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Large deviations for Poisson random measures and processes with independent increments

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Abstract. Large deviation principles are proved for rescaled Poisson random measures. As a consequence, Freidlin-Wentzell type large deviations results for processes with independent increments are obtained in situations where exponential moments are infinite.

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1. Introduction

Let $(p(\omega,t)\,;\, t\geq 0, \omega\in\Omega)$ be a stationary Poisson point process with values in a measurable space $\mathbf{A}$. For any $\varepsilon>0$, let us associate with it the Poisson random measure on $[0,T]\times\mathbf{A}$:

$$N^\varepsilon_\omega = \sum_{s\in D_\omega, \varepsilon s\leq T} \varepsilon\delta_{(\varepsilon s, p(\omega,s))}$$

where $T>0$, $\omega\in\Omega$, $\delta_{(t,a)}$ is the Dirac measure at $(t,a)\in[0,T]\times\mathbf{A}$ and D_ω is the domain of definition of $p(\omega,\cdot)$. We are interested in the random integrals $\int_{[0,T]\times\mathbf{A}}\theta\, dN^\varepsilon_\omega$ and the corresponding centred stochastic integrals $\int_{[0,T]\times\mathbf{A}}\theta\, d\widetilde{N}^\varepsilon_\omega$ where θ are measurable functions on $[0,T]\times\mathbf{A}$ ($\int\theta\, d\widetilde{N}^\varepsilon$ is the usual L^2-isometric extension of $\theta\mapsto\int\theta\, dN^\varepsilon - \mathbb{E}\int\theta\, dN^\varepsilon$).

In this article, large deviations (LD) results as ε tends to zero are proved for $\{\widetilde{N}^\varepsilon\}$ and $\{N^\varepsilon\}$ which are viewed as random linear forms on spaces of functions $\theta:[0,T]\times\mathbf{A}\to\mathbb{R}$. As a corollary, LD results are obtained for $\mathbb{R}^d$-valued processes $\{Y^\varepsilon\}$ defined for any $0\leq t\leq T$ by

$$Y^\varepsilon(t)=\varepsilon Y(t/\varepsilon)$$

where Y belongs to a class of $\mathbb{R}^d$-valued processes with independent increments (including all the Lévy processes: i.e. with stationary independent increments) without Gaussian component. Our results are of the following form

$$\varepsilon \log \mathbb{P}(N^\varepsilon \in A) \underset{\varepsilon\to 0}{\asymp} - I(A), \quad \varepsilon \log \mathbb{P}(\widetilde{N}^\varepsilon \in A) \underset{\varepsilon\to 0}{\asymp} - \tilde{I}(A) \quad \text{and} \quad \varepsilon \log \mathbb{P}(Y^\varepsilon \in B) \underset{\varepsilon\to 0}{\asymp} - J(B)$$

where A is a subset of the algebraic dual of some space of functions θ on $[0,T]\times\mathbf{A}$ and B is a subset of the space D of càdlàg paths $y:[0,T]\to\mathbb{R}^d$ with $y(0)=0$. A rigorous statement of such results is in terms of large deviation principles.

Conventions. We keep the conventions for LD results which were adopted in the book by A. Dembo and O. Zeitouni ([DeZ]). Let $\mathcal{X}$ be a Haussdorf topological space endowed with its Borel σ-field. A rate function is a $[0,\infty]$-valued lower semicontinuous function on $\mathcal{X}$. It is said to be a good rate function if its level sets are compact. A family $\{X^\varepsilon\}$ of $\mathcal{X}$-valued random variables is said to obey the large deviation principle (LDP) in $\mathcal{X}$ with rate function I if for each open subset $G\subset\mathcal{X}$ and each closed subset $F\subset\mathcal{X}$

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \le -\inf_{x\in F} I(x) \quad \text{and} \quad \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \ge -\inf_{x\in G} I(x)$$

If the lower bound holds for each open subset but the upper bound is only valid for compact subsets of $\mathcal{X}$, $\{X^\varepsilon\}$ is said to obey a weak LDP.

About the literature. Large deviation results for $\{N^\varepsilon\}$ acting on functions θ satisfying $\mathbb{E}\exp(\beta\int\theta\,dN)<\infty$ for all $\beta>0$, have recently been obtained by D. Florens and H. Pham ([FlP]) while LD results for $\{Y^\varepsilon\}$ have already been obtained by many authors. When Y is the Wiener process, the LDP for $\{Y^\varepsilon\}$ is given by Schilder's theorem ([Sch]). This has been extended by A. de Acosta ([Ac1], Theorem 1.2), ([Ac2], Theorem 6) to the case where Y is any Banach-valued Lévy process satisfying

$$\mathbb{E}\exp(\beta|Y(1)|)<\infty,\ \forall\beta>0. \tag{1.1}$$

A little sooner, R.S. Lipster and A.A. Pukhalskii (LiP) had obtained similar LD results for $\mathbb{R}^d$-valued normalized semimartingales under integrability assumptions of the type of (1.1). In this situation, the LDP holds in D with the uniform convergence topology. This result will be partially recovered later as a particular case of the Laplace principle by P. Dupuis and R.S. Ellis ([DuE], Theorem 10.2.6) for a class of Markov processes whose large deviations had already been investigated by A.D. Wentzell ([Wen], [FrW]). It seems that the first work in this direction involving jump processes is due to Borovkov ([Bor]) who proves a LDP under (1.1) for real compound Poisson processes.

Under the weaker assumption

$$\mathbb{E}\exp(\beta_o|Y(1)|)<\infty,\ \text{for some } \beta_o>0 \tag{1.2}$$

LDPs still hold for $\{Y^\varepsilon\}$ with weaker topologies on D. Assuming that the sample paths of Y have bounded variation, J. Lynch and J. Sethuraman ([LyS]) have obtained the LDP in dimension $d=1$

with the weak topology ($\lim_{n\to\infty} y_n = y$ if and only if $\lim_{n\to\infty} y_n(t) = y(t)$ for each $0 \le t \le T$ where y is continuous). In ([Ac1], Theorem 5.2), this result is extended to the finite dimensional case: $d < \infty$. Under (1.2), with $d = 1$ but dropping the bounded variation assumption, Mogulskii ([Mog], Theorems 2.5 & 2.7) proves the LD lower bound for the weak topology together with the LDP in the completion D_s of D with respect to some Skorokhod metric ($D \neq D_s$). The rate function I_s of this LDP on D_s is the greatest lower semicontinuous extension of an explicit function I on D, hence for any $A \subset D$, $\inf_{y\in \mathrm{cl}_D(A)} I(y) \ge \inf_{y\in \mathrm{cl}_{D_s}(A)} I_s(y)$ and $\inf_{y\in \mathrm{int}_D(A)} I(y) \le \inf_{y\in \mathrm{int}_{D_s}(A)} I_s(y)$. Therefore, the pullback of this LDP onto D isn't clear as it should require the identification of those $A \subset D$ which achieve equality in the above formulas.

Under the assumption that the sample paths have bounded variations and $\mathbb{E}|Y(1)| < \infty$, ([Ac1], Theorem 5.2) states a LD lower bound for the weak topology.

Under the only assumption $\mathbb{E}|Y(1)| < \infty$, A. de Acosta also obtains a possibly non-optimal LD lower bound for the uniform convergence topology ([Ac1], Theorem 5.1).

Presentation of the results. The aim of this article is to obtain LD results for $\{\widetilde{N}^\varepsilon\}$ and $\{N^\varepsilon\}$ and to improve in several directions the above LD results for $\{Y^\varepsilon\}$ in the situation where (1.1) doesn't hold. The bounded variation and ($d = 1$) assumptions are removed, the topologies are strengthened and (under an exponential integrability assumption of the type (1.2)) the requirement that the increments are stationary is removed. New alternate expressions for the LD rate functions are also derived.

We shall take advantage of the Lévy-Khinchin integral representation of $Y(1)$ and of duality results for Orlicz spaces associated with the log-Laplace transform of the Poisson law.

In Section 2, we compute $\log \mathbb{E}\exp(\int_{[0,T]\times\mathbf{A}} \theta\, d\widetilde{N})$ and $\log \mathbb{E}\exp(\int_{[0,T]\times\mathbf{A}} \theta\, dN)$ in terms of the characteristic measure of N, for a general function θ.

This log-Laplace evaluation is the first step to obtain in Section 3 weak LDPs for $\{\widetilde{N}^\varepsilon\}$ and $\{N^\varepsilon\}$ considered as random processes indexed by time-independent functions with no exponential integrability restrictions. Our proof mimics the usual approach to Cramér's theorem in $\mathbb{R}^d$ without exponential moments, via subadditivity arguments.

In Section 4, we consider $\{\widetilde{N}^\varepsilon\}$ (resp. $\{N^\varepsilon\}$) as random processes indexed by functions θ on $[0,T]\times\mathbf{A}$ such that $\mathbb{E}\exp(\beta\int_{[0,T]\times\mathbf{A}} \theta\, d\widetilde{N}) < \infty$ (resp. $\mathbb{E}\exp(\beta\int_{[0,T]\times\mathbf{A}} \theta\, dN) < \infty$), $\forall|\beta| \le \beta_o$, for some $\beta_o > 0$, and prove that they obey LDPs for the product topology.

Let M stand for the space of $\mathbb{R}^d$-valued (vector) measures on $[0,T]$ and $\sigma(D,M)$ be the topology of D weakened by M : $\lim_{n\to\infty} y_n = y$ if and only if for each $\mu \in M$, $\lim_{n\to\infty}\int_{[0,T]} y_n\, d\mu = \int_{[0,T]} y\, d\mu$. The following topologies on D are ordered as follows. The weak topology is weaker than the pointwise convergence topology which is weaker than $\sigma(D,M)$ which in turn is weaker than the uniform convergence topology. The Skorokhod topology is stronger than the weak topology and weaker than the uniform convergence one. It cannot be compared neither with the pointwise convergence topology nor with $\sigma(D,M)$. All these topologies, except the uniform one, generate the same Borel σ-field on D.

The results of Sections 3 and 4 are applied in Section 5 to prove LD results for $\{Y^\varepsilon\}$. With Y a general Lévy process on $\mathbb{R}^d$ (without Gaussian component), Theorem 5.1 states a weak LDP in D

with the topology of pointwise convergence. Under an assumption of the type of (1.2), Theorem 5.3 states a LDP for $\{Y^\varepsilon\}$ in $\sigma(D, M)$. Its proof is based on the contraction principle applied to the LDPs of Section 4. It relies on integration by parts formulas and a dual representation result which are derived in the Appendix.

In Section 6, explicit expressions for the rate functions of Section 5 are computed. They extend to the multidimensional case ($1 \leq d < \infty$) previous results of [LyS] and [Ac1]. Their derivation largely relies on a paper by R.T. Rockafellar ([Roc]).

As a consequence, we give in Theorem 6.3 a sufficient condition for the optimality of the LD lower bound in uniform convergence topology of [Ac1].

2. The log-Laplace transform of a Poisson random measure

Let $(\Omega, \mathcal{F}, I\!\!P)$ be a probability space, and R a σ-finite nonnegative measure on a standard measurable space $(\mathbf{U}, \mathcal{U})$. We consider the Poisson random measure M built on $(\Omega, \mathcal{F})$ and $(\mathbf{U}, \mathcal{U})$ with intensity R. Let $\mathcal{E}_R$ be the space of all the elementary functions on $(\mathbf{U}, \mathcal{U}) : f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{B_i}$ where $n \geq 1$, $\lambda_1, \dots, \lambda_n \in I\!\!R$ and $B_1, \dots, B_n$ are disjoint subsets in the class $\{B \in \mathcal{U}\,;\, R(B) < \infty\}$. As usual, the stochastic integral $f \in L^2(\mathbf{U}, R) \mapsto \int_{\mathbf{U}} f(u) \widetilde{M}(\cdot, du) \in L^2(\Omega, I\!\!P)$ is the unique isometric extension of $f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{B_i} \in \mathcal{E}_R \mapsto \left(\omega \mapsto \sum_{i=1}^{n} \lambda_i (M(\omega, B_i) - R(B_i)) \right) \in L^2(\Omega, I\!\!P)$.

Our aim is to compute the log-Laplace transform of the law of $\widetilde{M}$. Before stating this result at Proposition 2.2, we introduce some notations. Let us denote $\tilde{\rho}$ the log-Laplace transform of the centered Poisson law with parameter 1: $\tilde{\rho}(x) = \log I\!\!E e^{x(X - I\!\!E X)}, x \in I\!\!R$ where X is Poisson(1) distributed. We have

$$\tilde{\rho}(x) = e^x - x - 1, \quad x \in I\!\!R.$$

For any measurable function f on $\mathbf{U}$, we define the Luxemburg norm

$$\|f\|_\tau = \inf\{\alpha > 0\,;\, \int_{\mathbf{U}} \tau(f/\alpha)\, dR \leq 1\} \in [0, \infty],$$

which is associated with the Young function

$$\tau(x) := \tilde{\rho}(|x|) = e^{|x|} - |x| - 1, \quad x \in I\!\!R.$$

The corresponding Orlicz spaces are

$$\begin{aligned} L^\tau(\mathbf{U}, R) &:= \{f : \mathbf{U} \to I\!\!R, \text{measurable}, \|f\|_\tau < \infty\} \quad \text{and} \\ M^\tau(\mathbf{U}, R) &:= \left\{ f : \mathbf{U} \to I\!\!R, \text{measurable}, \int_{\mathbf{U}} \tau(\lambda f)\, dR < \infty, \forall \lambda > 0 \right\}. \end{aligned} \tag{2.1}$$

Notice that $M^\tau \subset L^\tau \subset L^2$ where, in general, the inclusions are strict. The spaces M^τ and L^τ are endowed with the norm $\|\cdot\|_\tau$, they are Banach spaces.

Lemma 2.1. *For any $f \in M^\tau(\mathbf{U}, R)$, we have*

$$\log \mathbb{E} \exp\left(\int_{\mathbf{U}} f\, d\widetilde{M}\right) = \int_{\mathbf{U}} \tilde\rho(f)\, dR \in [0, +\infty[.$$

<u>Proof</u>. By an easy computation, in restriction to the elementary functions, we obtain

$$\log \mathbb{E} \exp\left(\int_{\mathbf{U}} f\, d\widetilde{M}\right) = \int_{\mathbf{U}} \tilde\rho(f)\, dR, \quad \forall f \in \mathcal{E}_R. \tag{2.2}$$

We want to extend this identity to $M^\tau(\mathbf{U}, R)$. This will follow from a continuity-density argument. As $\mathcal{E}_R$ is $\|\cdot\|_\tau$-dense in $M^\tau(\mathbf{U}, R)$ (see [RaR]), it remains to check that the functions: $G_1(f) = \log \mathbb{E} \exp\left(\int_{\mathbf{U}} f\, d\widetilde{M}\right)$ and $G_2(f) = \int_{\mathbf{U}} \tilde\rho(f)\, dR$ are $\|\cdot\|_\tau$-continuous on $M^\tau(\mathbf{U}, R)$.
As $\tilde\rho$ is a convex function and R is nonnegative, G_2 is also convex. Its effective domain is the whole space $M^\tau(\mathbf{U}, R)$ and it is bounded above on the unit ball of $M^\tau(\mathbf{U}, R)$. Indeed, for any $f \in M^\tau$, with $\|f\|_\tau \le 1$, we have $G_2(f) \le \int_{\mathbf{U}} \tau(f)\, dR \le 1$ (notice that $\tilde\rho(x) \le \tau(x), \forall x \in \mathbb{R}$). Therefore, G_2 is continuous on $M^\tau(\mathbf{U}, R)$.
Let us show now that G_1 is also continuous on $M^\tau(\mathbf{U}, R)$. We have for some $C > 0$,

$$\|\int_{\mathbf{U}} f\, d\widetilde{M}\|_{\tau, \mathbb{P}} \le C\|f\|_{\tau, R}, \forall f \in \mathcal{E}_R, \tag{2.3}$$

since for any $f \in \mathcal{E}_R$

$$\begin{aligned}
\|\int_{\mathbf{U}} f\, d\widetilde{M}\|_{\tau,\mathbb{P}} &:= \inf\left\{b > 0\,;\, \mathbb{E}\tilde\rho\left(\frac{|\int_{\mathbf{U}} f\, d\widetilde{M}|}{b}\right) \le 1\right\} \\
&\le \inf\left\{b > 0\,;\, \mathbb{E}\exp\left(\frac{\int_{\mathbf{U}} f\, d\widetilde{M}}{b}\right) + \mathbb{E}\exp\left(\frac{-\int_{\mathbf{U}} f\, d\widetilde{M}}{b}\right) \le 3\right\} \\
&= \inf\left\{b > 0\,;\, \exp\left(\int_{\mathbf{U}} \tilde\rho\left(\frac{f}{b}\right) dR\right) + \exp\left(\int_{\mathbf{U}} \tilde\rho\left(-\frac{f}{b}\right) dR\right) \le 3\right\} \\
&\le \inf\left\{b > 0\,;\, \int_{\mathbf{U}} \tilde\rho\left(\frac{|f|}{b}\right) dR \le \log(3/2)\right\} \\
&\le \|f\|_{\tau,R}/\log(3/2).
\end{aligned}$$

