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On Large Deviations for Particle Systems Associated with Spatially Homogeneous Boltzmann Type Equations

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Abstract. *We consider a dynamical interacting particle system whose empirical distribution tends to the solution of a spatially homogeneous Boltzmann type equation, as the number of particles tends to infinity. These laws of large numbers were proved for the Maxwellian molecules by H. Tanaka ([Ta1]) and for the hard spheres by A.S. Sznitman ([Sz1]). In the present paper we investigate the corresponding large deviations: the large deviation upper bound is obtained and, using convex analysis, a non-variational formulation of the rate function is given. Our results hold for Maxwellian molecules with a cutoff potential and for hard spheres.*

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1. Introduction

The aim of this paper is to study large deviations for a large particle system associated with a spatially homogeneous Boltzmann type equation.

About Boltzmann equation. Let $u_t(x,z)$ stand for the density of molecules of a gas at time t, with location $x \in IR^3$ and velocity $z \in IR^3$. We have

$$u_t(x,z) \geq 0 \quad \text{and} \quad \int_{IR^3 \times IR^3} u_t(x,z)\, dxdz = 1, \forall\, 0 \leq t \leq T.$$

The evolution of a dilute gas is well described by the following Boltzmann equation

$$(1.1)\quad \partial_t u_t + z \cdot \nabla_x u_t = \int_{IR^3 \times S_2} \{u_t(x,z^*)\, u_t(x,z'^*) - u_t(x,z)u_t(x,z')\}\, q(|z-z'|,(z-z')\cdot n) dz' dn$$

where

$$z^* = z - [(z - z') \cdot n]\, n$$
$$z'^* = z' - [(z' - z) \cdot n]\, n\,,$$

(z, z') and (z^*, z'^*) respectively standing for the incident and resulting velocities of a biparticle performing a collision which is described by means of a parameter $n \in S_2$: the unit sphere of $I\!R^3$. This leads us to

$$\left\{\begin{array}{rl} z^* + z'^* & = z + z' \\ |z^* - z'^*| & = |z - z'| \\ (z^* - z'^*) \cdot n & = (z - z') \cdot n \end{array}\right. \tag{1.2}$$

and to the conservation of kinetic energy:

$$|z^*|^2 + |z'^*|^2 = |z|^2 + |z'|^2. \tag{1.3}$$

In the special case where

$$q(z, z') := \int_{S_2} q(|z - z'|, (z - z') \cdot n)\, dn < \infty,$$

$q(z, z')$ is the mean intensity of the collisions with incident velocities (z, z') while

$$\frac{q(|z - z'|, (z - z') \cdot n)}{q(z, z')}\, dn$$

describes the distribution of these random-like collisions. It is precisely the integration with respect to dn which yields the growth of the entropy (Boltzmann H-theorem).

If the initial distribution is spatially homogeneous, that is $\nabla_x u_0 \equiv 0$, this homogeneity is preserved as time runs: $\nabla_x u_t \equiv 0, \forall\, t > 0$ and (1.1) becomes

$$\partial_t u_t(z) = \int_{I\!R^3 \times S_2} \{u_t\,(z^*)\, u_t\,(z'^*) - u_t(z) u_t(z')\}\, q(|z - z'|, (z - z') \cdot n)\, dz' dn \tag{1.4}$$

which is the spatially homogeneous Boltzmann equation. Many results about equations (1.1) and (1.4) are collected in the book of C. Cercignani ([Cer]). The global existence of a solution to the Cauchy problem for (1.1) has been obtained by R. di Perna and P.L. Lions in [dPL].

Taking (1.2) into account, one can interpreat equation (1.4) as a flow equation.

L. Arkeryd and A.S. Sznitmann have proved strong ([Ark]) and weak ([Sz1]) existence and uniqueness of the solutions of (1.4) in the hard spheres case which corresponds to $q(z, z', n) = |(z - z') \cdot n|$.

H. Tanaka ([Ta1]) also proved the weak existence and uniqueness of the solutions of (1.4) in the case of Maxwellian molecules which corresponds to $q(z, z', n) = \psi(|\cos(z - z', n)|)$ with $\psi(u) = O_{u \to 0}(u^{-3/2})$.

