell) = \begin{cases} \int_\Omega \gamma^* \left(\frac{d\ell}{dR}\right) dR & \text{if } \ell \ll R \\ +\infty & \text{otherwise,} \end{cases}$$

where $\ell \ll R$ means that there exists a measurable function $\frac{d\ell}{dR} : \Omega \to \mathbb{R}$ such that $\ell = \frac{d\ell}{dR} \cdot R$.

<u>Proof</u>. Because of Lemma 6.1 and Theorem 4.7, one can reduce the proof to the case where γ is an even function. Let γ be even.

If $\gamma \equiv 0$, the result is clear.

Let γ be such that $\gamma \not\equiv 0$. When γ satisfies $\gamma(x) = 0 \Leftrightarrow x = 0$, we have $U' = L_{\gamma^*}$ (see [RaR], Theorem 7, p. 110). Then, the conclusion follows from Theorem 5.2.

But under (6.1), γ may be such that $\{0\} \neq \gamma^{-1}(\{0\})$. We present a general proof for this proposition. We are going to show that for any $\ell \in U^*$ such that $\Phi^*(\ell) < \infty$, we have:

(a) ℓ is σ-additive on $\mathcal{A}$,

(b) ℓ is a measure acting on U,

(c) ℓ is absolutely continuous with respect to R.

By Proposition 4.4, it is enough to prove (a), (b) and (c) for $\ell \geq 0$.

Let us show (a). For any nonincreasing sequence $(A_n)_{n\geq 1}$ in $\mathcal{A}$ with $\bigcap_{n\geq 1} A_n = \emptyset$ and any $\lambda > 0$, we have

$$\langle \ell, \lambda \mathbb{1}_{A_n} \rangle - \int_\Omega \gamma(\lambda \mathbb{1}_{A_n})\, dR \leq \Phi^*(\ell), \ \forall n \geq 1.$$

Hence, for any $\lambda > 0$:

$$\begin{aligned} \lambda \limsup_{n\to\infty} \langle \ell, \mathbb{1}_{A_n} \rangle &\leq \gamma(\lambda) \lim_{n\to\infty} R(A_n) + \Phi^*(\ell) \\ &= \Phi^*(\ell) \end{aligned}$$

which implies that $\lim_{n\to\infty} \langle \ell, \mathbb{1}_{A_n} \rangle = 0$. Notice that the assumption *dom* $\gamma = \mathbb{R}$ is essential for the proof of (a).

Let us show (b). We denote by $\mathcal{S}$ the space of all the simple functions. For any nonnegative function $u \in U$ and any $\lambda > 0$, there exists $v_\lambda \in \mathcal{S}$ with $0 \leq v_\lambda \leq u$ such that $\int_\Omega \gamma(\lambda(u - v_\lambda))\, dR \leq 1$. But,

$$\langle \ell, \lambda(u - v_\lambda) \rangle - \int_\Omega \gamma(\lambda(u - v_\lambda))\, dR \leq \Phi^*(\ell).$$

Hence, for any $\lambda > 0$, there exists $v_\lambda \in \mathcal{S}, 0 \leq v_\lambda \leq u$ such that

$$0 \leq \lambda \langle \ell, (u - v_\lambda) \rangle \leq \Phi^*(\ell) + 1$$

which leads us to: $\inf_{v\in\mathcal{S}, 0\leq v\leq u} \langle \ell, (u - v) \rangle = 0$. Together with (a), this proves (b).

Let us show (c). For any $A \in \mathcal{A}$ such that $R(A) = 0$ and any $\lambda > 0$, we have:

$$\lambda\langle \ell, \mathbb{I}_A\rangle = \lambda\langle \ell, \mathbb{I}_A\rangle - \int_\Omega \gamma(\lambda \mathbb{I}_A)\, dR \leq \Phi^*(\ell).$$

Therefore, $\langle \ell, \mathbb{I}_A\rangle = 0$. Together with (b), this proves (c).
On the other hand, $U \subset L_\gamma$ and by Lemma 2.1 and Theorem 5.1, we have: *dom* $\Phi^* \subset L_\gamma{}' = L_{\gamma^*} \oplus L^{\mathrm{s}}_\gamma$. Together with (c), we obtain

$$\text{dom } \Phi^* \subset L_{\gamma^*}. \tag{6.4}$$

One concludes the proof of the proposition with Theorem 5.2. ■

Let us define the cone $\mathbf{W_\Phi}$ which consists of all the elements $\xi \in U^{**}$ such that $\xi \in U^{\mathrm{bb}}$ (this implies that ξ_+ and ξ_- exist) and in restriction to $L_{\gamma_+^*}$, ξ_+ is continuous: $\xi_{+|L_{\gamma_+^*}} \in L'_{\gamma_+^*}$ and in restriction to $L_{\gamma_-^*}$, ξ_- is continuous: $\xi_{-|L_{\gamma_-^*}} \in L'_{\gamma_-^*}$.