The first inequality follows from: $\tilde\rho(|x|) \le \tilde\rho(x) + \tilde\rho(-x) = e^x + e^{-x} - 2$, the following equality follows from (2.2), the second inequality follows from: $\max(\tilde\rho(x), \tilde\rho(-x)) \le \tilde\rho(|x|)$ and the third one from: $\tau(\alpha x) \le \alpha\tau(x)$ when $0 \le \alpha \le 1$. The inequality (2.3) means that $f \in (\mathcal{E}_R, \|\cdot\|_{\tau,R}) \mapsto \int_{\mathbf{U}} f\, d\widetilde{M} \in (M^\tau(\Omega, \mathbb{P}), \|\cdot\|_{\tau,\mathbb{P}})$ is continuous. Let I_τ denote its continuous extension to $M^\tau(\mathbf{U}, R)$. For any $f \in M^\tau(\mathbf{U}, R)$, there exists a sequence $(f_n)_{n\ge1}$ in $\mathcal{E}_R$ such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $M^\tau(\mathbf{U}, R)$. But this implies that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^2(\mathbf{U}, R)$. Therefore, $I_\tau(f) = \lim_{n\to\infty} \int_{\mathbf{U}} f_n\, d\widetilde{M} = \int_{\mathbf{U}} f\, d\widetilde{M}$. We have just proved that the stochastic integral $f \in M^\tau(\mathbf{U}, R) \mapsto \int_{\mathbf{U}} f\, d\widetilde{M} \in M^\tau(\Omega, \mathbb{P})$ is continuous. In particular, (2.3) extends to: $\|\int_{\mathbf{U}} f\, d\widetilde{M}\|_{\tau,\mathbb{P}} \le \|f\|_{\tau,R}, \forall f \in M^\tau(\mathbf{U}, R)$ and for any $f \in M^\tau(\mathbf{U}, R)$ such that$\|f\|_{\tau,R} \le 1/4$, we get $\mathbb{E}\tilde\rho(2|\int_{\mathbf{U}} f\, d\widetilde{M}|) \le 1$. As, $e^x \le \tilde\rho(2|x|) + 3, \forall x \in \mathbb{R}$, it follows that

$\mathbb{E}\exp(\int_{\mathbf{U}} f\, d\widetilde{M}) \leq \mathbb{E}\tilde{\rho}(2|\int_{\mathbf{U}} f\, d\widetilde{M}|) + 3 \leq 4$. Therefore, the convex function $f \in M^\tau(\mathbf{U}, R) \mapsto \mathbb{E}\exp(\int_{\mathbf{U}} f\, d\widetilde{M})$ is bounded above on the open ball $\{f \in M^\tau(\mathbf{U}, R)\,;\, \|f\|_{\tau,R} < 1/4\}$, so that it is continuous on the interior of its effective domain. But, its effective domain is the whole space $M^\tau(\mathbf{U}, R)$. Hence, $f \in M^\tau(\mathbf{U}, R) \mapsto \mathbb{E}\exp(\int_{\mathbf{U}} f\, d\widetilde{M})$ is continuous and so is its logarithm G_1. This completes the proof of the lemma. ■

Let us denote the log-Laplace transform of the Poisson(1) law:

$$\rho(x) = e^x - 1,\ x \in \mathbb{R}.$$

Proposition 2.2.
(a) For any $f \in L^2(\mathbf{U}, R)$, we have

$$\log \mathbb{E}\exp\left(\int_{\mathbf{U}} f\, d\widetilde{M}\right) = \int_{\mathbf{U}} \tilde{\rho}(f)\, dR \in [0, +\infty].$$

(b) For any $f \in L^1(\mathbf{U}, R)$, we have

$$\log \mathbb{E}\exp\left(\int_{\mathbf{U}} f\, dM\right) = \int_{\mathbf{U}} \rho(f)\, dR \in]-\infty, +\infty].$$

(c) If R is a bounded measure, for any measurable function f on $\mathbf{U}$, we have

$$\log \mathbb{E}\exp\left(\int_{\mathbf{U}} f\, dM\right) = \int_{\mathbf{U}} \rho(f)\, dR \in]-\infty, +\infty].$$

Remarks. In (b), the meaning of $\int_{\mathbf{U}} \rho(f)\, dR$ is $\int_{\mathbf{U}} \tilde{\rho}(f)\, dR + \int_{\mathbf{U}} f\, dR \in [0, \infty] + \mathbb{R} =]-\infty, \infty]$.
In (c), the meaning of $\int_{\mathbf{U}} \rho(f)\, dR$ is $\int_{\mathbf{U}} \rho(f_+)\, dR + \int_{\mathbf{U}} \rho(-f_-)\, dR \in [0, \infty] +]-R(\mathbf{U}), 0] \subset]-\infty, \infty]$.

Proof. Let us begin with (a). For any $f \in L^2(\mathbf{U}, R)$ and $k \geq 1$, let us put $f_k = \mathbb{1}_{\{1/k \leq |f| \leq k\}} f$. We have $\lim_{k\to\infty} f_k = f$ pointwise and $\lim_{k\to\infty} \int_{\mathbf{U}} f_k\, d\widetilde{M} = \int_{\mathbf{U}} f\, d\widetilde{M}$ in $L^2(\Omega, \mathbb{P})$. Hence, possibly extracting a subsequence, we get $\lim_{k\to\infty} \int_{\mathbf{U}} f_k\, d\widetilde{M} = \int_{\mathbf{U}} f\, d\widetilde{M}$, $\mathbb{P}$-a.s. As f_k belongs to $M^\tau(\mathbf{U}, R)$, it follows from Lemma 2.1 that

$$\log \mathbb{E}\exp\left(\int_{\mathbf{U}} f_k\, d\widetilde{M}\right) = \int_{\mathbf{U}} \tilde{\rho}(f_k)\, dR = \int_{\{1/k \leq |f| \leq k\}} \tilde{\rho}(f)\, dR,\ \forall k \geq 1.$$

Using Fatou's lemma and the monotone convergence theorem:

$$\begin{aligned}\log \mathbb{E}\exp\left(\int_{\mathbf{U}} f\, d\widetilde{M}\right) &= \log \mathbb{E} \lim_{k\to\infty} \exp\left(\int_{\mathbf{U}} f_k\, d\widetilde{M}\right)\\ &\leq \liminf_{k\to\infty} \log \mathbb{E}\exp\left(\int_{\mathbf{U}} f_k\, d\widetilde{M}\right) = \lim_{k\to\infty} \int_{\{1/k \leq |f| \leq k\}} \tilde{\rho}(f)\, dR = \int_{\mathbf{U}} \tilde{\rho}(f)\, dR.\end{aligned}$$

Now, let us prove the converse inequality. Let f, g stand in $L^2(\mathbf{U}, R)$. By Jensen's inequality, $\log \mathbb{E}\exp\left(\int_{\mathbf{U}} f\, d\widetilde{M}\right) \geq \mathbb{E}\int_{\mathbf{U}} f\, d\widetilde{M} = 0$. If $fg = 0$, then $\int_{\mathbf{U}} f\, d\widetilde{M}$ and $\int_{\mathbf{U}} g\, d\widetilde{M}$ are independent and $\log \mathbb{E} \exp\left(\int_{\mathbf{U}} (f+g)\, d\widetilde{M}\right) = \log \mathbb{E} \exp\left(\int_{\mathbf{U}} f\, d\widetilde{M}\right) + \log \mathbb{E} \exp\left(\int_{\mathbf{U}} g\, d\widetilde{M}\right)$. As, for any $k \geq 1$, we have $f_k(f - f_k) = 0$, it follows that

$$\begin{aligned}\log \mathbb{E} \exp\left(\int_{\mathbf{U}} f\, d\widetilde{M}\right) &= \log \mathbb{E} \exp\left(\int_{\mathbf{U}} f_k\, d\widetilde{M}\right) + \log \mathbb{E} \exp\left(\int_{\mathbf{U}} (f - f_k)\, d\widetilde{M}\right) \\ &\geq \log \mathbb{E} \exp\left(\int_{\mathbf{U}} f_k\, d\widetilde{M}\right) = \int_{\{1/k \leq |f| \leq k\}} \tilde{\rho}(f)\, dR.\end{aligned}$$

Letting k tend to infinity, we obtain the desired inequality and (a).

Let us prove (b) and (c).

In the situation (b), $\mathbb{E} \int_{\mathbf{U}} |f|\, dM = \int_{\mathbf{U}} |f|\, dR < \infty$. Therefore, $\int_{\mathbf{U}} f\, dM$ is almost surely an absolutely convergent series.

In the situation (c), $\int_{\mathbf{U}} f\, dM$ is almost surely the sum of finitely many terms.

Under (b) or (c), for any $k \geq 1$, we have $f_k \in L^2(\mathbf{U}, R) \cap L^1(\mathbf{U}, R)$. Hence, $\int_{\mathbf{U}} f_k\, dM = \int_{\mathbf{U}} f_k\, d\widetilde{M} + \int_{\mathbf{U}} f_k\, dR$ and, with Lemma 2.1: $\log \mathbb{E} \exp \int_{\mathbf{U}} f_k\, dM = \int_{\mathbf{U}} \rho(f_k)\, dR \in \mathbb{R}$.

Let $f \geq 0$. Then,

$$\begin{aligned}\log \mathbb{E} \exp \int_{\mathbf{U}} f\, dM &= \lim_{k\to\infty} \log \mathbb{E} \exp \int_{\mathbf{U}} f_k\, dM \quad \text{(monotone convergence)} \\ &= \lim_{k\to\infty} \int_{\mathbf{U}} \rho(f_k)\, dR \\ &= \int_{\mathbf{U}} \rho(f)\, dR \quad \text{(monotone convergence)}\end{aligned}$$

Let $f \leq 0$. One obtains similar equalities, invoking the dominated convergence theorem instead of the monotone convergence theorem for the first equality. Indeed, we have $0 < \exp \int_{\mathbf{U}} f_k\, dM \leq 1$, $\forall k \geq 1$. Therefore,

$$\log \mathbb{E} \int_{\mathbf{U}} f_+\, dM = \int_{\mathbf{U}} \rho(f_+)\, dR \quad \text{and} \quad \log \mathbb{E} \int_{\mathbf{U}} f_-\, dM = \int_{\mathbf{U}} \rho(f_-)\, dR \tag{2.4}$$

On the other hand, since $\int_{\mathbf{U}} f_+\, dM$ and $\int_{\mathbf{U}} f_-\, dM$ are independent random variables, we obtain

$$\log \mathbb{E} \int_{\mathbf{U}} f\, dM = \log \mathbb{E} \exp \int_{\mathbf{U}} f_+\, dM + \log \mathbb{E} \exp(-\int_{\mathbf{U}} f_-\, dM).$$

Together with (2.4), this gives the announced results. ■

3. Large deviations for Poisson random measures without integrability conditions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ a right continuous filtration of the $\mathbb{P}$-complete σ-field $\mathcal{F}$ and $(\mathbf{A}, \mathcal{A})$ a standard measurable space. Let us consider an $\mathbf{A}$-valued point process $(p_t)_{t\geq 0}$ built on $(\Omega, \mathcal{F})$. It defines a $\mathbb{N} \cup \{\infty\}$-valued counting random measure on $[0, \infty[\times \mathbf{A}$ endowed with the product σ-field $\mathcal{B}([0, \infty[) \times \mathcal{A}$:

$$N(\omega; dtda) = \sum_{s \in D_\omega} \delta_{(s, p_s(\omega))}(dtda), \quad \omega \in \Omega$$

where D_ω is the countable definition domain of $t \mapsto p_t(\omega)$. This process is assumed to be $(\mathcal{F}_t)_{t\geq 0}$-adapted in the sense that for any $A \in \mathcal{A}$, $N(\omega, [0,t] \times A)$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted.

Let Λ be a nonnegative σ-finite measure on $(\mathbf{A}, \mathcal{A})$. We suppose that p is a stationary Poisson process with characteristic measure Λ, this means that N is a Poisson random measure with intensity measure

$$R(dtda) = dt\Lambda(da).$$

The associated stochastic integral is denoted $\int_{[0,\infty[\times\mathbf{A}} g\, d\widetilde{N}$, $g \in L^2([0,\infty[\times\mathbf{A}, R)$.

We fix the terminal time $T > 0$. For all $\varepsilon > 0$, let us consider the ε-normalized measure on $[0,T]\times\mathbf{A}$

$$N^\varepsilon(\omega, dtda) = \sum_{s\in D_\omega, s\leq T/\varepsilon} \varepsilon\delta_{(\varepsilon s, p_s(\omega))}(dtda), \quad \omega \in \Omega$$

For any $A \in \mathcal{A}$ such that $\Lambda(A) < \infty$ and any $\varepsilon > 0, 0 \leq t \leq T$ the law of $N^\varepsilon([0,t] \times A)$ is $\varepsilon\mathcal{P}(\frac{t}{\varepsilon}\Lambda(A))$, in particular $I\!\!E N^\varepsilon([0,t] \times A) = t\Lambda(A) = I\!\!E N([0,t] \times A)$ and $\mathrm{Var}(N^\varepsilon([0,t] \times A)) = \varepsilon t\Lambda(A) = \varepsilon \mathrm{Var}(N([0,t] \times A)$.

Let us denote L_0 the space of all the measurable functions $\varphi : \mathbf{A} \mapsto I\!R$, $L_1 = L^1(\mathbf{A}, \Lambda)$ and $L_2 = L^2(\mathbf{A}, \Lambda)$, where Λ-almost everywhere equal functions are identified. For any integer $d \geq 1$, L_0^d, L_1^d and L_2^d are the corresponding spaces of functions $\varphi : \mathbf{A} \mapsto I\!R^d$.

For any $\theta : [0,T] \times \mathbf{A} \mapsto I\!R^d$, provided that the integrals below are meaningful, one defines

$$\begin{aligned}
\langle \widetilde{N}^\varepsilon, \theta\rangle &= (\langle \widetilde{N}^\varepsilon, \theta_k\rangle)_{k\leq d} \text{ with } \langle \widetilde{N}^\varepsilon, \theta_k\rangle = \varepsilon \int_{[0,T/\varepsilon]\times\mathbf{A}} \theta_k(\varepsilon t, a)\, \widetilde{N}(dtda),\ k \leq d \\
\langle N^\varepsilon, \theta\rangle &= (\langle N^\varepsilon, \theta_k\rangle)_{k\leq d} \text{ with } \langle N^\varepsilon, \theta_k\rangle = \varepsilon \int_{[0,T/\varepsilon]\times\mathbf{A}} \theta_k(\varepsilon t, a)\, N(dtda),\ k \leq d.
\end{aligned}$$

For any $\varphi \in L_0^d, 0 \leq t \leq T, y \in I\!R^d$, let us denote

$$\begin{aligned}
\langle \widetilde{N}_t^\varepsilon, \varphi\rangle &= \langle \widetilde{N}^\varepsilon, 1\!\!1_{[0,t]} \otimes \varphi\rangle \\
\langle N_t^\varepsilon, \varphi\rangle &= \langle N^\varepsilon, 1\!\!1_{[0,t]} \otimes \varphi\rangle \\
\tilde{I}^{\varphi,t}(y) &= \sup_{\lambda\in I\!R^d} \left\{ \lambda \cdot y - t \int_{\mathbf{A}} \tilde{\rho}(\lambda \cdot \varphi)\, d\Lambda \right\} \\
I^{\varphi,t}(y) &= \sup_{\lambda\in I\!R^d} \left\{ \lambda \cdot y - t \int_{\mathbf{A}} \rho(\lambda \cdot \varphi)\, d\Lambda \right\}.
\end{aligned}$$

Proposition 3.1. *Let us fix $0 \leq t \leq T, \varphi \in L_0^d$.*

(a) if $\varphi \in L_2^d$, $\{\langle \widetilde{N}_t^\varepsilon, \varphi\rangle\}$ satisfies a weak LDP in $I\!R^d$ with rate function $\tilde{I}^{\varphi,t}$.

(b) if $\varphi \in L_1^d$, $\{\langle N_t^\varepsilon, \varphi\rangle\}$ satisfies a weak LDP in $I\!R^d$ with rate function $I^{\varphi,t}$.