In both cases, difficulties of the probabilistic approach mostly come from

$$\sup_{z, z'} q(z, z') = +\infty, \tag{1.5}$$

while choosing an analytic approach, one has to deal with the non-linearity (almost quadratic) of equation (1.1).

Carrying out the change of variables

$$\begin{cases} (z,z',n) & \to (z,z',\Delta,\Delta') \\ q(|z-z'|,(z-z')\cdot n)\,dn & \to \mathcal{L}(z,z',d\Delta d\Delta') \end{cases} \quad \text{with} \quad \begin{cases} \Delta(z,z',n) & = [(z-z')\cdot n]\,n \\ \Delta'(z,z',n) & = [(z'-z)\cdot n]\,n \end{cases},$$

equation (1.4) becomes

$$\partial_t u_t = A(u_t)^* u_t \tag{1.6}$$

where

$$A(u_t)f(z) = \int_{(\Delta)} \{f(z+\Delta)-f(z)\} \left(\int_{(z',\Delta')} \mathcal{L}(z,z',\cdot\times d\Delta')\,u_t(dz') \right)(d\Delta).$$

The Boltzmann-McKean particle system. Let us take a collision (Lévy) kernel $\mathcal{L}$, we want to build a Markov particle system whose empirical measure, as the number N of particles tends to infinity, approaches a weak solution of (1.6). Let us denote $z^N = (z_1,\dots,z_N) \in (I\!R^d)^N$ any configuration, the set of all the jumps of the biparticules is $E = (I\!R^d)^2 \setminus \{(0,0)\}$, $M_1(I\!R^d)$ stands for the set of all probability measures built on the Borel σ-field of $I\!R^d$.

Let $X^N = (X_i^N)_{1\le i\le N}$ be the Markov process on $(I\!R^d)^N$ whose generator is defined, for all $\Phi \in C_o^1((I\!R^d)^N)$, by

$$A_N\Phi(z^N) = \frac{1}{N-1} \sum_{1\le i<j\le N} \int_E \{\Phi(z^N+\Delta_{(i)}+\Delta'_{(j)}) - \Phi(z^N)\}\,\mathcal{L}(z_i,z_j,d\Delta d\Delta') \tag{1.7}$$

where for all $\Delta \in I\!R^d,\ 1\le i\le N$, $\Delta_{(i)} = (0,...,0,\ \underset{\underset{i^{\text{th}}}{\uparrow}}{\Delta},0,...,0) \in (I\!R^d)^N$. The empirical measure of z^N is defined by $\bar{z}^N = \dfrac{1}{N}\displaystyle\sum_{1\le i\le N} \delta_{z_i} \in M_1(I\!R^d)$ and

$$\overline{X}^N : t\in[0,T] \mapsto \overline{X}^N(t) = \frac{1}{N}\sum_{1\le i\le N} \delta_{X_i^N(t)} \in M_1(I\!R^d)$$

is the empirical process of the particle system. If $\overline{X}^N(0) \xrightarrow[n\to\infty]{\mathcal{L}} u_o$, then

$$\overline{X}^N(\cdot) \xrightarrow[n\to\infty]{\mathcal{L}} u(\cdot) \tag{1.8}$$

where $u(\cdot)$ is a weak solution of (1.6), with initial condition $u(0) = u_o$. The proof relies upon

$$A_N f(\langle\varphi,\cdot\rangle)(\bar{z}^N) = f'(\langle\varphi,\bar{z}^N\rangle)\,\langle A(\bar{z}^N)\varphi,\bar{z}^N\rangle + O_{f,\varphi}\left(\frac{1}{N}\right).$$

A.S. Sznitmann ([Sz1]) has proved this law of large numbers in the hard spheres case. A consequence of (1.8) is the propagation of chaos which in its weaker form may approximately be stated as follows.

If for any $N \geq 1$, the law of $(X_i^N(0))_{1\leq i\leq N}$ (at time $t=0$) is $u_o^{\otimes N}$, then for any $k\geq 1$ and any $t>0$, $u(t)^{\otimes k}$ is the limiting law of $(X_i^N(t))_{1\leq i\leq k}$. On this subject see [Sz3].