For any $\xi \in W_\Phi$, we denote by

$$\begin{aligned} \xi_+^{\mathrm{ac}} &= \left(\xi_{+|L_{\gamma_+^*}}\right)^{\mathrm{ac}} \in L_{\gamma_+}, \quad \xi_-^{\mathrm{ac}} = \left(\xi_{-|L_{\gamma_-^*}}\right)^{\mathrm{ac}} \in L_{\gamma_-}, \quad \text{and} \quad \xi^{\mathrm{ac}} = \xi_+^{\mathrm{ac}} - \xi_-^{\mathrm{ac}} \\ \xi_+^{\mathrm{s}} &= \left(\xi_{+|L_{\gamma_+^*}}\right)^{\mathrm{s}} \in L^{\mathrm{s}}_{\gamma_+^*}, \quad \xi_-^{\mathrm{s}} = \left(\xi_{-|L_{\gamma_-^*}}\right)^{\mathrm{s}} \in L^{\mathrm{s}}_{\gamma_-^*}, \quad \text{and} \quad \xi^{\mathrm{s}} = \xi_+^{\mathrm{s}} - \xi_-^{\mathrm{s}} \end{aligned}$$

To any element ξ of W_Φ, one can uniquely associate a projected element

$$\pi(\xi) = \xi_+^{\mathrm{ac}} - \xi_-^{\mathrm{ac}} + \xi_+^{\mathrm{s}} - \xi_-^{\mathrm{s}} = \xi^{\mathrm{ac}} + \xi^{\mathrm{s}} \tag{6.5}$$

with $\xi_+^{\mathrm{ac}}, \xi_-^{\mathrm{ac}}, \xi_+^{\mathrm{s}}, \xi_-^{\mathrm{s}} \geq 0$, $\xi_+^{\mathrm{ac}}, \xi_-^{\mathrm{ac}}$ absolutely continuous, $\xi_+^{\mathrm{s}}, \xi_-^{\mathrm{s}}$ singular and $\xi^{\mathrm{ac}} \wedge \xi^{\mathrm{s}} = \xi_+^{\mathrm{ac}} \wedge \xi_-^{\mathrm{ac}} = \xi_+^{\mathrm{s}} \wedge \xi_-^{\mathrm{s}} = 0$.
The uniqueness of the decomposition (6.5) comes from the uniqueness of the decomposition $\xi = \xi_+ - \xi_-$ in a Riesz space together with the uniqueness of the decomposition (5.2).

Theorem 6.3. *Under the assumptions (6.1), for any $\xi \in U^{**}$, taking the decomposition (6.5) into account, we have*

$$\overline{\Phi}(\xi) = \begin{cases} \int_\Omega \gamma\left(\xi^{\mathrm{ac}}\right)\, dR + \sup\left\{\langle \xi^{\mathrm{s}}, u\rangle\,;\, f, \int_\Omega \gamma^*(f)\, dR < \infty\right\} & \text{if } \xi \in W_\Phi \\ +\infty & \text{otherwise.} \end{cases}$$

Remark. In Theorem 6.3, it is meant that for any $\xi \in W_\Phi$, $\sup\left\{\langle \xi^{\mathrm{s}}, u\rangle\,;\, f, \int_\Omega \gamma^*(f)\, dR\right\} = \sup\left\{\langle \xi_+^{\mathrm{s}}, f\rangle\,;\, f \geq 0, \int_\Omega \gamma^*(f)\, dR < \infty\right\} + \sup\left\{\langle \xi_-^{\mathrm{s}}, |f|\rangle\,;\, f \leq 0, \int_\Omega \gamma^*(f)\, dR < \infty\right\}$ where the duality brackets $\langle {\xi^{\mathrm{s}}}_\pm, f\rangle$ are well-defined.