(c) if $\varphi \in L_0^d$ and $\Lambda(\mathbf{A}) < \infty$, $\{\langle N_t^\varepsilon, \varphi\rangle\}$ satisfies a weak LDP in $I\!R^d$ with rate function $I^{\varphi,t}$.

Proof. Let us begin with (b). Let $\varepsilon > 0$ and C be an open convex subset of $I\!R^d$. We define $Z^\varepsilon = \langle N_t^\varepsilon, \varphi\rangle$ and $\Gamma(1/\varepsilon) = -\log I\!\!P(Z^\varepsilon \in C) \in [0,\infty]$. Our proof follows step by step, the proof of Cramér's theorem for the empirical mean $\overline{X}$ of a random sample in $I\!R^d$, without any integrability condition (see [DeZ], Theorem 6.1.3), with Z^ε in the part of $\overline{X}$. We are going to check that Γ is sub-additive, i.e.

$$\Gamma(1/\varepsilon + 1/\varepsilon^*) \leq \Gamma(1/\varepsilon) + \Gamma(1/\varepsilon^*),\ \forall \varepsilon, \varepsilon^* > 0 \tag{3.1}$$

and that

$$\text{for any } 0 < \varepsilon_0 < \varepsilon_1,\ \Gamma(1/\varepsilon_1) < \infty \Longrightarrow \sup_{\varepsilon \in [\varepsilon_0, \varepsilon_1]} \Gamma(1/\varepsilon) < \infty \tag{3.2}$$

Following ([DeZ], Lemma 6.1.11), these two results lead us to the following

Lemma. *For any U open convex subset of $\mathbb{R}^d$, the limit $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(Z^\varepsilon \in U)$ exists as an extended real number.*

Once this result is established, the remainder of the proof of Proposition 3.1 is analogous to the proof of ([DeZ], Theorem 6.1.3). Hence, one gets a weak LDP for $\{Z^\varepsilon\}$ the rate function of which is the convex conjugate $G^*(z)$ of

$$G(\lambda) := \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \exp\left(\frac{1}{\varepsilon}\langle \lambda, Z^\varepsilon\rangle\right),\ \lambda \in \mathbb{R}^d.$$

By Proposition 2.1.b, we have

$$\begin{aligned}
\varepsilon \log \mathbb{E} \exp\left(\frac{1}{\varepsilon}\langle \lambda, Z^\varepsilon\rangle\right) &= \varepsilon \log \mathbb{E} \exp\Big(\int_{[0,\infty[\times \mathbf{A}} \mathbb{1}_{]0,t/\varepsilon]}(s)\lambda \cdot \varphi(a)\, N(dsda)\Big) \\
&= \varepsilon \int_{]0,t/\varepsilon]\times \mathbf{A}} \rho(\lambda \cdot \varphi(a))\, ds\Lambda(da) \\
&= t \int_{\mathbf{A}} \rho(\lambda \cdot \varphi(a))\, \Lambda(da)
\end{aligned}$$

for any $\varepsilon > 0$. Therefore, the limit $G(\lambda) = t\int_{\mathbf{A}} \rho\,(\lambda \cdot \varphi(a))\ \Lambda(da) \in]-\infty, +\infty]$ exists. We finally obtain the right statement, noticing that : $I^{\varphi,t} = G^*$. It remains to prove (3.1) and (3.2).

Let us show (3.1). For any $0 < \varepsilon < \varepsilon'$, let us denote $\varepsilon^* = \varepsilon\varepsilon'/(\varepsilon' - \varepsilon)$ and $Z^\varepsilon_{\varepsilon'} = \varepsilon^* \int_{[0,\infty[\times \mathbf{A}} \mathbb{1}_{]t/\varepsilon', t/\varepsilon]} \otimes \varphi\, dN$. Remark that $Z^\varepsilon = Z^\varepsilon_{\varepsilon_o}$ with $\varepsilon_o = \infty$ and that

$$\begin{aligned}
&Z^\varepsilon = (\varepsilon/\varepsilon')Z^{\varepsilon'} + (\varepsilon/\varepsilon^*)Z^\varepsilon_{\varepsilon'} \text{ with } (\varepsilon/\varepsilon') + (\varepsilon/\varepsilon^*) = 1 \\
&Z^\varepsilon_{\varepsilon'} \overset{\mathcal{L}}{=} Z^{\varepsilon^*} \\
&Z^{\varepsilon'} \text{ and } Z^\varepsilon_{\varepsilon'} \text{ are independent.}
\end{aligned}$$

Taking these remarks into account together with the convexity of C, for any $\varepsilon', \varepsilon^* > 0$ with $\varepsilon = (1/\varepsilon' + 1/\varepsilon^*)^{-1}$, one obtains:

$$\begin{aligned}
\mathbb{P}(Z^{\varepsilon'} \in C)\mathbb{P}(Z^{\varepsilon^*} \in C) &= \mathbb{P}(Z^{\varepsilon'} \in C \text{ and } Z^\varepsilon_{\varepsilon'} \in C) \\
&\leq \mathbb{P}((\varepsilon/\varepsilon')Z^{\varepsilon'} + (\varepsilon/\varepsilon^*)Z^\varepsilon_{\varepsilon'} \in C) \\
&= \mathbb{P}(Z^\varepsilon \in C)
\end{aligned}$$

which proves (3.1).

The proof of (3.2) relies upon the following result:

Lemma 3.2. *Let $0<\varepsilon_0<\varepsilon_1$.*
Under the condition (a), $\{\langle \widetilde{N}^\varepsilon_t, \varphi\rangle\,;\,\varepsilon_0\le\varepsilon\le\varepsilon_1\}$ is tight in $\mathbb{R}^d$.
Under the condition (b) or (c), $\{\langle N^\varepsilon_t, \varphi\rangle\,;\,\varepsilon_0\le\varepsilon\le\varepsilon_1\}$ is tight in $\mathbb{R}^d$.

Its proof is postponed after Proposition 3.1's proof.

From this lemma, as in ([DeZ], Lemma 6.1.14), one deduces that for any convex open subset U of $\mathbb{R}^d$ satisfying $\mathbb{P}(Z^{\varepsilon_1}\in U)>0$ for some $\varepsilon_1>0$ (i.e. $\Gamma(1/\varepsilon_1)<\infty$), there exists $V\subset U$ and $b>0$ such that for ε small enough

$$\mathbb{P}(Z^\varepsilon\in U)\ge \mathbb{P}(Z^{\varepsilon_1}\in V)^l\,\mathbb{P}(|Z^\eta|\le b)$$

with $\mathbb{P}(Z^{\varepsilon_1}\in V)>0$, $\mathbb{P}(|Z^\eta|\le b)>0$ and $1/\varepsilon=(l/\varepsilon_1)+(1/\eta)$, $l\in\mathbb{N},\eta>\varepsilon_1$. From which (3.2) follows.

The proof of the proposition under conditions (c) and (a) follows the same line, invoking the corresponding statements in Proposition 2.2 and Lemma 3.2. ∎

Proof of Lemma 3.2. Since $\mathbb{P}(\max_{k\le d}|\langle N^\varepsilon_t,\varphi_k\rangle|\ge b)\le\sum_{k\le d}\mathbb{P}(|\langle N^\varepsilon_t,\varphi_k\rangle|\ge b)$, it is enough to prove the lemma with $d=1$.
Under condition (a): $\varphi\in L_2$, for any $b>0$, by Chebychev's inequality:

$$\inf_{0<\varepsilon\le 1}\mathbb{P}(|\langle \widetilde{N}^\varepsilon_t,\varphi\rangle|\le b)\ge 1-\frac{T\int_{\mathbf{A}}\varphi^2\,d\Lambda}{b^2}\underset{b\to\infty}{\longrightarrow}1.$$

Under condition (b): $\varphi\in L_1$. Similarly, we get

$$\inf_{0<\varepsilon\le 1}\mathbb{P}(|Z^\varepsilon|\le b)\ge 1-\frac{T\int_{\mathbf{A}}|\varphi|\,d\Lambda}{b}\underset{b\to\infty}{\longrightarrow}1.$$

Under condition (c): $\varphi\in L_0$ and $\Lambda(\mathbf{A})<\infty$. Let $0<\varepsilon_0\le\varepsilon\le\varepsilon'$. Since $Z^\varepsilon-Z^{\varepsilon'}=(\varepsilon/\varepsilon'-1)Z^{\varepsilon'}+\varepsilon\int_{]t/\varepsilon',t/\varepsilon]\times\mathbf{A}}\varphi\,dN$, for any $\delta>0$, we get

$$\begin{aligned}\mathbb{P}(|Z^\varepsilon-Z^{\varepsilon'}|>\delta)&\le\mathbb{P}\Big(N(]t/\varepsilon',t/\varepsilon]\times\mathbf{A})\ge 1\Big)+\mathbb{P}\Big((\varepsilon/\varepsilon'-1)|Z^{\varepsilon'}|>\delta/2\Big)\\&\le\Big[1-\exp\big(-\Lambda(\mathbf{A})t(1/\varepsilon-1/\varepsilon')\big)\Big]+\mathbb{P}\Big((\varepsilon/\varepsilon'-1)\int_{]0,t/\varepsilon_0]\times\mathbf{A}}|\varphi|\,dN>\delta/2\Big)\end{aligned}$$

It comes out that $\varepsilon\mapsto Z^\varepsilon$ is continuous in probability on $[\varepsilon_0,\infty[$. It is a fortiori weakly continuous, so that the image $\{\mathrm{Law}(Z^\varepsilon)\,;\,\varepsilon_0\le\varepsilon\le\varepsilon_1\}$ of the compact set $[\varepsilon_0,\varepsilon_1]$ is weakly compact in the set of probability measures on $\mathbb{R}^d$. ∎

Let $L_2^\sharp$, $L_1^\sharp$ and $L_0^\sharp$ stand for the algebraic dual spaces of L_2, L_1 and L_0, endowed with the product topologies $\sigma(L_2^\sharp,L_2)$, $\sigma(L_1^\sharp,L_1)$ and $\sigma(L_0^\sharp,L_0)$.
Let us fix $t=T$. One can see $\{\widetilde{N}^\varepsilon_T\}$ as a random element on $L_2^\sharp$ and $\{N^\varepsilon_T\}$ as a random element on $L_0^\sharp$ (when $\Lambda(\mathbf{A})<\infty$) or $L_1^\sharp$.

(3.3) Remark. As a stochastic integral, $\langle \widetilde{N}^\varepsilon_T, \varphi\rangle$ (resp. $\langle N^\varepsilon_T, \varphi\rangle$) is defined almost surely. Hence, at most countably many $\langle \widetilde{N}^\varepsilon_T, \varphi\rangle$ (resp. $\langle N^\varepsilon_T, \varphi\rangle$) where φ is the varying index, can be considered simultaneously. But, this is the case for the events of interest since for any Borel subset A of $L_2^\sharp$ (resp. $L_1^\sharp$ and $L_0^\sharp$) : $\mathbb{1}_{\{\widetilde{N}^\varepsilon_T \in A\}}$ is the pointwise limit of measurable cylinder functions of the form $f(\langle \widetilde{N}^\varepsilon_T, \varphi_1\rangle, \ldots, \langle \widetilde{N}^\varepsilon_T, \varphi_p\rangle)$.

We define

$$I_2^T(q) = \sup_{\varphi\in L_2} \left\{ \langle q, \varphi\rangle - T\int_{\mathbf{A}} \tilde\rho(\varphi)\, d\Lambda \right\}, \quad q \in L_2^\sharp$$
$$I_1^T(q) = \sup_{\varphi\in L_1} \left\{ \langle q, \varphi\rangle - T\int_{\mathbf{A}} \rho(\varphi)\, d\Lambda \right\}, \quad q \in L_1^\sharp$$
$$I_0^T(q) = \sup_{\varphi\in L_0} \left\{ \langle q, \varphi\rangle - T\int_{\mathbf{A}} \rho(\varphi)\, d\Lambda \right\}, \quad q \in L_0^\sharp$$

Proposition 3.3.

(a) $\{\widetilde{N}^\varepsilon_T\}$ *satisfies a weak LDP in* $L_2^\sharp$ *with rate function* I_2^T.

(b) $\{N^\varepsilon_T\}$ *satisfies a weak LDP in* $L_1^\sharp$ *with rate function* I_1^T.

(c) *if* $\Lambda(\mathbf{A}) < \infty$, $\{N^\varepsilon_T\}$ *satisfies a weak LDP in* $L_0^\sharp$ *with rate function* I_0^T.

Proof. Invoking Proposition 3.1 and Proposition A.2, one gets the weak LDP for $\{\widetilde{N}^\varepsilon_T\}$ with rate function $\sup_{d\geq 1, \varphi\in L_2^d, \lambda\in\mathbb{R}^d} \left\{ \lambda\cdot\langle q,\varphi\rangle - T\int_{\mathbf{A}} \tilde\rho(\lambda\cdot\varphi)\, d\Lambda\right\} = I_2^T(q)$. The proofs of (b) and (c) are similar. ■

4. Large deviations for Poisson random measures with exponential moments

Some Orlicz spaces. In Section 2, the Young function $\tau(x) = \tilde\rho(|x|)$ has been introduced. Similarly, let us associate with ρ the Young function $\sigma(x) = \rho(|x|) = e^{|x|} - 1, x \in \mathbb{R}$. As in (2.1), the corresponding Orlicz spaces are $L^\sigma(\mathbf{U}, R)$ and $M^\sigma(\mathbf{U}, R)$ endowed with the norm $\|f\|_\sigma = \inf\{\alpha > 0\,;\, \int_{\mathbf{U}} \sigma(|f|/\alpha)\, dR \leq 1\}$.

Notice that $L^\tau \subset L^2$, $L^\sigma = L^\tau \cap L^1$ and if $R(\mathbf{U}) < \infty$, then $L^\tau = L^\sigma \subset L^2 \subset L^1$. We also have

$$\theta \in L^\tau \iff \mathbb{1}_{(|\theta|\leq 1)}\theta \in L^2 \text{ and } \exists \lambda_o > 0, \int_{\{|\theta|\geq 1\}} e^{\lambda_o|\theta|}\, dR < \infty$$
$$\theta \in L^\sigma \iff \mathbb{1}_{(|\theta|\leq 1)}\theta \in L^1 \text{ and } \exists \lambda_o > 0, \int_{\{|\theta|\geq 1\}} e^{\lambda_o|\theta|}\, dR < \infty$$
$$\theta \in L^\tau = L^\sigma \iff \exists \lambda_o > 0, \int_{\mathbf{U}} e^{\lambda_o|\theta|}\, dR < \infty, \qquad \text{if } R(\mathbf{U}) < \infty.$$

Let us denote $\overline{\Lambda}(dtda) = dt\Lambda(da)$ which is a measure on $[0,T]\times\mathbf{A}$.

For any Young function γ, M^γ is the space of functions f such that for all $\lambda > 0$, $\int \gamma(\lambda|f|)\, dR < \infty$. We write: $L_\gamma(\Lambda) = L^\gamma(\mathbf{A}, \Lambda)$, $M_\gamma(\Lambda) = M^\gamma(\mathbf{A},\Lambda)$ and $L_\gamma(\overline{\Lambda}) = L^\gamma([0,T]\times\mathbf{A}, \overline{\Lambda})$, $M_\gamma(\overline{\Lambda}) = M^\gamma([0,T]\times\mathbf{A}, \overline{\Lambda})$. Their d-dimensional analogues are $L^d_\gamma(\Lambda)$, $M^d_\gamma(\Lambda)$, $L^d_\gamma(\overline{\Lambda})$ and $M^d_\gamma(\overline{\Lambda})$.

Let us introduce two function spaces on $[0,T]\times\mathbf{A}$:

$$\tilde{\mathcal{L}}=\{\theta:[0,T]\times\mathbf{A}\mapsto\mathbb{R}\,;\,\theta(t,a)=\alpha(t,a)+\beta(t,a)\varphi(a),\alpha\in M_\tau(\overline{\Lambda}),\beta\in L_\infty(\overline{\Lambda}),\varphi\in L_\tau(\Lambda)\}$$
$$\mathcal{L}=\{\theta:[0,T]\times\mathbf{A}\mapsto\mathbb{R}\,;\,\theta(t,a)=\alpha(t,a)+\beta(t,a)\varphi(a),\alpha\in M_\sigma(\overline{\Lambda}),\beta\in L_\infty(\overline{\Lambda}),\varphi\in L_\sigma(\Lambda)\}$$

Their algebraic dual spaces $\tilde{\mathcal{L}}^\sharp$ and $\mathcal{L}^\sharp$ are endowed with their pointwise convergence topologies $\sigma(\tilde{\mathcal{L}}^\sharp,\tilde{\mathcal{L}})$ and $\sigma(\mathcal{L}^\sharp,\mathcal{L})$.