Some literature connected with the Boltzmann-McKean system. M. Kac ([Kac]), H.P. McKean ([MK1]) and H. Tanaka ([Ta1]) have obtained the law of large numbers (1.8) in the case of Maxwellian molecules and A.S. Sznitman ([Sz1]) has proved a result of a stronger type, dealing with $\widehat{X}^N = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{X_i^N(\cdot)}$ instead of $\overline{X}^N$, in a rather general setting including the case of the hard spheres. R. Ferland and G. Giroux ([FGi]) have obtained related results for a Boltzmann equation with cutoff.

The fluctuations around the limit (1.8) have been obtained in different cases by (among others) M. Kac ([Kac]), H.P. McKean ([MK2]) and K. Uchiyama ([Uc1], [Uc2]).

The aim of this paper is to obtain a large deviation upper bound associated with (1.8) and to study its rate function $I(\cdot)$. Therefore, we shall get estimates of the type

$$\frac{1}{N}\log I\!P\left(\overline{X}^N \in A\right) \underset{N\to\infty}{\asymp} -\inf_{\mu\in A} I(\mu)$$

for some subsets A of $D([0,T],M_1(I\!R^d))$. Many large deviations results for Markovian systems of interacting particles already exist in the literature. Let us mention in the case of weakly interacting diffusions the paper of D.A. Dawson and J. Gärtner ([DaG]) which extends previous results of H. Tanaka ([Ta2]) and A.S. Sznitman ([Sz2]) and a paper of C. Kipnis and S. Olla ([KiO]) for the hydrodynamical limit of independent Brownian motion. In the case of pure jump processes, F. Comets ([Com]) proved a large deviation principle for the Curie-Weiss model on the torus. His results have been extended by the author ([Léo]) and by M. Sugiura ([Sug]) to a larger class of weakly interacting particle systems with jumps. In their article: [BAB], G. Ben Arous and M. Brunaud derived fluctuation results from a large deviation principle by means of a Laplace method. In all these papers, the large deviation principles are obtained by taking advantage of an absolute continuity of the laws of the interacting particle systems with respect to some laws of independent (non-interacting) particle systems, so that the problem to be solved is to transfer Sanov's theorem.

Presentation of our results. At section 2, we obtain a large deviation upper bound for the empirical process of a general particle system which includes the Boltzmann-McKean system described above. This result is stated in theorem 2.1. The corresponding rate function is given in an abstract variational form.

At section 3, we give a more explicit expression of this rate function, in theorem 3.1, in the case of the Boltzmann-McKean system, under assumptions which include the hard spheres and the Maxwellian molecules with a cutoff potential. During this proof, the main troubles one has to deal with are the following ones.

(1.9.i) Because a system lead by equation (1.7) performs simultaneous jumps, there is no noninteracting (i.i.d.) reference (dominating in the sense of absolute continuity) particle system. Indeed, two independent Poisson processes never jump at the same time.

(1.9.ii) Because of (1.2) and (1.3), the cone generated by the support of the measure $\mathcal{L}(z,z',\cdot)$ may not be the whole space. In other words, the diffusion with jumps X^N may be degenerated.

(1.9.iii) The intensity of the jumps is unbounded (see (1.5)).

Although given in a less abstract form than in theorem 2.1, the rate function has still a variational form, since it appears as the Legendre transform of some log-Laplace functional. The remainder of the paper is devoted to the obtention of a non-variational expression of the rate function. This result is stated at section 7, in theorem 7.1.

At section 4, it is explained why the usual approach for the computation of "Gaussian" rate functions fails when applied to our "Poissonian" situation. It is proposed to introduce some made-to-measure Orlicz function spaces associated with $u \mapsto e^u - u - 1$: the log-Laplace transform of the centered Poisson law with parameter 1, and to take advantage of the Legendre polar form of the rate function. The problem is then to obtain an extension $\overline{\Gamma}$ of a functional Γ of the form: $f \mapsto \int (e^f - f - 1)\, d\Lambda$.