Proof. Because of Lemma 6.1 and Theorem 4.7, one can reduce the proof to the case where γ is an even function. Let γ be even.
If $\gamma \equiv 0$, the result is clear. If $\gamma \not\equiv 0$, we have (see (6.4)): *dom* $\Phi^* \subset L_{\gamma^*}$, and by Theorem 5.1, we obtain

$$L'_{\gamma^*} = L_\gamma \oplus L^{\mathrm{s}}_{\gamma^*}.$$

It follows from Theorems 4.7 and 5.2 that $\overline{\Phi}(\xi) = \infty$ if $\xi \notin W_\Phi$ and that when $\xi \in W_\Phi$, $\overline{\Phi}(\xi) = \int_\Omega \gamma\left(\xi^{\mathrm{ac}}\right) dR + \sup\left\{\langle \xi^{\mathrm{s}}_+, f\rangle\,;\, f \geq 0, \int_\Omega \gamma^*(f)\, dR < \infty\right\} + \sup\left\{\langle \xi^{\mathrm{s}}_-, |f|\rangle\,;\, f \leq 0, \int_\Omega \gamma^*(f)\, dR < \infty\right\}$ which yields the desired result. ■

In order to consider the extension $\overline{\Psi}$ of Ψ : the restriction of Φ to $V \subset U$, we have to consider the subset of W_Φ :

$$W_\Psi = \sigma(W_\Phi, dom\ \Phi^*)\text{-closure of } V \text{ in } W_\Phi.$$

To any element ξ of W_Ψ, one can uniquely associate a projected element

$$\pi(\xi) = \xi^{\mathrm{ac}} + \xi^{\mathrm{s}} \tag{6.6}$$

with a natural notation. The uniqueness of the decomposition (6.6) comes from the uniqueness of (6.5).
In ([Lé2], Lemma 3.1), it is shown that if $\lim_{x\to\infty} \gamma(x) = \infty$, R is bounded and $\xi \in W_\Psi$, then ξ^{ac} is the limit of a $\sigma(L_1, L_\infty)$-convergent sequence in V. Under an additional assumption if $R(\Omega) = \infty$, a similar statement is also proved.

Theorem 6.4. *Under the assumptions (6.1), for any $\xi \in V^{**}$, taking the decomposition (6.6) into account, we have*

$$\overline{\Psi}(\xi) = \begin{cases} \int_\Omega \gamma\left(\xi^{\mathrm{ac}}\right) dR + \sup\left\{\langle \xi^{\mathrm{s}}, f\rangle\,;\, f, \int_\Omega \gamma^*(f)\, dR < \infty\right\} & \textit{if } \xi \in W_\Psi \\ +\infty & \textit{otherwise.} \end{cases}$$

Remark. The topology $\sigma(W_\Phi, dom\ \Phi^*)$ which is needed for the definition of W_Ψ may not be Hausdorff in the general case, but by Lemma 4.6.b, $\overline{\Psi}(\xi) = \overline{\Psi}(\xi')$ whenever ξ and ξ' are not separated by the topology.

Proof. The convex integral functional Φ satisfies condition (2.4) with the Orlicz gauge norm associated with the Young function $\max(\gamma_+, \gamma_-)$. By Corollary 2.4, Φ^* is $\sigma(U^*, U)$-inf-compact. In addition, by Theorem 5.2, Φ is $\sigma(U, U^*)$-lower semicontinuous and convex. Therefore, the assumptions of Proposition 3.3 are fulfilled. One concludes with Proposition 3.3, Theorem 4.7 and Theorem 6.3. ■

A direct consequence. In [Lé2], Theorem 6.4 is the first step for a proof of a general result of minimization of convex integral functionals of the type (5.3) under linear constraints. In this subsection, we only indicate a consequence of Theorem 6.4, which does not require any additional work.

Since $\Psi^* = \overline{\Psi}^*$, under (6.1), for any $\ell \in V^*$: $\sup_{v\in V}\{\langle \ell, v\rangle - \Psi(v)\} = \sup_{v\in W_\Psi}\{\langle \ell, \xi\rangle - \overline{\Psi}(\xi)\}$ and

$$\begin{aligned} &\sup\left\{\langle \ell, v\rangle - \int_\Omega \gamma(v)\, dR\,;\, v = v_+ - v_-, v_+ \in V_+^o, v_- \in V_-^o\right\} \\ =&\sup\left\{\langle \ell, v\rangle - \int_\Omega \gamma(v)\, dR\,;\, v \in V\right\} \\ =&\sup\left\{\langle \ell, \xi\rangle - \int_\Omega \gamma(\xi)\, dR\,;\, \xi = \xi_+ - \xi_-, \xi_+ \in \overline{V_+}, \xi_- \in \overline{V_-}\right\} \end{aligned} \tag{6.7}$$

where $\overline{V_\pm}$ is the $\sigma(L_{\gamma_\pm}, L_{\gamma_\pm^*})$-closure of V in $L_{\gamma_\pm}$ and $V_\pm^o$ is any $\sigma(V, L_{\gamma_\pm^*})$-dense subset of V.