We are going to encounter the rate functions

$$\tilde{I}(n)=\sup_{\theta\in\tilde{\mathcal{L}}}\left\{\langle n,\theta\rangle-\int_{[0,T]\times\mathbf{A}}\tilde{\rho}(\theta)\,dtd\Lambda\right\},\quad n\in\tilde{\mathcal{L}}^\sharp$$
$$I(n)=\sup_{\theta\in\mathcal{L}}\left\{\langle n,\theta\rangle-\int_{[0,T]\times\mathbf{A}}\rho(\theta)\,dtd\Lambda\right\},\quad n\in\mathcal{L}^\sharp$$

Let us state the main result of this section.

Theorem 4.1.

(a) $\{\widetilde{N}^\varepsilon\}$ *satisfies the LDP in* $\tilde{\mathcal{L}}^\sharp$ *with the good rate function* $\tilde{I}$.

(b) $\{N^\varepsilon\}$ *satisfies the LDP in* $\mathcal{L}^\sharp$ *with the good rate function* I.

For measurability considerations, see remark (3.3).

Remark. If $\tilde{\mathcal{L}}$ (resp. $\mathcal{L}$) is replaced by $M_\tau(\overline{\Lambda})$ (resp. $M_\sigma(\overline{\Lambda})$), this weaker result can be proved easily by means of Gärtner-Ellis' theorem, since in this situation the log-Laplace transform which is given in Lemma 2.1 is a steep function.

The basic technical result for the proof of Theorem 4.1 is the following lemma.

Lemma 4.2. (Exponential estimates). *Let* $d\geq 1$, $R\geq 0$, $\varepsilon>0$.

(a) *For any* $\theta\in L^d_\tau(\overline{\Lambda})$, $\mathbb{P}(\max_{k\leq d}|\langle\widetilde{N}^\varepsilon,\theta_k\rangle|>R)\leq 2d\exp\left[-\frac{1}{\varepsilon}\left(\frac{R}{\max_{k\leq d}\|\theta_k\|_\tau}-1\right)\right]$

(b) *For any* $\theta\in L^d_\sigma(\overline{\Lambda})$, $\mathbb{P}(\max_{k\leq d}|\langle N^\varepsilon,\theta_k\rangle|>R)\leq 2d\exp\left[-\frac{1}{\varepsilon}\left(\frac{R}{\max_{k\leq d}\|\theta_k\|_\sigma}-1\right)\right]$

Proof of Lemma 4.2. The proofs of (a) and (b) are similar. Let us prove (b). Let θ stand in $L_\sigma(\overline{\Lambda})$, (with $d=1$). For any $\lambda,\varepsilon>0$, by Proposition 2.2.b, we have

$$\begin{aligned}\mathbb{P}(\langle N^\varepsilon,\theta\rangle>R)&\leq e^{-\frac{\lambda R}{\varepsilon}}\mathbb{E}\exp\left(\frac{\lambda}{\varepsilon}\langle N^\varepsilon,\theta\rangle\right)\\&=\exp\left[\frac{1}{\varepsilon}\left(-\lambda R+\int_{[0,T]\times\mathbf{A}}\rho(\lambda\theta)\,d\overline{\Lambda}\right)\right]\\&\leq\exp\left[\frac{1}{\varepsilon}\left(-\lambda R+\int_{[0,T]\times\mathbf{A}}\rho(\lambda|\theta|)\,d\overline{\Lambda}\right)\right].\end{aligned}$$

Choosing $\lambda=1/\|\theta\|_\sigma$, one obtains $\mathbb{P}(\langle N^\varepsilon,\theta\rangle>R)\leq\exp[\frac{1}{\varepsilon}(-R/\|\theta\|_\sigma+1)]$.

One concludes with $I\!\!P(\max_{k\le d}|\langle N^\varepsilon,\theta\rangle|>R)\le\sum_{k\le d}[I\!\!P(\langle N^\varepsilon,\theta_k\rangle>R)+I\!\!P(\langle N^\varepsilon,-\theta_k\rangle>R)]$. ■

Proof of Theorem 4.1. The proofs of (a) and (b) are similar. Let us prove (b).

Step 0. The goodness of the rate function is a consequence of Banach-Alaoglu theorem applied to the dual pairing (L^σ, L^{σ^*}), see ([Léo], Corollary 2.2) for more details.

Step 1. Let $d\ge 1, 0\le t\le T, \varphi\in L^d_\sigma$ be fixed. Letting R tend to infinity in Lemma 4.2.b, one sees that $\{\langle N^\varepsilon_t,\varphi\rangle\}$ is exponentially tight. Because of Proposition 3.1.b, it follows that $\{\langle N^\varepsilon_t,\varphi\rangle\}$ satisfies the full LDP in $\mathbb{R}^d$ with rate function $I^{\varphi,t}$. As a supremum of continuous affine functions, $I^{\varphi,t}$ is convex and lower semicontinuous. One deduces from the exponential tightness and the lower bound that $I^{\varphi,t}$ is also a good rate function (see [DeZ], Lemma 1.2.18).

Step2. Let $d\ge 1, J\ge 1, 0=t_0\le t_1\le\cdots\le t_J\le T$ and $\varphi_1,\dots,\varphi_J\in L^d_\sigma$ be fixed. Taking the independence of the increments into account, the result obtained in Step 1 together with Lemma A.1 lead us to the LDP for $\{(\langle N^\varepsilon, 1\!\!\mathrm{I}_{]t_j,t_{j+1}]}\otimes\varphi_j\rangle)_{0\le j\le J-1}\}$ in $\mathbb{R}^{dJ}$ with the rate function $(y_1,\dots,y_J)\mapsto\sum_{0\le j\le J-1}I^{\varphi_j,t_{j+1}-t_j}(y_j)$.
It follows from the contraction principle applied with the continuous transformation $S(y_1,\dots,y_J)=\sum_j y_j$, that $\{\langle N^\varepsilon,\sum_{0\le j\le J-1}1\!\!\mathrm{I}_{]t_j,t_{j+1}]}\otimes\varphi_j\rangle\}$ obeys the LDP in $\mathbb{R}^d$ with rate function $\mathcal{I}_2(y)=\inf\{\sum_{0\le j\le J-1}I^{\varphi_j,t_{j+1}-t_j}(y_j)\,;\,y_1,\dots,y_J$ such that $\sum_j y_j=y\}$. Since S is linear, $\mathcal{I}_2$ is still convex (its epigraph is the linear transform of a convex epigraph). Since S is continuous, $\mathcal{I}_2$ is still a good rate function.

Step 3. By Lemma 4.2.b, we get the following exponential approximation estimate. For any $\theta,\eta\in L^d_\sigma(\overline{\Lambda})$, with $R=\delta>0$ arbitrarily small, $\limsup_{\varepsilon\to 0}\varepsilon\log I\!\!P(\max_{k\le d}|\langle N^\varepsilon,\theta_k-\eta_k\rangle|>\delta)\le -\frac{\delta}{\max_{k\le d}\|\theta_k-\eta_k\|_\sigma}+1$. It follows that for any sequence $(\theta^n)_{n\ge 1}$ $\|\cdot\|_\sigma$-converging to θ in $L^d_\sigma(\overline{\Lambda})$, the sequence $\langle N^\varepsilon,\theta^n\rangle_{n\ge 1}$ is an exponential approximation (see [DeZ], Definition 4.2.14) of $\langle N^\varepsilon,\theta\rangle$:

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log I\!\!P(\max_{k\le d}|\langle N^\varepsilon,\theta^n_k-\theta_k\rangle|>\delta)\le-\lim_{n\to\infty}\frac{\delta}{\max_{k\le d}\|\theta^n_k-\theta_k\|_\sigma}+1=-\infty.$$

But, any function in $\mathcal{L}^d$ is the $\|\cdot\|_\sigma$-limit of simple functions of the form $\sum_{0\le j\le J-1}1\!\!\mathrm{I}_{]t_j,t_{j+1}]}\otimes\varphi_j$ with $\varphi_j\in L^d_\sigma$: the simple functions are dense in M^σ and $\|\beta^n\varphi-\beta\varphi\|_\sigma\le\|\beta^n-\beta\|_\infty\|\varphi\|_\sigma$. Hence, by ([DeZ], Theorem 4.2.16.a), the LDP of Step 2 yields a weak LDP for $\{\langle N^\varepsilon,\theta\rangle\}$ for any $\theta\in\mathcal{L}^d$ with a convex lower semicontinuous rate function $\mathcal{I}_3$ (the explicit form of $\mathcal{I}_3$ is given in [DeZ], Theorem 4.2.16.a). Again, letting R tend to infinity in Lemma 4.2.b, one sees that $\{\langle N^\varepsilon,\theta\rangle\}$ is exponentially tight; this proves that $\{\langle N^\varepsilon,\theta\rangle\}$ satisfies the full LDP in $\mathbb{R}^d$ with the good ([DeZ], Lemma 1.2.18) rate function $\mathcal{I}_3$.

Step 4. As a direct consequence of the Dawson-Gärtner's theorem on projective limits of LDPs ([DeZ], Theorem 4.6.9), we obtain a LDP for $\{N^\varepsilon\}$ in $\mathcal{L}^\sharp$ with a convex good rate function $\mathcal{I}_4$.

Step 5. It remains to compute $\mathcal{I}_4$. Since it is a good rate function, the conditions of Laplace-Varadhan theorem ([DeZ], Theorem 4.3.1) are fulfilled. Together with Proposition 2.2.b, this theorem states that for any $\theta\in\mathcal{L}$ such that there exists $\gamma>1$ with $\int_{[0,T]\times\mathbf{A}}\rho(\gamma|\theta|)\,dtd\Lambda<\infty$, we have

$$H(\theta):=\lim_{\varepsilon\to 0}\varepsilon\log I\!\!E\exp\left(\frac{1}{\varepsilon}\langle N^\varepsilon,\theta\rangle\right)=\sup_{n\in\mathcal{L}^\sharp}\{\langle n,\theta\rangle-\mathcal{I}_4(n)\}:=G(\theta). \tag{4.1}$$

By Proposition 2.2.b, $H(\theta) = \int_{[0,T]\times\mathbf{A}} \rho(\theta)\, dtd\Lambda$. With $\theta_l = \min(\theta, l), l \geq 1$, we get

$$\lim_{l\to\infty} H(\theta_l) = H(\theta) \leq \infty. \tag{4.2}$$

Taking advantage of $\rho' \geq 0$ and keeping track of the abstract form of $\mathcal{I}_4$ expressed in terms of limits, sup and inf of $\mathcal{I}_2$'s, one can show that $\mathcal{I}_4(n) < \infty$ implies that $n \geq 0$. Consequently, $G(\theta) = \sup_{n\geq 0}\{\langle n, \theta\rangle - \mathcal{I}_4(n)\}$, so that $G(\theta_l) \leq G(\theta), \forall l \geq 1$.
With (4.1): $H(\theta_l) = G(\theta_l), \forall l \geq 1$, by (4.2), we get: $H(\theta) \leq G(\theta), \forall \theta \in \mathcal{L}$. In particular, $H(\theta) = \infty \Longrightarrow G(\theta) = \infty$.
Now, let us pick $\theta \in \mathcal{L}$ such that $H(\theta) < \infty$. If θ stands in the intrinsic core of the effective domain of H, there exists $\gamma > 1$ such that $\int_{[0,T]\times\mathbf{A}} \rho(\gamma|\theta|)\, dtd\Lambda < \infty$. With (4.1), we obtain $H(\theta) = G(\theta)$.
If θ stands on the intrinsic boundary of the effective domain of H and $H(\theta) < \infty$, since $\gamma \in \mathbb{R} \mapsto H(\gamma\theta) \in]-\infty, \infty]$ and $\gamma \in \mathbb{R} \mapsto G(\gamma\theta) \in]-\infty,\infty]$ are convex lower semicontinuous, considering the limits as $\gamma \uparrow 1$ and the equalities $H(\theta_l) = G(\theta_l), \forall l \geq 1$, we see that $H(\theta) = G(\theta)$.
We have just proved that

$$\sup_{n\in\mathcal{L}^\sharp} \{\langle n, \theta\rangle - \mathcal{I}_4(n)\} = \int_{[0,T]\times\mathbf{A}} \rho(\theta)\, dtd\Lambda, \forall \theta \in \mathcal{L}.$$

As $\mathcal{I}_4$ is convex and $\sigma(\mathcal{L}^\sharp, \mathcal{L})$-lower semicontinuous, this proves that

$$\mathcal{I}_4(n) = \sup_{\theta\in\mathcal{L}} \left\{ \langle n, \theta\rangle - \int_{[0,T]\times\mathbf{A}} \rho(\theta)\, dtd\Lambda \right\}$$

which is the desired result. ■

As Λ is σ-finite, there exists a sequence $(A_k)_{k\geq 1}$ of measurable subsets of $\mathbf{A}$ such that $\cup_{k\geq 1} A_k = \mathbf{A}$ and $\Lambda(A_k) < \infty$, $\forall k \geq 1$. A continuous linear form n on $L_\tau(\overline{\Lambda})$ is said to be singular with respect to $\overline{\Lambda}$ if there exists a nonincreasing sequence $(B_l)_{l\geq 1}$ of measurable subsets of $[0,T]\times\mathbf{A}$ such that for all $k \geq 1$, $\lim_{l\to\infty} \overline{\Lambda}(([0,T]\times A_k)\cap B_l) = 0$ and for all $\theta \in L_\tau(\overline{\Lambda})$, $\langle n, \mathbb{1}_{([0,T]\times A_k)\setminus B_l}\theta\rangle = 0$.
The topological dual space of $(M_\tau(\overline{\Lambda}), \|\cdot\|_\tau)$ is $L_{\tau^*}(\overline{\Lambda})$ where τ^* is the convex conjugate of τ (see [RaR]). While, the topological dual space $L_\tau^\star(\overline{\Lambda})$ of $(L_\tau(\overline{\Lambda}), \|\cdot\|_\tau)$ is $L_{\tau^*}(\overline{\Lambda}) \oplus L_\tau^s(\overline{\Lambda})$ where $L_\tau^s(\overline{\Lambda})$ is the space of all continuous forms which are singular with respect to $\overline{\Lambda}$ (see [Léo], Theorem 5.8).
This means that any $n \in L_\tau^\star(\overline{\Lambda})$ can be uniquely decomposed as

$$n = n_a + n_s \tag{4.3}$$

where $n_a \in L_{\tau^*}(\overline{\Lambda})$ and $n_s \in L_\tau^s(\overline{\Lambda})$. Similar results hold with the Young function σ instead of τ.
The convex conjugates of $\tilde{\rho}$ and ρ are given for any $x \in \mathbb{R}$ by

$$\tilde{\rho}^*(x) = \begin{cases} (x+1)\log(x+1) - x & \text{if } x > -1 \\ 1 & \text{if } x = -1 \\ +\infty & \text{if } x < -1 \end{cases} \quad \text{and} \quad \rho^*(x) = \begin{cases} x\log x - x + 1 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ +\infty & \text{if } x < 0 \end{cases}$$

Proposition 4.3.

(a) *For any n in $\tilde{\mathcal{L}}^\sharp$, we have*

$$\tilde{I}(n) = \begin{cases} \int \tilde{\rho}^*(\frac{dn_a}{d\overline{\Lambda}})\, d\overline{\Lambda} + \sup\{\langle n_s, \theta\rangle\,;\, \theta \in L_\tau(\overline{\Lambda}), \int \tilde{\rho}(\theta)\, d\overline{\Lambda} < \infty\} & \text{if } n \in L^\star_\tau(\overline{\Lambda}) \\ +\infty & \text{otherwise} \end{cases}$$

where $n = n_a + n_s$ is the decomposition (4.3).

(b) *For any n in $\mathcal{L}^\sharp$, we have*

$$I(n) = \begin{cases} \int \rho^*(\frac{dn_a}{d\overline{\Lambda}})\, d\overline{\Lambda} + \sup\{\langle n_s, \theta\rangle\,;\, \theta \in L_\sigma(\overline{\Lambda}), \int \rho(\theta)\, d\overline{\Lambda} < \infty\} & \text{if } n \in L^\star_\sigma(\overline{\Lambda}) \\ +\infty & \text{otherwise} \end{cases}$$

Proof. See ([Léo], Theorem 6.2). ■

5. Large deviations for processes with independent increments

Let us fix the dimension $d \geq 1$. We are going to deduce from our previous results, LDPs for the càdlàg $\mathbb{R}^d$-valued processes with non-stationary independent increments

$$(5.1)\quad Y^\varepsilon(t) \overset{v}{=} c(t) + \int_{[0,t]\times\mathbf{A}} \theta_o(t,a)\mathbb{1}_{\{|\theta_o(t,a)|\leq 1\}}\, \widetilde{N}^\varepsilon(dsda) + \int_{[0,t]\times\mathbf{A}} \theta_o(t,a)\mathbb{1}_{\{|\theta_o(t,a)|>1\}}\, N^\varepsilon(dsda)$$

where θ_o is a fixed measurable $\mathbb{R}^d$-valued function on $[0,T]\times\mathbf{A}$ with some required integrability property, c is a deterministic càdlàg path (with $c_0 = 0$) and $\overset{v}{=}$ means that the processes are càdlàg versions. Let $D = \{x \in D([0,T],\mathbb{R}^d)\,;\, x_0 = 0\}$ stand for the set of the sample paths of these processes.

Under the assumptions that $\theta_o\mathbb{1}_{\{|\theta_o|\leq 1\}}$ belongs to $L_2(\overline{\Lambda})$, $\overline{\Lambda}(|\theta_o| > 1) < \infty$ and $c \in D$, (5.1) is the general expression for a (ε-normalized) $\mathbb{R}^d$-valued process with non-stationary independent increments (see [JaS]).

The law of $Y^\varepsilon - c$ is a probability measure on D solution to the martingale problem associated with the generator given for any $g \in C_o^{1,1}(]0,T[\times\mathbb{R}^d)$ by:

$$\begin{aligned} \partial_t g(t,z) &+ \frac{1}{\varepsilon}\int_{\mathbf{A}} [g(t,z+\varepsilon\theta_o(t,a)) - g(t,z) - \varepsilon\theta_o(t,a)\cdot\nabla_z g(t,z)]\mathbb{1}_{\{|\theta_o(t,a)|\leq 1\}}\Lambda(da) \\ &+ \frac{1}{\varepsilon}\int_{\mathbf{A}} [g(t,z+\varepsilon\theta_o(t,a)) - g(t,z)]\mathbb{1}_{\{|\theta_o(t,a)|>1\}}\Lambda(da). \end{aligned}$$

Its Lévy measure is $\frac{1}{\varepsilon}\overline{\Lambda}\circ(\varepsilon\theta_o)^{-1}$. The special cases

$$(5.2)\qquad \widetilde{X}^\varepsilon_t \overset{v}{=} \langle \widetilde{N}^\varepsilon, \mathbb{1}_{[0,t]}\theta_o\rangle \text{ and } X^\varepsilon_t \overset{v}{=} \langle N^\varepsilon, \mathbb{1}_{[0,t]}\theta_o\rangle,\ t\in[0,T]$$

correspond respectively to the situations where $c(t) = -\int_{[0,t]\times\mathbf{A}} \theta_o\mathbb{1}_{|\theta_o|>1}\, d\overline{\Lambda}$ with $\theta_o\mathbb{1}_{|\theta_o|>1}$ in $L_1(\overline{\Lambda})$ and $c(t) = \int_{[0,t]\times\mathbf{A}} \theta_o\mathbb{1}_{|\theta_o|\leq 1}\, d\overline{\Lambda}$ with $\theta_o\mathbb{1}_{|\theta_o|\leq 1}$ in $L_1(\overline{\Lambda})$.

Without integrability assumptions. If θ_o doesn't depend on t : $\theta_o(t,a) = \varphi_o(a)$,

$$(5.3)\qquad Y^\varepsilon(t) \overset{v}{=} t\alpha + \int_{[0,t]\times\mathbf{A}} \varphi_o(a)\mathbb{1}_{\{|\varphi_o(a)|\leq 1\}}\, \widetilde{N}^\varepsilon(dsda) + \int_{[0,t]\times\mathbf{A}} \varphi_o(a)\mathbb{1}_{\{|\varphi_o(a)|>1\}}\, N^\varepsilon(dsda)$$

with $\alpha \in \mathbb{R}^d$, is a general Lévy process without Gaussian component.

For any $\lambda \in \mathbb{R}^d$, we define

$$G_{\varphi_o}(\lambda) = \int_{\mathbf{A}} \left[\tilde{\rho}(\lambda \cdot \varphi_o(a))\mathbb{1}_{\{|\varphi_o(a)|\leq 1\}} + \rho(\lambda \cdot \varphi_o(a))\mathbb{1}_{\{|\varphi_o(a)|>1\}}\right] \Lambda(da),\ \lambda \in \mathbb{R}^d.$$

Let S stand for the space of the $\mathbb{R}^d$-valued simple functions on $[0,T] : \gamma(t) = \sum_{1\leq l\leq L} \gamma_l \mathbb{1}_{]t_{l-1},t_l]}(t)$, with $L \geq 1$, $\gamma_l \in \mathbb{R}^d, l \leq L$ and $0 = t_0 < t_1 < \cdots < t_L = T$. The relevent rate function is

$$J_S^{\varphi_o}(y) = \sup_{\gamma\in S}\left\{\int_{[0,T]} \gamma \cdot (dy - \alpha dt) - \int_{[0,T]} G_{\varphi_o}(\gamma_t)\, dt\right\},\ y \in D$$

where $\int_{[0,T]} \gamma \cdot dx$ has the obvious meaning: $\sum_l \gamma_l \cdot [x(t_l) - x(t_{l-1})]$, for any $\gamma \in S, x \in D$.

Theorem 5.1. *Let us define Y^ε as in (5.3). We assume that $\varphi_o\mathbb{1}_{\{|\varphi_o|\leq 1\}} \in L_2^d$ and $\Lambda(|\varphi_o| > 1) < \infty$. Then, the family of Lévy processes $\{Y^\varepsilon\}$ obeys a weak LDP in D endowed with the topology of pointwise convergence with the rate function $J_S^{\varphi_o}$.*

<u>Proof</u>. This is not a direct consequence of the previous results but a consequence of their proofs. The deterministic part $t\alpha$ is treated by means of the contraction principle. We take $\alpha = 0$ in the remainder of the proof.

The starting point is similar to Proposition 3.1 where one considers $Z^\varepsilon = \langle \widetilde{N}_t^\varepsilon, \varphi\mathbb{1}_{|\varphi|\leq 1}\rangle + \langle N_t^\varepsilon, \varphi\mathbb{1}_{|\varphi|>1}\rangle$. As a sum of independent random variables, it gives $\varepsilon \log \mathbb{E} \exp\left(\frac{1}{\varepsilon}\langle \lambda, Z^\varepsilon\rangle\right) = t\int_{\mathbf{A}}[\tilde{\rho}(\lambda\cdot\varphi)\mathbb{1}_{|\varphi|\leq 1} + \rho(\lambda\cdot\varphi)\mathbb{1}_{|\varphi|>1}]\, d\Lambda$.

Let $0 = t_0 < t_1 < \cdots < t_L = T$. The random vectors $Y_{t_l}^\varepsilon - Y_{t_{l-1}}^\varepsilon, l \leq L$ are independent and $Y_{t_l}^\varepsilon - Y_{t_{l-1}}^\varepsilon \overset{\mathcal{L}}{=} Y_{t_l - t_{l-1}}^\varepsilon$. Combining a slight modification of Proposition 3.1 with Proposition A.1, we obtain a weak LDP for $(Y_{t_l}^\varepsilon - Y_{t_{l-1}}^\varepsilon)_{l\leq L}$ with the rate function $\sum_{l\leq L}\sup_{\gamma_l\in\mathbb{R}^d}\left\{\gamma_l\cdot u_l - (t_l - t_{l-1})\int_{\mathbf{A}} G_{\varphi_o}(\gamma_l)\, d\Lambda\right\}$ where $u_l \in \mathbb{R}^d, 1 \leq l \leq L$.

As the application $F : (u_1, \ldots, u_L) \mapsto (u_1, u_1 + u_2, \ldots, u_1 + \cdots + u_L)$ is one-one and bicontinuous, by the contraction principle we obtain a weak LDP for $(Y_{t_l}^\varepsilon)_{l\leq L} = F(Y_{t_1}^\varepsilon - Y_{t_0}^\varepsilon, \ldots, Y_{t_L}^\varepsilon - Y_{t_{L-1}}^\varepsilon)$ with rate function

$$\begin{aligned}
&\sum_{l\leq L}(t_l - t_{l-1})\sup_{\gamma_l\in\mathbb{R}^d}\left\{\gamma_l\cdot\frac{z_l - z_{l-1}}{t_l - t_{l-1}} - G_{\varphi_o}(\gamma_l)\right\}\\
=&\sum_{l\leq L}\sup_{\lambda_l\in\mathbb{R}^d}\left\{\lambda_l\cdot z_l - (t_l - t_{l-1})G_{\varphi_o}\Big(\sum_{i\geq l}\lambda_i\Big)\right\}\\
=&\sup_{\lambda_1,\ldots,\lambda_L\in\mathbb{R}^d}\left\{\sum_{l\leq L}\lambda_l\cdot z_l - \sum_{l\leq L}(t_l - t_{l-1})G_{\varphi_o}\Big(\sum_{i\geq l}\lambda_i\Big)\right\}
\end{aligned}$$

(with $\lambda_l = \gamma_l - \gamma_{l+1}$ and $\gamma_{L+1} = 0$).

One concludes with Proposition A.2 which states a weak LDP for projective limits. ■

(5.4) Remark. Under the assumption: $\varphi_o 1\!\!1_{\{|\varphi_o|\le 1\}} \in L_1^d$, the sample paths x belong to the space V_r of right continuous paths with bounded variations. Their generalized derivatives $\dot{x}$ belong to M : the set of bounded $\mathbb{R}^d$-valued measures on $[0,T]$. Since $x(0)=0$, $\dot{x}$ is an unambiguous description of x. Identifying x and $\dot{x}$, one is allowed to consider the usual weak topology on M; transfered on V_r, it is still called the weak topology. It is weaker than the pointwise convergence topology.
In [Ac1], the large deviation lower bound is proved for the weak topology assuming that $\varphi_o 1\!\!1_{\{|\varphi_o|\le 1\}} \in L_1^d$.

With integrability assumptions. Let $\tilde{\mathcal{L}}^d$ (resp. $\mathcal{L}^d$) be the space of $\mathbb{R}^d$-valued functions on $[0,T]\times \mathbf{A}$ with their components in $\tilde{\mathcal{L}}$ (resp. $\mathcal{L}$). In this subsection, θ_o is a fixed function in $\tilde{\mathcal{L}}^d$ and we consider Y^ε as defined in (5.1).

For any $\gamma \in B$: the space of bounded measurable $\mathbb{R}^d$-valued functions on $[0,T]$, we define

$$\langle \dot{Y}^\varepsilon, \gamma\rangle := \int_{[0,T]} \gamma(t)\cdot dY_t^\varepsilon = \langle \widetilde{N}^\varepsilon, \gamma\cdot(\theta_o 1\!\!1_{\{|\theta_o|\le 1\}})\rangle + \langle N^\varepsilon, \gamma\cdot(\theta_o 1\!\!1_{\{|\theta_o|>1\}})\rangle \tag{5.5}$$

(in this definition we state $c=0$ in (5.1)). This invites us to consider $\dot{Y}^\varepsilon$ as a random element in the algebraic dual space $B^\sharp$ of B; see remark (3.3) for measurability considerations.

Let V_ℓ (resp. V_r) be the space of left (resp. right) continuous $\mathbb{R}^d$-valued functions on $[0,T]$ with bounded variations. We denote $V_\ell^\sharp$ the algebraic dual space of V_ℓ and $V_\ell^\star$ is its subspace of continuous elements for the uniform convergence topology.
Let us denote M the space of bounded $\mathbb{R}^d$-valued measures on $[0,T]$.

Let $B^\star$ be the topological dual space of $L^\infty_{\mathbb{R}^d}([0,T],dt)$. Clearly, $M\subset B^\sharp$. We shall consider the $*$-weak topologies $\sigma(B^\sharp,B)$, $\sigma(B^\star,B)$ and $\sigma(M,V_\ell)$.

Notice that if $\theta_o\in\mathcal{L}^d$, then $\dot{Y}^\varepsilon$ stands almost surely in M.

The rate functions of interest are given by

$$I_B(\xi) = \sup_{\gamma\in B}\left\{\langle \xi,\gamma\rangle - \int_{[0,T]} G_{\theta_o}(t,\gamma_t)\,dt\right\},\quad \xi\in B^\sharp$$
$$I_V(\nu) = \sup_{\gamma\in V_\ell}\left\{\langle \nu,\gamma\rangle - \int_{[0,T]} G_{\theta_o}(t,\gamma_t)\,dt\right\},\quad \nu\in V_\ell^\sharp$$

where, for any $t\in[0,T], \lambda\in\mathbb{R}^d$,

$$G_{\theta_o}(t,\lambda) = \int_{\mathbf{A}} \left[\tilde{\rho}(\lambda\cdot\theta_o(t,a))1\!\!1_{\{|\theta_o(t,a)|\le 1\}} + \rho(\lambda\cdot\theta_o(t,a))1\!\!1_{\{|\theta_o(t,a)|>1\}}\right]\,\Lambda(da).$$

Theorem 5.2. *Let us fix θ_o and define $\dot{Y}^\varepsilon$ as in (5.5).*

(a) if $\theta_o\in\tilde{\mathcal{L}}^d$, $\{\dot{Y}^\varepsilon\}$ satisfies the LDP in $B^\sharp$ for the topology $\sigma(B^\sharp,B)$ with the good rate function I_B.

(b) if $\theta_o\in\mathcal{L}^d$, $\{\dot{Y}^\varepsilon\}$ satisfies the LDP in $B^\star$ for the topology $\sigma(B^\star,B)$ with the good rate function I_B (restricted to $B^\star$).

(c) *if $\theta_o \in \mathcal{L}^d$, $\{\dot{Y}^\varepsilon\}$ satisfies the LDP in M for the topology $\sigma(M, V_\ell)$ with the good rate function I_V (restricted to M).*

Proof. Noticing that $\theta_o{\cdot}\gamma \in \tilde{\mathcal{L}}$ (resp. $\in \mathcal{L}$) whenever $\gamma \in B$ and $\theta_o \in \tilde{\mathcal{L}}^d$ (resp. $\in \mathcal{L}^d$), following step by step the proof of Theorem 4.1, replacing $\theta(t,a)$ by $\theta_o(t,a)\cdot\gamma(t)$ and $\langle n, \theta\rangle$ by $\langle n, \theta_o\cdot\gamma\rangle = \langle \xi, \gamma\rangle$, one proves the LDP of (a) and two weaker versions of the LDP of (b): in $\sigma(B^\sharp, B)$ with rate function I_B and in $\sigma(V_\ell^\sharp, V_\ell)$ with rate function I_V. Noticing that $\dot{Y}^\varepsilon$ is M-valued when $\theta_o \in \mathcal{L}^d$, one completes the proofs of (b) and (c) with Lemma 6.4 and ([DeZ], Lemma 4.1.5.b). ■

Let us introduce the linear application $\Psi : V_\ell^\sharp \to M^\sharp$ defined for any $\xi \in V_\ell^\sharp$ by

$$\langle \Psi(\xi), \mu\rangle_{M^\sharp, M} = \langle \xi, \mu([\cdot, T])\rangle_{V_\ell^\sharp, V_\ell}, \ \forall \mu \in M. \tag{5.6}$$

We are going to transfer the LDP for $\dot{Y}^\varepsilon$ to the LDP for Y^ε, by means of the application Ψ. We keep the notations $\dot{Y}^\varepsilon$ for its restriction to $V_\ell \subset B$. Considering $\dot{Y}^\varepsilon$ as a random element in $V_\ell^\sharp$, Proposition B.1.b and (App.2) of the appendix provide us with

$$\Psi(\dot{Y}^\varepsilon) = Y^\varepsilon - c. \tag{5.7}$$

where Y^ε is seen as a random element in $D \subset M^\sharp$.

The space D is now endowed with the $*$-weak topology $\sigma(D, M)$: a sequence $(x_n)_{n\geq 1}$ converges towards x in $\sigma(D, M)$ if and only if $\lim_{n\to\infty} \int_{[0,T]} x_n(t)\cdot\mu(dt) = \int_{[0,T]} x(t)\cdot\mu(dt), \forall \mu \in M$. Considering Dirac measures, one shows that this topology is stronger than the topology of the pointwise convergence. In restriction to V_r, it is also stronger than the weak topology (see remark (5.4)) which is $\sigma(V_r, M_{\mathrm{na}})$ where $M_{\mathrm{na}} \subset M$ is the set of nonatomic measures.