Notice that, θ being a Young function, the space $\mathcal{S}$ of simple functions is dense in L_{θ^*} when θ^* satisfies the Δ_2-condition: $\theta^*(2t) \leq K\theta^*(t), \forall t$, for some $K > 0$; or when $L_{\theta^*} = L_\infty \cap L_p$ $(1 \leq p \leq \infty)$. In this situation, by Goldstine's lemma (see (2.3)) and Ascoli's theorem, we see that the $\sigma(L_\theta, L_{\theta^*})$-closure and the $\sigma(L_\theta, \mathcal{S})$-closure of V in L_θ are equal.

Let us consider the case of the entropy: I_τ^*, where $\gamma = \tau$ is given by

$$\tau(x) = e^x - x - 1, \quad x \in \mathbb{R}.$$

This gives $\tau_+(x) = e^{|x|} - |x| - 1$, $\tau_-(x) = e^{-|x|} + |x| - 1$, $\tau_+^*(y) = (|y|+1)\log(|y|+1) - |y|$ and $\tau_-^*(y) = \begin{cases} (1-|y|)\log(1-|y|) + |y| & \text{if } |y| < 1 \\ 1 & \text{if } |y| = 1 \\ +\infty & \text{if } |y| > 1 \end{cases}$. By (6.7) and the previous remark, with $V = M_{\tau_+} \cap M_{\tau_-}$, for any $\ell \in V^*$, we have

$$\begin{aligned} I_\tau^*(\ell) &= \sup\left\{ \langle \ell, v\rangle - \int_\Omega \tau(v)\, dR\, ;\, v \in \mathcal{S} \right\} \\ &= \sup\left\{ \langle \ell, \xi\rangle - \int_\Omega \tau(\xi)\, dR\, ;\, \xi = \xi_+ - \xi_-, \xi_+ \in L_{\tau_+}, \xi_- \in L_{\tau_-} \right\}, \end{aligned}$$

since L_{τ_+} is included in the bidual space of M_{τ_+}, L_{τ_-} is the bidual space of M_{τ_-} and $\mathcal{S}$ is dense in M_{τ_+} and M_{τ_-}.

If $\mathcal{A}$ is the Borel σ-field of a metric space Ω and if there exists an R-localizing sequence of closed sets, then the space $C_{b,o}(\Omega)$, of the continuous bounded functions v satisfying $R(v \neq 0) < \infty$, is $\sigma(M_{\tau_\pm}, \mathcal{S})$-dense, so that

$$I_\tau^*(\ell) = \sup\left\{ \langle \ell, v\rangle - \int_\Omega \tau(v)\, dR\, ;\, v \in C_{b,o}(\Omega) \right\}.$$

If $\mathcal{A}$ is the Borel σ-field of a Polish space Ω and if there exists an R-localizing sequence of closed sets, then the space $C_c(\Omega)$, of the continuous functions with a compact support is $\sigma(M_{\tau_\pm}, \mathcal{S})$-dense, so that

$$I_\tau^*(\ell) = \sup\left\{ \langle \ell, v\rangle - \int_\Omega \tau(v)\, dR\, ;\, v \in C_c(\Omega) \right\}.$$

If R is a (nonnegative) Radon measure on $\Omega = \mathbb{R}^d$, then the space $C_c^\infty(\mathbb{R}^d)$, of the infinitely differentiable functions with a compact support is $\sigma(C_c(\mathbb{R}^d), \mathcal{S})$-dense, so that

$$I_\tau^*(\ell) = \sup\left\{ \langle \ell, v\rangle - \int_{\mathbb{R}^d} \tau(v)\, dR\, ;\, v \in C_c^\infty(\mathbb{R}^d) \right\}.$$

Erratum. In [Lé1], an expression for $\overline{\Psi}$ in the special case where $\gamma = \tau$ is given in Theorem 6.6 and Corollary 6.7 (with a different method of proof), but the contribution of the singular component is missing. Fortunately, this does not affect the correctness of the remaining results of [Lé1].

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