We are going to prove that the rate function for $\{Y^\varepsilon\}$ is given for any $y \in D$ by

$$J_M^{\theta_o}(y) = \begin{cases} \sup_{\mu\in M} \left\{ \int_{[0,T]} (y_t - c_t)\cdot\mu(dt) - \int_{[0,T]} G_{\theta_o}(t, \mu[t,T])\, dt \right\} & \text{if } y - c \in V_r \\ +\infty & \text{otherwise} \end{cases}$$

Theorem 5.3. *Let us fix θ_o in $\tilde{\mathcal{L}}^d$ and define Y^ε as in (5.1). Then, $\{Y^\varepsilon\}$ satisfies the LDP in D for the topology $\sigma(D, M)$ with the good rate function $J_M^{\theta_o}$.*

Proof. A weaker version of Theorem 5.2.c is the LDP for $\{\dot{Y}^\varepsilon\}$ in $V_\ell^\sharp$ for the topology $\sigma(V_\ell^\sharp, V_\ell)$ with rate function I_V. As Ψ is continuous for $\sigma(V_\ell^\sharp, V_\ell)$ and $\sigma(M^\sharp, M)$, the contraction principle and (5.7) lead us to the LDP for $\{Y^\varepsilon\}$ in $\sigma(M^\sharp, M)$ with the good rate function $J(y) = J_o(y - c)$ where $J_o(x) = \inf\{I_V(\nu)\,;\, \nu \in V_\ell^\sharp, \Psi(\nu) = x\}, x \in M^\sharp$.

Suppose that $x \in M^\sharp$ satisfies $J_o(x) < \infty$. Then, there exists $\nu \in V_\ell^\sharp$ such that $I_V(\nu) < \infty$ and $\Psi(\nu) = x$. By Lemma 6.4, ν belongs to M and by Lemma B.2 in the appendix, $\Psi(\nu) = x$ implies that x belongs to V_r and $\nu = \dot{x}$. Therefore, for any $x \in M^\sharp$, $J_o(x) = I_V(\dot{x})$ if $x \in V_r$ and $J_o(x) = \infty$ otherwise. According to (App.1):

$$J_M^{\theta_o}(x) = I_V(\dot{x}), \ \forall x \in V_r \tag{5.8}$$

This completes the proof of the theorem. ■

Remark. A slight modification of the proof of Theorem 4.1 combined with a projective limit approach to the pointwise convergence topology, allows us to state the LDP for $\{Y^\varepsilon\}$ in D with the pointwise convergence topology and the rate function:

$$y \mapsto \sup_{\mu \in M_S} \left\{ \int_{[0,T]} (y_t - c_t) \cdot \mu(dt) - \int_{[0,T]} G_{\theta_o}(t, \mu[t,T])\, dt \right\}$$

where M_S is the space finitely supported $\mathbb{R}^d$-valued measures on $[0,T]$.

6. The rate functions

In this section, we give alternate expressions for the rate functions of Section 5.

Without exponential integrability assumptions. We are going to work with the rate function $J_S^{\varphi_o}$ of the weak LDP for the Lévy processes $\{Y^\varepsilon\}$ defined at (5.3). With $\varepsilon = 1$ and $t = 1$ we get

$$Y := Y^1(1) = \alpha + \int_{[0,1]\times\mathbf{A}} \varphi_o \mathbb{1}_{\{|\varphi_o|\le 1\}}\, d\widetilde{N} + \int_{[0,1]\times\mathbf{A}} \varphi_o \mathbb{1}_{\{|\varphi_o|>1\}}\, dN$$

the log-Laplace transform of which is denoted, for any $\lambda \in \mathbb{R}^d$, by

$$H(\lambda) = \log \mathbb{E} \exp(\lambda \cdot Y) \in]-\infty, \infty].$$

We also write *dom* $H = \{\lambda \in \mathbb{R}^d\,;\, H(\lambda) < \infty\}$ its effective domain and *intdom* H the interior of *dom* H in $\mathbb{R}^d$.

Lemma 6.1.

(a) For any $y \in D$, $J_S^{\varphi_o}(y) = \sup_{\gamma\in S} \left\{ \int_{[0,T]} \gamma \cdot dy - \int_{[0,T]} H(\gamma_t)\, dt \right\}.$

(b) *Assuming that intdom H is not empty, for any $y \in V_r$,*

$$J_S^{\varphi_o}(y) = \sup_{\gamma\in C} \left\{ \int_{[0,T]} \gamma_t \cdot \dot{y}(dt) - \int_{[0,T]} H(\gamma_t)\, dt \right\}.$$

Remarks. The expression in (b) already appeared in [Ac1].
Possibly modifying the dimension d, one may always assume that *intdom* H is not empty without loss of generality.

Proof. (a) Because of Proposition 2.2, for any $\lambda \in \mathbb{R}^d$,
$G_{\varphi_o}(\lambda) = \log \mathbb{E} \exp(\lambda \cdot \int_{[0,T]\times\mathbf{A}} \varphi_o \mathbb{1}_{\{|\varphi_o|\le 1\}}\, d\widetilde{N}) + \log \mathbb{E} \exp(\lambda \cdot \int_{[0,T]\times\mathbf{A}} \varphi_o \mathbb{1}_{\{|\varphi_o|>1\}}\, dN)$. Since these variables are independent, we get: $\alpha \cdot \lambda + G_{\varphi_o}(\lambda) = H(\lambda)$. From which the result follows.

(b) For any $y \in V_r$, we have $J_S^{\varphi_o}(y) = I_S^{\varphi_o}(\dot{y})$ with

$$\text{(6.1)} \qquad I_S^{\varphi_o}(\nu) = \sup_{\gamma\in S} \left\{ \int_{[0,T]} \gamma_t \cdot \nu(dt) - \int_{[0,T]} H(\gamma_t)\, dt \right\},\ \nu \in M.$$

Let us show that in the above identity: $\sup_{\gamma\in S}=\sup_{\gamma\in C}$.

We shall need the following preliminary result. Let $(u_n)_{n\geq 1}$ be a sequence of functions on $[0,T]$ such that $0\leq u_n(t)\leq 1, \forall t\in[0,T]$ and $\lim_{n\to\infty}u_n(t)=1$, dt-almost everywhere. Then, for any measurable $\mathbb{R}^d$-valued function γ on $[0,T]$ with $\int_{[0,T]}H(\gamma_t)\,dt<\infty$, we have

$$\lim_{n\to\infty}\int_{[0,T]}H(u_n(t)\gamma_t)\,dt=\int_{[0,T]}H(\gamma_t)\,dt \tag{6.2}$$

Because of $0\leq\tilde\rho(uv)\leq\tilde\rho(v), \forall 0\leq u\leq 1, v\in\mathbb{R}$; $0\leq\rho(uv)\leq\rho(v)$, $\forall 0\leq u\leq 1, v\geq 0$ and $-1\leq\rho(uv)\leq 0$, $0\leq u\leq 1, v\leq 0$, thanks to the dominated convergence theorem, both the integrals of the right hand side of the following identity converge:

$$\begin{aligned}\int_{[0,T]}H(u_n(t)\gamma_t)\,dt=&\int_{[0,T]\times\mathbf{A}}[\tilde\rho(u_n\gamma\cdot\varphi_o)\mathbb{1}_{\{|\varphi_o|\leq 1\}}+\rho(u_n\gamma\cdot\varphi_o)\mathbb{1}_{\{|\varphi_o|>1,\gamma\cdot\varphi_o\geq 0\}}]\,d\bar\Lambda\\&+\int_{[0,T]\times\mathbf{A}}\rho(u_n\gamma\cdot\varphi_o)\mathbb{1}_{\{|\varphi_o|>1,\gamma\cdot\varphi_o<0\}}\,d\bar\Lambda\end{aligned}$$

This proves (6.2).

Notice that H is convex (Hölder's inequality) and lower semicontinuous (Fatou's lemma) on $\mathbb{R}^d$. Hence, it is continuous on *intdom* H.

Let us show that $\sup_{\gamma\in S}\geq\sup_{\gamma\in C}$. As $\sup_{\gamma\in C}=\sup_{\gamma\in C,\int H(\gamma)<\infty}$, we restrict our attention on $\gamma\in C$ such that $\int_{[0,T]}H(\gamma_t)\,dt<\infty$. The sequence $\gamma_n=(1-1/n)\gamma$ uniformly approximates γ with its values in *intdom* H. By (6.2), for any $\nu\in M$, $\lim_{n\to\infty}(\int_{[0,T]}\gamma_n\,d\nu-\int_{[0,T]}H(\gamma_n(t))\,dt)=\int_{[0,T]}\gamma\,d\nu-\int_{[0,T]}H(\gamma(t))\,dt$. Therefore, $\sup_{\gamma\in C}=\sup_{\gamma\in C,\int H(\gamma)<\infty,\gamma([0,T])\subset \mathit{intdom}\, H}$. Let $\gamma\in C$ be such that $\int_{[0,T]}H(\gamma_t)\,dt<\infty$ and $\gamma([0,T])\subset$ *intdom* H. As it is uniformly continuous, it is approximated by $\gamma_n\in S$ with $\gamma_n([0,T])\subset$ *intdom* H. But, H is uniformly continuous on a compact neighbourhood of $\gamma([0,T])$, consequently $H\circ\gamma_n$ uniformly converges to $H\circ\gamma$. It follows that $\lim_{n\to\infty}(\int_{[0,T]}\gamma_n\,d\nu-\int_{[0,T]}H(\gamma_n(t))\,dt)=\int_{[0,T]}\gamma\,d\nu-\int_{[0,T]}H(\gamma(t))\,dt$ and $\sup_{\gamma\in S}\geq\sup_{\gamma\in C}$.

Now, let us prove that $\sup_{\gamma\in C}\geq\sup_{\gamma\in S}$. For all $0\leq a<a+1/n\leq b-1/n<b\leq T$, $n\geq 1$, we define the function $\chi^n_{a,b}(t)=\mathbb{1}_{]a,a+1/n]}(t)n(t-a)+\mathbb{1}_{]a+1/n,b-1/n]}(t)+\mathbb{1}_{]b-1/n,b]}(t)n(b-t)$, $0\leq t\leq T$. It is a continuous function with $0\leq\chi^n_{a,b}\leq 1$ and $\lim_{n\to\infty}\chi^n_{a,b}=\mathbb{1}_{]a,b[}$. As $\sup_{\gamma\in S}=\sup_{\gamma\in S,\int H(\gamma)<\infty}$, we consider $\gamma\in S$ such that $\int_{[0,T]}H(\gamma_t)\,dt<\infty$. We approximate $\gamma=\sum_l\lambda_l\mathbb{1}_{]t_{l-1},t_l]}$ dt-almost everywhere by $\gamma_n=\sum_l\lambda_l\chi^n_{t_{l-1},t_l}$ for large enough n. Taking (6.2) into account, we obtain $\lim_{n\to\infty}\int_{[0,T]}\gamma_n\,d\nu-\int_{[0,T]}H(\gamma_n(t))\,dt=\int_{[0,T]}\gamma\,d\nu-\int_{[0,T]}H(\gamma(t))\,dt$. This proves the inequality $\sup_{\gamma\in C}\geq\sup_{\gamma\in S}$ and completes the proof of the lemma. ■

We introduce the Lagrangian associated with Y :

$$L_a(v)=\sup_{\lambda\in\mathbb{R}^d}\{\lambda\cdot v-H(\lambda)\}\in[0,\infty],\ v\in\mathbb{R}^d$$

and its recession function

$$L_s(w)=\lim_{u\to\infty}\frac{L_a(uw)}{u}\in[0,\infty],\ w\in\mathbb{R}^d.$$

We define $C_{\Lambda\circ\varphi_o^{-1}} \subset \mathbb{R}^d$ as the closed convex cone with vertex 0 generated by the support of the image measure $\Lambda \circ \varphi_o^{-1} \in M$. It is a closed convex cone. We say that a closed convex cone C is acute if $C \setminus \{0\}$ is a subset of an open half-space of the vector space spanned by C.

It is assumed below that $\varphi_o \mathbb{1}_{\{|\varphi_o|\le 1\}} \in L_1^d$. In this situation, one can rewrite (5.3)

$$Y^\varepsilon(t) = t\beta + \int_{[0,t]\times \mathbf{A}} \varphi_o(a)\, N^\varepsilon(dsda) \tag{6.3}$$

with $\beta = \alpha - \int_{\mathbf{A}} \varphi_o \mathbb{1}_{\{|\varphi_o|\le 1\}}\, d\Lambda \in \mathbb{R}^d$, and Y^ε has bounded variation sample paths (see remark (5.4)).

Theorem 6.2. *Let us assume that $\varphi_o \mathbb{1}_{\{|\varphi_o|\le 1\}} \in L_1^d$, $\Lambda(|\varphi_o| > 1) < \infty$ and that $C_{\Lambda\circ\varphi_o^{-1}}$ is acute. Then, the rate function of the weak LDP satisfied by $\{Y^\varepsilon\}$ defined at (6.3), is given for any $y \in D$, by*

$$J_S^{\varphi_o}(y) = \begin{cases} \int_{[0,T]} L_a\left(\frac{d\dot y_a}{dt}(t)\right) dt + \int_{[0,T]} L_s\left(\frac{d\dot y_s}{d\mu}(t)\right) \mu(dt) & \text{if } y \in V_r \\ & \text{and } \dot y \text{ is } (\beta + C_{\Lambda\circ\varphi_o^{-1}})\text{-valued} \\ +\infty & \text{otherwise} \end{cases}$$

where $\dot y = \dot y_a + \dot y_s$ is the decomposition in absolutely continuous and singular parts of $\dot y \in M$ with respect to the Lebesgue measure dt on $[0,T]$ and μ is any bounded nonnegative measure on $[0,T]$ such that $\dot y_s$ is absolutely continuous with respect to μ (e.g. $\mu = |\dot y_s|$).

Proof. Let us first check that the effective domain of $J_S^{\varphi_o}$ is a subset of V_r. With $C^* = \{\lambda \in \mathbb{R}^d\,;\, \lambda\cdot\varphi_o \le 0, \Lambda\text{-a.e.}\}$, we see that

$$C_{\Lambda\circ\varphi_o^{-1}} = \{v \in \mathbb{R}^d\,;\, v\cdot\lambda \le 0, \forall \lambda \in C^*\} \tag{6.4}$$

and

$$C_{\Lambda\circ\varphi_o^{-1}} \text{ is acute if and only if } C^* \text{ spans } \mathbb{R}^d. \tag{6.5}$$

Let us take $y \in D$ such that $J_S^{\varphi_o}(y) < \infty$. For any $\lambda_o \in \mathbb{R}^d$, $0 \le a < b \le T$, $u \in \mathbb{R}$, choosing $\gamma = u\lambda_o \mathbb{1}_{]a,b]}$ in $\sup_{\gamma\in S}$ and denoting $F_{\varphi_o}(\lambda) = \int_{\mathbf{A}} \rho(\lambda\cdot\varphi_o)\, d\Lambda = \int_{\mathbf{A}} \lambda\cdot\varphi_o \mathbb{1}_{\{|\varphi_o|\le 1\}}\, d\Lambda + G_{\varphi_o}(\lambda)$, we get $\sup_{u\in\mathbb{R}}\{u\lambda_o\cdot(\frac{y_b-y_a}{b-a} - \beta) - F_{\varphi_o}(u\lambda_o)\} = \sup_{u\in\mathbb{R}}\{u\lambda_o\cdot(\frac{y_b-y_a}{b-a}) - H(u\lambda_o)\} \le \frac{J_S^{\varphi_o}(y)}{b-a} < \infty$. Therefore, $\lambda_o\cdot(\frac{y_b-y_a}{b-a} - \beta)$ stands in the closure of $\{\frac{d}{du}F_{\varphi_o}(u\lambda_o) = \int_{\mathbf{A}} \lambda_o\cdot\varphi_o e^{u\lambda_o\cdot\varphi_o}\, d\Lambda\,;\, u \in \mathbb{R}\}$. It comes out that

$$\lambda_o\cdot\left(\frac{y_b - y_a}{b-a} - \beta\right) \le 0, \forall \lambda_o \in C^*. \tag{6.6}$$

Hence, for any $\lambda_o \in C^*$, $t \mapsto \lambda_o\cdot(y_t - t\beta)$ is a nonincreasing function. As, thanks to (6.5), C^* spans $\mathbb{R}^d$, it follows that y has bounded variation.

On the other hand, (6.6) and (6.4) lead us to $\frac{y_b-y_a}{b-a} \in \beta + C_{\Lambda\circ\varphi_o^{-1}}, \forall 0 \le a < b \le T$. This implies that $\dot y$ is a $(\beta + C_{\Lambda\circ\varphi_o^{-1}})$-valued measure.

As by (6.5), *intdom* H is nonempty, we can invoke Lemma 6.1.b: $\forall y \in V_r$, $J_S^{\varphi_o}(y) = I_S^{\varphi_o}(\dot{y}) = \sup_{\gamma \in C}\{\int_{[0,T]} \gamma_t \cdot \dot{y}_t \, dt - \int_{[0,T]} H(\gamma_t)\, dt\}$, $\dot{y} \in M$ (see (6.1)). Now, the expression of $J_S^{\varphi_o}$ which is stated in the theorem is a result of R. T. Rockafellar: ([Roc], Theorem 6). Notice that the assumptions of ([Roc], Theorem 6) are satisfied: since H doesn't depend on t, it is convex and lower semicontinuous and *intdom* $H \neq \emptyset$. $\blacksquare$

Under the only assumption $I\!\!E|Y| < \infty$, A. de Acosta ([Ac1], Thm. 5.1) has proved a LD lower bound for $\{Y^\varepsilon\}$ with the uniform topology and the rate function given for any $y \in D$ by $J_S^{\varphi_o}(y)$ if y is continuous and $+\infty$ otherwise. This lower bound rate function may not be optimal in the sense that it may not match the upper bound rate function (for compact sets). The second part of the next corollary states a sufficient condition for this uniform lower bound to be optimal.

Corollary 6.3. *Let us assume that in addition to the hypotheses of Theorem 6.2, dom H is a cone. Then, for any $y \in D$,*

$$J_S^{\varphi_o}(y) = \begin{cases} \int_{[0,T]} L_a\left(\frac{d\dot{y}}{dt}(t)\right) dt & \text{if } y \text{ is absolutely continuous} \\ & \text{and } \dot{y} \text{ is } (\beta + C_{\Lambda \circ \varphi_o^{-1}})\text{-valued} \\ +\infty & \text{otherwise} \end{cases}$$

Let us assume now that in addition to the hypotheses of Theorem 6.2, dom H is a cone and $I\!\!E|Y| < \infty$. Then, $\{Y^\varepsilon\}$ obeys the LD lower bound on V_r endowed with the uniform convergence topology with rate function $J_S^{\varphi_o}$ as above.

Proof. The first statement is a direct consequence of ([Roc], Corollary 1A). The second statement is a direct consequence of ([Ac1], Thm. 5.1). $\blacksquare$

With exponential integrability assumptions. During the proofs of Theorems 5.2 and 5.3, the following result has been used.

Lemma 6.4. *Let θ_o be in $\tilde{\mathcal{L}}^d$. Then, dom $I_B \subset B^\star$ and dom $I_V \subset M$.*

Proof. Let us show that *dom* $I_B \subset B^\star$. Let us take $\xi \in B^\sharp$. For any $\gamma \in B$, $\alpha \in I\!\!R$, we have $\int_{[0,T]} G_{\theta_o}(t, \alpha\gamma_t)\, dt = \int_{[0,T]\times \mathbf{A}} \tilde{\rho}(\alpha\gamma \cdot \theta_o)\, d\overline{\Lambda} + \int_{[0,T]\times \mathbf{A}} \alpha\gamma \cdot \theta_o \mathbb{1}_{\{|\theta_o|>1\}}\, d\overline{\Lambda}$ $\leq \int_{[0,T]\times \mathbf{A}} \tau(|\alpha|\gamma \cdot \theta_o)\, d\overline{\Lambda} + |\alpha| \int_{[0,T]\times \mathbf{A}} |\gamma \cdot \theta_o \mathbb{1}_{\{|\theta_o|>1\}}|\, d\overline{\Lambda}$. As, $\alpha\langle \xi, \gamma\rangle \leq \int_{[0,T]} G_{\theta_o}(t, \alpha\gamma_t)\, dt + I_B(\xi)$, with $\alpha = \pm \frac{1}{\|\gamma\cdot\theta_o\|_{\tau,\overline{\Lambda}} + \|\gamma\cdot\theta_o \mathbb{1}_{\{|\theta_o|>1\}}\|_{1,\overline{\Lambda}}}$, we get

$$\begin{aligned} |\langle \xi, \gamma\rangle| &\leq (2 + I_B(\xi))(\|\gamma \cdot \theta_o\|_{\tau,\overline{\Lambda}} + \|\gamma \cdot \theta_o \mathbb{1}_{\{|\theta_o|>1\}}\|_{1,\overline{\Lambda}}) \\ &\leq 2(2 + I_B(\xi))\|\theta_o\|_{1,\overline{\Lambda}}\|\gamma\|_\infty \end{aligned}$$

with $\|\gamma\|_\infty = dt\text{-ess}\sup_{t\in[0,T]}|\gamma_t|$. Therefore, $\xi \in B^\star$ whenever $I_B(\xi) < \infty$.

Let us prove: *dom* $I_V \subset M$. We obtain similarly, for any $\nu \in V_\ell^\sharp$

$$|\langle \nu, \gamma\rangle| \leq 2(2 + I_V(\nu))\|\theta_o\|_{1,\overline{\Lambda}}\|\gamma\|, \forall \gamma \in V_\ell$$

with $\|\gamma\| = \sup_{t\in[0,T]} |\gamma_t|$. Therefore, if $I_V(\nu) < \infty$, then $\nu \in V_\ell^\star$. But, $V_\ell^\star \simeq M$ by Theorem B.3. This completes the proof of the lemma. $\blacksquare$

We introduce some notions and notations which will be useful to state the next result. For any $t \in [0,T]$,

$$Z^{(t)} = \int_{[0,1]\times \mathbf{A}} \theta_o(t,a) \mathbb{1}_{\{|\theta_o(t,a)|\le 1\}} \widetilde{N}(dsda) + \int_{[0,1]\times \mathbf{A}} \theta_o(t,a) \mathbb{1}_{\{|\theta_o(t,a)|> 1\}} N(dsda).$$

Its log-laplace transform is

$$\log \mathbb{E} \exp(\lambda \cdot Z^{(t)}) = G_{\theta_o}(t,\lambda), \ \lambda \in \mathbb{R}^d.$$

We define the associated Lagrangian

$$L_a(t,v) = \sup_{\lambda\in\mathbb{R}^d} \{\lambda\cdot v - G_{\theta_o}(t,\lambda)\} \in [0,\infty], \ v \in \mathbb{R}^d$$

and its recession function

$$L_s(t,w) = \lim_{u\to\infty} \frac{L_a(t,uw)}{u} \in [0,\infty], \ w \in \mathbb{R}^d.$$

Any element $\xi \in B^\star$ can uniquely be decomposed as

$$\xi = \xi_a + \xi_s \tag{6.7}$$

where $\xi_a \in M$ is absolutely continuous with respect to dt and ξ_s is dt-singular. This means that there exists a nonincreasing sequence $(T_k)_{k\ge1}$ of Borel subsets of $[0,T]$ such that $\lim_{k\to\infty}\int_{T_k} dt = 0$ and for all $\gamma\in B$, $\langle \xi_s, \gamma\mathbb{1}_{[0,T]\setminus T_k}\rangle = 0$ (see for instance [CaV]).

Theorem 6.5. *Let θ_o be in $\tilde{\mathcal{L}}^d$.*

(a) *For any $\xi \in B^\sharp$,*

$$I_B(\xi) = \begin{cases} \int_{[0,T]} L_a(t, \frac{d\xi_a}{dt}(t))\,dt + \sup\{\langle \xi_s,\gamma\rangle\,;\, \gamma\in B, \int_{[0,T]} G_{\theta_o}(t,\gamma_t)\,dt < \infty\} & \text{if } \xi\in B^\star \\ +\infty & \text{otherwise} \end{cases}$$

where $\xi = \xi_a + \xi_s$ is the decomposition (6.7).

(b) *Let us assume in addition that $\Gamma_{\theta_o} := \{(t,\lambda)\in[0,T]\times\mathbb{R}^d\,;\, \lambda \in \text{ intdom}(G_{\theta_o}(t,\cdot))\}$ satisfies $\Gamma_{\theta_o} = \text{ int cl } \Gamma_{\theta_o}$. Then, for any $\nu \in V_\ell^\sharp$,*

$$I_V(\nu) = \begin{cases} \int_{[0,T]} L_a(t, \frac{d\nu_a}{dt}(t))\,dt + \int_{[0,T]} L_s(t, \frac{d\nu_s}{d\mu}(t))\,\mu(dt) & \text{if } \nu \in M \\ +\infty & \text{otherwise} \end{cases}$$

where $\nu = \nu_a + \nu_s$ is the decomposition of the measure $\nu\in M$ in absolutely continuous and singular parts with respect to dt and μ is any nonnegative measure on $[0,T]$ with respect to which ν_s is absolutely continuous.

(c) *Under the assumptions of (b), we also have for any $y\in D$, $J_M^{\theta_o}(y) = {J_M^{\theta_o}}'(y-c)$ with*

$${J_M^{\theta_o}}'(x) = \begin{cases} \int_{[0,T]} L_a(t, \frac{d\dot{x}_a}{dt}(t))\,dt + \int_{[0,T]} L_s(t, \frac{d\dot{x}_s}{d\mu}(t))\,\mu(dt) & \text{if } x \in V_r \\ +\infty & \text{otherwise} \end{cases}$$

where $\dot{x} = \dot{x}_a + \dot{x}_s$ is the decomposition of the measure $\dot{x} \in M$ in absolutely continuous and singular parts with respect to dt and μ is any nonnegative measure on $[0,T]$ with respect to which $\dot{x}_s$ is absolutely continuous.

Proof. (a) We have shown in Lemma 6.4 that *dom* $I_B \subset B^\star$. Hence, the expression for $I_B(\xi)$ is given in ([Roc], Theorem 1). Indeed, the assumptions of this theorem are satisfied since $G_{\theta_o}(\cdot, 0) \equiv 0 \in L^1(dt)$ and with $\gamma_t^* = \mathbb{E}Z^{(t)}$, we get $\dot{\gamma}^* \in L^1(dt)$ and $t \mapsto L_a(t, \dot{\gamma}_t^*) = 0 \in L^1(dt)$.

(b) We have shown in Lemma 6.4 that *dom* $I_V \subset M$. As in Lemma 6.1, one can prove that for all $\nu \in M$, $I_V(\nu) = \sup_{\gamma \in V_\ell}\{\} = \sup_{\gamma \in C}\{\}$. Hence, the expression for $I_V(\nu)$ is given in ([Roc], Theorem 5). Indeed, the assumptions of this theorem are satisfied: $\theta_o \in \tilde{\mathcal{L}}^d$ implies that the interior of *dom* $G_{\theta_o}(t,\cdot)$ is nonempty for all $t \in [0,T]$ and the assumption $\Gamma_{\theta_o} =$ *int cl* Γ_{θ_o} means that the multifunction $t \mapsto$ *dom* $G_{\theta_o}(t,\cdot)$ is fully lower semicontinuous (see ([Roc], Lemma 2).

(c) It is deduced from (b) by means of (5.8). ■

Remarks. If θ_o belongs to $M_\tau(\overline{\Lambda})$: i.e. $\theta_o \mathbb{1}_{\{|\theta_o| \le 1\}} \in L_2(\overline{\Lambda})$ and $\int_{[0,T]\times\mathbf{A}} \exp(\beta|\theta_o|)\mathbb{1}_{\{|\theta_o|>1\}}\, d\overline{\Lambda} < \infty$, $\forall \beta > 0$, then $\int_{[0,T]} G_{\theta_o}(t, \gamma_t)\, dt < \infty$, $\forall \gamma \in B$ and *dom* $I_B \subset L^1(dt)$. Similarly, in this case $J_M^{\theta_o}(y) < \infty$ implies that $y - c$ is absolutely continuous.
If θ_o doesn't depend on t, the assumption on Γ_{θ_o} in (b) and (c) is satisfied.
In the case $d = 1$, the expression (c) has been derived in [LyS] and [Ac1] via martingale methods.

We conclude with a dual equality. Let us define $\tilde{L}_a(v) = \sup_{\lambda \in \mathbb{R}^d}\{\lambda \cdot v - \int_{[0,T]\times\mathbf{A}} \tilde{\rho}(\lambda \cdot \theta_o)\, d\overline{\Lambda}\} \in [0,\infty]$, $v \in \mathbb{R}^d$ and its recession function $\tilde{L}_s(w) = \lim_{u\to\infty} \tilde{L}_a(uw)/u \in [0,\infty]$, $w \in \mathbb{R}^d$.

Corollary 6.6. *Let θ_o stand in $\tilde{\mathcal{L}}^d$. Then, $\{\widetilde{X}^\varepsilon\}$ (defined at (5.2)) obeys the LDP in $\sigma(D,M)$ with the good rate function given for any $y \in D$ by*

$$\tilde{J}_M^{\theta_o}(y) = \begin{cases} \int_{[0,T]} \tilde{L}_a(t, \frac{d\dot{y}_a}{dt}(t))\, dt + \int_{[0,T]} \tilde{L}_s(t, \frac{d\dot{y}_s}{d\mu}(t))\, \mu(dt) & \textit{if } y \in V_r \\ +\infty & \textit{otherwise} \end{cases}$$

where $\dot{y} = \dot{y}_a + \dot{y}_s$ is the decomposition of the measure $\dot{y} \in M$ in absolutely continuous and singular parts with respect to dt and μ is any nonnegative measure on $[0,T]$ with respect to which $\dot{y}_s$ is absolutely continuous.

Moreover, for any $y \in D$,

$$\begin{aligned} \tilde{J}_M^{\theta_o}(y) &= \sup_{\mu \in M} \left\{ \int_{[0,T]} y_t \cdot \mu(dt) - \int_{[0,T]\times\mathbf{A}} \tilde{\rho}(\mu([t,T]) \cdot \theta_o(t,a))\, dt\Lambda(da) \right\} \\ &= \inf \left\{ \tilde{I}(n)\, ;\, n \in \tilde{\mathcal{L}}^\sharp, \langle n, \mu([\cdot,T]) \cdot \theta_o \rangle = \int_{[0,T]} y_t \cdot \mu(dt), \forall \mu \in M \right\} \end{aligned}$$

Proof. As $\widetilde{X}^\varepsilon$ corresponds to Y^ε with $c_t = -\int_{[0,t]\times\mathbf{A}} \theta_o(s,a)\mathbb{1}_{\{|\theta_o(s,a)|>1\}}\, ds\Lambda(da)$, the LDP is Theorem 5.3 where $\tilde{J}_M^{\theta_o}(y) = \sup_{\mu\in M} \Big\{ \int_{[0,T]} [y_t + \int_{[0,t]\times\mathbf{A}} \theta_o(s,a)\mathbb{1}_{\{|\theta_o(s,a)|>1\}}\, ds\Lambda(da)] \cdot \mu(dt)$ $- \int_{[0,T]} G_{\theta_o}(t, \mu([t,T])\, dt \Big\}$ (use (App.1) to transform this identity) is expressed by means of Theorem 6.5.c. Because of the uniqueness of the rate function of a LDP (see [DeZ], Lemma

4.1.4) and thanks to the contraction principle applied to Theorem 4.1 with the (weakly) continuous application $\Phi : \tilde{\mathcal{L}}^\sharp \to M^\sharp$ defined for any $\nu \in \tilde{\mathcal{L}}^\sharp$ by $\langle \Phi(n), \mu\rangle = \langle n, \mu([\cdot, T]) \cdot \theta_o\rangle$, $\forall \mu \in M$, we obtain the last equality. ■

Appendix

A. Two abstract results.

Proposition A.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be regular Haussdorf topological spaces. We assume that $\{\mu_\varepsilon\}$ and $\{\nu_\varepsilon\}$ satisfy weak LDPs on $\mathcal{X}$ and $\mathcal{Y}$ with lower semicontinuous rate functions I and J. Then, $\{\mu_\varepsilon \otimes \nu_\varepsilon\}$ satisfies a weak LDP on $\mathcal{X} \times \mathcal{Y}$ with rate function $I \oplus J$. Moreover, $I \oplus J$ is lower semicontinuous on $\mathcal{X} \times \mathcal{Y}$.*

Proof. This result is a slight modification of ([LyS], Lemma 2.8). ■

The result of Dawson and Gärtner ([DaG]) about projective limits of LDPs is usually stated for full LDPs with good rate functions (see [DeZ], Theorem 4.6.1). In our context, projective limits of weak LDPs with lower semicontinuous rate functions are needed.

Let $(J, \leq)$ be a partially ordered right filtering family and $\mathcal{X} = \varprojlim \mathcal{Y}_j$ be the projective limit of the topological spaces $(\mathcal{Y}_j)_{j\in J}$. The canonical projections $p_j : \mathcal{X} \mapsto \mathcal{Y}_j$, $j \in J$, are continuous.

Proposition A.2. *Let $\{\mu_\varepsilon\}$ be a family of probability measures on $\mathcal{X} = \varprojlim \mathcal{Y}_j$, endowed with its Borel σ-field, such that for any $j \in J$, $\{\mu_\varepsilon \circ p_j^{-1}\}$ satisfies a weak LDP with rate function I_j. Suppose that for any $j \in J$, $I_j : \mathcal{Y}_j \mapsto [0, \infty]$ is lower semicontinuous and that for any $i \leq j$ in J : $I_i \circ p_i \leq I_j \circ p_j$. Then, $\{\mu_\varepsilon\}$ satisfies a weak LDP with rate function $I(x) = \sup_{j\in J} I_j \circ p_j(x)$, $x \in \mathcal{X}$.*

Proof. Let us prove the lower bound. For any open subset G of $\mathcal{X}$ and any $x \in G$, there exists $j \in J$ and an open subset V_j of $\mathcal{Y}_j$ such that $x \in p_j^{-1}(V_j) \subset G$. Therefore,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \geq \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon \circ p_j^{-1}(V_j) \geq -I_j(V_j) \geq -I_j \circ p_j(x) \geq -\sup_{j\in J} I_j \circ p_j(x)$$

which gives the desired lower bound.

Let us prove the weak upper bound. Let C be a compact subset of $\mathcal{X}$. Since p_j is continuous, $p_j(C)$ is a compact subset of $\mathcal{Y}_j$. It follows from the weak upper bound in $\mathcal{Y}_j$ that for any $j \in J$

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(C) \leq \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon \circ p_j^{-1}(p_j(C)) \leq -\inf_{x\in C} I_j \circ p_j(x).$$

Hence, $\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(C) \leq -\sup_{j\in J} \inf_{x\in C} I_j \circ p_j(x)$. One completes the proof of this upper bound with the next lemma. ■

Lemma. *Let $(J, \leq)$ be a partially ordered right filtering family and $\{f_j, j \in J\}$ a family of lower semicontinuous functions on a topological space $\mathcal{X}$ which is nondecreasing: $\forall i, j \in J$, $i \leq j \Longrightarrow f_i \leq f_j$. Then, for any compact subset C of $\mathcal{X}$, we have*

$$\sup_{j\in J} \inf_{x\in C} f_j(x) = \inf_{x\in C} \sup_{j\in J} f_j(x).$$

Proof. Let us give a proof of this standard result. Denote $A = \sup_{j\in J}\inf_{x\in C} f_j(x)$ and $B = \inf_{x\in C}\sup_{j\in J} f_j(x)$. Clearly, $A \le B$. Let us show that $B \le A$. If $A = \infty$, the result is immediate. Suppose now that $A < \infty$. Since f_j is lower semicontinuous, C is compact and $A < \infty$, $D_j := \{(x, y) \in C \times \mathbb{R}\,;\, f_j(x) \le t \le A\}$ is a compact subset of $\mathcal{X} \times \mathbb{R}$. As f_j is lower semicontinuous, it attains its infimum at some point x^* of the compact C, and because of definition of $A : f_j(x^*) = \inf_{x\in C} f_j(x) \le A$. Consequenly, D_j is non-empty.
It follows from our assumptions that $\{D_j\}_{j\in J}$ is a right filtering decreasing family on non-empty compact sets. Therefore, $\bigcap_{j\in J} D_j$ is non-empty. This means that there exists $(x_o, t_o) \in C \times \mathbb{R}$ such that $\sup_{j\in J} f_j(x_o) \le t_o \le A$. Finally, $B \le A$. This completes the proof of the lemma. ■

B. Some duality results. We study functional spaces which are used in Section 5 to prove the LDP for $\{Y^\varepsilon\}$ under exponential integrability assumptions.

Let V_ℓ (resp. V_r) be the space of left (resp. right) continuous $\mathbb{R}^d$-valued functions on $[0, T]$ with bounded variations. We denote $V_\ell^\sharp$ the algebraic dual space of V_ℓ and $V_\ell^\star$ is its subspace of continuous elements for the uniform convergence topology.
Let us denote M the space of bounded $\mathbb{R}^d$-valued measures on $[0, T]$.

Proposition B.1. (Integration by parts).

(a) *Deterministic formula. For any $\mu \in M$ and $\nu \in V_\ell^\star$, we have $\mu([\cdot, T]) \in V_\ell$, $\nu([0, \cdot]) \in V_r$ and*

$$\int_{[0,T]} \nu([0, t]) \cdot \mu(dt) = \langle \mu([\cdot, T]), \nu\rangle_{V_\ell, V_\ell^\star}.$$

where $\nu([0, \cdot])$ stands for $\langle \mathbb{1}_{[0,\cdot]}, \nu\rangle$.

(b) *Stochastic formula. For any $\mu \in M$ and $\theta_o \in L_2^d(\overline{\Lambda})$ we have*

$$\int_{[0,T]} \widetilde{X}_t^\varepsilon \cdot \mu(dt) = \int_{[0,T]\times\mathbf{A}} \mu([t, T]) \cdot \theta_o(t, a)\, \widetilde{N}^\varepsilon(dtda) \quad \text{in } L^2(\Omega, I\!P).$$

where $\widetilde{X}^\varepsilon$ is defined at (5.2).

Remark. Let us notice that if $\mu, \nu \in M$:

$$\text{(App.1)}\quad \int_{[0,T]} \nu([0, t])\, \mu(dt) = \int_{[0,T]^2} \mathbb{1}_{\{0\le s\le t\le T\}}\, \nu(ds)\mu(dt) = \int_{[0,T]} \mu([s, T])\, \nu(ds), \forall \mu, \nu \in M.$$

As a consequence, in view of (5.2), we also have for any $\mu \in M$ and $\theta_o \in L_1^d(\overline{\Lambda})$

$$\text{(App.2)}\quad \int_{[0,T]} X_t^\varepsilon \cdot \mu(dt) = \int_{[0,T]\times\mathbf{A}} \mu([t, T]) \cdot \theta_o(t, a)\, N^\varepsilon(dtda) \quad \text{almost surely.}$$

Proof. As the proof easily reduces to the dimension $d = 1$, we take $d = 1$.

Let us prove (a). Any $\nu \in V_\ell^\star$ is relatively bounded (with respect to the natural order on the Riesz space V_ℓ). Therefore, it admits a Jordan decomposition $\nu = \nu_+ - \nu_-$. As a consequence, $\nu([0, \cdot])$ belongs to V_r. Similarly, since $\mu = \mu_+ - \mu_-$, $\mu([\cdot, T])$ belongs to V_ℓ. These Jordan decompositions allow us to reduce the proof to the case where μ and ν are nonnegative.

Let us take $\mu \in M$, $\nu \in V_\ell^\star$, $\mu, \nu \geq 0$ and denote $A(t) = \mu([t,T]), t \in [0,T]$. It is a nonincreasing function in V_ℓ. Hence, it can be uniformly approximated by a sequence $(A_n)_{n\geq 1}$ of simple functions of V_ℓ. Let $\mu_n \in M$ be defined by $\mu_n([\cdot,T]) = A_n$. An easy computation gives, for any $n \geq 1$,

$$\int_{[0,T]} \nu([0,t]) \cdot \mu_n(dt) = \langle A_n, \nu\rangle_{V_\ell, V_\ell^\star}. \tag{App.3}$$

As ν is continuous with respect to the uniform topology on V_ℓ, we get $\lim_{n\to\infty}\langle A_n,\nu\rangle = \langle A,\nu\rangle$. Since $\nu([0,\cdot])$ is right continuous and nondecreasing it can be decomposed as $\nu([0,\cdot]) = f_{\mathrm{cont}} + f_{\mathrm{jump}}$ where f_{cont} is continuous and f_{jump} is its jump part: $f_{\mathrm{jump}} = \sum_k \beta_k \mathbb{1}_{U_k}$, where $\beta_k > 0$ and the U_k's are of the form $U_k = [s,t[\cap[0,T]$. As μ_n weakly tends to μ, we get $\lim_{n\to\infty}\int_{[0,T]} f_{\mathrm{cont}}\, d\mu_n = \int_{[0,T]} f_{\mathrm{cont}}\, d\mu$. We also have: $\int_{[0,T]} f_{\mathrm{jump}}\, d\mu_n = \sum_k \beta_k \mu_n(U_k)$ and $\mu_n(U_k) = A_n(t) - A_n(s) \underset{n\to\infty}{\longrightarrow} A(t) - A(s) = \mu(U_k)$. It follows that $\lim_{n\to\infty}\int_{[0,T]} f_{\mathrm{jump}}\, d\mu_n = \int_{[0,T]} f_{\mathrm{jump}}\, d\mu$. Putting these convergences together with (App.3), we obtain the desired identity.

Let us prove (b). For simplicity, the proof is written with $\varepsilon = 1$. Let θ^n stand in $\mathcal{E}_{\overline{\Lambda}}$, see the begining of Section 2 for the definition of $\mathcal{E}_R$. Because of (App.1), we have almost surely

$$W(\mu([\cdot,T])\theta^n) = \int_{[0,T]} W(\mathbb{1}_{[0,t]}\theta^n)\, \mu(dt) \tag{App.4}$$

where W is the stochastic integral with respect to $\widetilde{N}$ which is defined as the isometry from $L_2(\overline{\Lambda})$ to $L^2(I\!\!P)$.

Let θ_o belong to $L_2(\overline{\Lambda})$. Consider the approximation $\theta^n \underset{n\to\infty}{\longrightarrow} \theta_o$ in $L_2(\overline{\Lambda})$, $\theta^n \in \mathcal{E}_{\overline{\Lambda}}$. Since, $\int_{[0,T]\times\mathbf{A}} |\mu([\cdot,T])(\theta^n - \theta_o)|^2\, d\overline{\Lambda} \leq |\mu|([0,T])^2 \|\theta^n - \theta_o\|^2_{2,\overline{\Lambda}} \underset{n\to\infty}{\longrightarrow} 0$, we get

$$W(\mu([\cdot,T])\theta^n) \underset{n\to\infty}{\longrightarrow} W(\mu([\cdot,T])\theta_o) \quad \text{in } L^2(I\!\!P). \tag{App.5}$$

We also have $I\!\!E\left|\int_{[0,T]}\{W(\mathbb{1}_{[0,t]}\theta^n) - W(\mathbb{1}_{[0,t]}\theta_o)\}\,\mu(dt)\right|^2 \leq \int_{[0,T]}\{I\!\!E|W(\mathbb{1}_{[0,t]}(\theta^n - \theta_o))|^2\}\,|\mu|(dt) \leq |\mu|([0,T])\|\theta^n - \theta_o\|^2_{2,\overline{\Lambda}} \underset{n\to\infty}{\longrightarrow} 0$. Therefore,

$$\int_{[0,T]} W(\mathbb{1}_{[0,t]}\theta^n)\,\mu(dt) \underset{n\to\infty}{\longrightarrow} \int_{[0,T]} W(\mathbb{1}_{[0,t]}\theta_o)\,\mu(dt) \quad \text{in } L^2(I\!\!P).$$

This convergence together with (5.2), (App.4) and (App.5) completes the proof of the proposition.
■

There is a one-one correspondence between V_r and M : for any $x \in V_r$, there exists a unique $m \in M$ such that $x = m([0,\cdot])$ and for any $m \in M$, $x = m([0,\cdot])$ stands in V_r. We denote

$$x = \dot{x}([0,\cdot]),\ x \in V_r, \dot{x} \in M.$$

Lemma B.2. *The application Ψ defined at (5.6) is a linear injection. Its restriction to $V_\ell^\star$ is one-one from $V_\ell^\star$ to $V_r \subset M^\sharp$. Moreover,*

$$\Psi(\nu) = \nu([0,\cdot]),\ \forall \nu \in V_\ell^\star$$
$$\Psi^{-1}(x) = r_{V_\ell}(\dot{x}),\ \forall x \in V_r$$

where $r_{V_\ell}(m)$ is the restriction to V_ℓ of the measure $m \in M$.

Proof. For any $\xi \in V_\ell^\sharp$, $\Psi(\xi) = 0 \Leftrightarrow \langle \xi, \mu([\cdot, T]\rangle = 0, \forall \mu \in M \Leftrightarrow \xi = 0$, since $\mu([\cdot, T])$ describes V_ℓ when μ describes M. Hence, Ψ is an injection.

For any $\mu \in M, \nu \in V_\ell^\star$, we have $\langle \Psi(\nu), \mu \rangle = \langle \mu([\cdot, T]), \nu \rangle = \int_{[0,T]} \nu([0, t]) \cdot \mu(dt)$, where the first equality is the definition of Ψ and the second one is Proposition B.1.a. This is $\Psi(\nu) = \nu([0, \cdot])$ which stands in V_r as proved in Proposition B.1.

Similarly, for any $x \in V_r, \mu \in M$, $\langle \Psi(r_{V_\ell}(\dot{x})), \mu \rangle = \int_{[0,T]} \mu([t, T]) \cdot \dot{x}(dt) = \int_{[0,T]} x_t \cdot \mu(dt)$, which is $\Psi^{-1}(x) = r_{V_\ell}(\dot{x})$. ■

This lemma together with the basic identity $\nu = \Psi^{-1} \circ \Psi(\nu), \nu \in V_\ell^\star$, leads us to the following corollary.

Theorem B.3. (Representation of $V_\ell^\star$). *For any $\nu \in V_\ell^\star$, there exists a unique $m_\nu \in M$ such that*

$$\langle \nu, \gamma \rangle = \int_{[0,T]} \gamma \cdot dm_\nu, \ \forall \gamma \in V_\ell.$$

Conversely, the restriction $r_{V_\ell}(m)$ of any $m \in M$ to V_ℓ stands in $V_\ell^\star$.

In other words, $V_\ell^\star$ is isomorphic to M.

References

[Ac1] A. de Acosta. *Large deviations for vector-valued Lévy processes*. Stoch. Proc. and Appl. 51, (1994), 75–115.

[Ac2] A. de Acosta. *Exponential Tightness and Projective Systems in Large deviation Theory*. Festschrift for Lucien Le Cam, Research Papers in Probability and Statistics. Eds. D. Pollard, E. Torgersen, G.L. Yang. Springer, 143–156.

[Bor] A.A. Borovkov. *Boundary-value problems for random walks and lage deviations in function spaces*. Theory Probab. Appl. 12 (1967), 575–595.

[CaV] C. Castaing and M. Valadier. *Convex analysis and measurable multifunctions*. Lecture Notes in Mathematics 580. (1977), Springer, Berlin.

[DaG] D.A. Dawson and J. Gärtner. *Large deviations from the McKean-Vlasov limit for weakly interacting diffusions.* Stochastics and Stoch. Rep., Vol. 20, (1987), p 247–308.

[DeZ] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications.* (1993), Jones and Bartlett Publishers.

[DuE] P. Dupuis and R.S. Ellis. *A Weak Convergence Approach to the Theory of Large Deviations.* (1997), Wiley Series in Probability and Statistics, Wiley Interscience.

[FlP] D. Florens and H. Pham. *Large deviation probabilities in estimation of Poisson random measures.* Stoch. Proc. and Appl. 76, (1998), 117–139.

[FrW] M.I. Freidlin and A.D. Wentzell. *Random Perturbations of Dynamical Systems*. Grundlehren der math. Wissenschaft 260, Springer Verlag, (1984).

[JaS] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer Verlag, (1987).

[Léo] C. Léonard. *Convex conjugates of integral functionals*. (1996), Prépublication de l'Université d'Orsay.

[LiP] R.S. Lipster and A.A. Pukhalskii. *Limit theorems on large deviations for semimartingales.* Stochastics and Stochastic Reports, Vol. 38, (1992), 201–249.

[LyS] J. Lynch and J. Sethuraman. *Large Deviations for Processes with Independent Increments*. Ann. Probab. , Vol. 15, n° 2 (1987), 610–627.

[Mog] A.A. Mogulskii. *Large Deviations for Processes with Independent Increments.* Ann. Probab. 21, No. 1, (1993), 202–215.

[RaR] M. M. Rao and Z. D. Ren. *Theory of Orlicz Spaces.* (1991), Marcel Dekker Inc. , New-York.

[Roc] R. T. Rockafellar. *Integrals which are convex functionals, II.* Pacific J. Math. Vol. 39, No 2, (1971), 439–469.

[Sch] M. Schilder. *Some asymptotics formulae for Wiener integrals*, Amer. Math. Soc. 125, (1966), 63–85.

[Wen] A.D. Wentzell. *Rough limit theorems on large deviations for Markov stochastic processes.* *I* : Theory Probab. Appl. 21, n° 2, (1976), 227–242 ; *II* : Theory Probab. Appl. 21, n° 3, (1976), 499–512 ; *III* : Theory Probab. Appl. 24, n° 4, (1979), 675–692 ; *IV* : Theory Probab. Appl. 27, n° 2, (1982), 215–234.