General results about Legendre polarity are derived at section 5. They will be used at section 6 to compute the extension $\overline{\Gamma}$ in theorem 6.6 and its corollary 6.7. Heavy technical troubles appear at this stage, in account of the bad behaviour of $u \mapsto e^u - u - 1$. Indeed, $e^u - u - 1$ grows too fast when $u \geq 0$ and decreases too slowly when $u \leq 0$, so that the Orlicz spaces under consideration are not reflexive. This is in contrast with the Gaussian situation where $e^u - u - 1$ is replaced by $u^2/2$, and the corresponding function spaces are L^2-spaces.

Finally, at section 7, the results of the previous sections are applied to obtain the non-variational formulation of the rate function.

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2. An upper bound for a general particle system

In the present section, we prove an abstract large deviation upper bound for the empirical process of a general particle system. The main results of the section are theorem 2.1, its corollary 2.2 and proposition 2.3. They will be the starting point for the proof of a large deviation upper bound for the Boltzmann-McKean system at section 3.

Statement of the results. The state space of the particles is a Polish space $\mathcal{Z}$ endowed with its Borel σ-field $\mathcal{B}(\mathcal{Z})$. For all $N \geq 1$, the N-particle system is described by the càdlàg stochastic process

$$\left(X_i^N\right)_{1 \leq i \leq N} : \Omega \longrightarrow D([0,T], \mathcal{Z}^N)$$

where $D([0,T], \mathcal{Z}^N)$ is the space of all càdlàg paths from $[0,T]$ to $\mathcal{Z}^N$ equipped with its canonical σ-field. In theorem 2.1 and its corollary 2.2 below, we state a large deviation upper bound for the empirical processes

$$\overline{X}^N : t \in [0,T] \mapsto \overline{X}^N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i^N(t)} \in M_1(\mathcal{Z})$$

as N tends to infinity.

Instead of considering $\overline{X}^N(\omega)$ as an element of $D([0,T], M_1(\mathcal{Z}))$, we shall view $\overline{X}^N(\omega)(dtdz) = (\frac{1}{N} \sum_{i=1}^N \delta_{X_i^N(\omega,t)}(dz))dt$ as a measure on $[0,T] \times \mathcal{Z}$. This random measure, still denoted by $\overline{X}^N$, will be called a *relaxed path.*

As a definition, a relaxed path μ is a nonnegative bounded measure on $[0,T]\times\mathcal{Z}$ such that its marginal projection on $[0,T]$: $\mu_{[0,T]}(dt) = \mu(dt\times\mathcal{Z})$ is the Lebesgue measure on $[0,T]$. Indeed, a regular version $(\mu_t(dz))_{0\le t\le T}$ of the desintegration $\mu(dtdz)=\mu_t(dz)\mu_{[0,T]}(dt)=\mu_t(dz)dt$ is a $M_1(\mathcal{Z})$-valued path. Such a regular version exists since $[0,T]\times\mathcal{Z}$ is Polish, moreover there exists at most one càdlàg version of $(\mu_t)_{0\le t\le T}$. Conversely, if $(\mu_t)_{0\le t\le T}$ is a measurable $M_1(\mathcal{Z})$-valued path, then $\mu_t(dz)dt$ is a relaxed path.

To control the behaviour of the particle at infinity (in $\mathcal{Z}$), let us introduce a $[1,+\infty[$-valued continuous function φ on $\mathcal{Z}$:

$$\varphi : \mathcal{Z} \longrightarrow [1,+\infty[.$$

Let M_φ be the set of all probability measures on $\mathcal{Z}$ which integrate φ :

$$M_\varphi = \{\xi\in M_1(\mathcal{Z})\,;\ \int_{\mathcal{Z}}\varphi\,d\xi<+\infty\}$$

and C_φ be the space of all continuous functions $g:\mathcal{Z}\longrightarrow I\!R$ such that $\sup\left\{\dfrac{|g(z)|}{\varphi(z)}\,;\ z\in\mathcal{Z}\right\}<+\infty$. Our set of measure-valued paths is

$$D_{M_\varphi}=\{\mu\,;\ \mu(\cdot)\,:\ t\in[0,T]\mapsto\mu(t)\in M_\varphi \text{ such that for any } g\in C_\varphi, t\in[0,T]\mapsto\int_{\mathcal{Z}}g(z)\,\mu(t;dz) \text{ is càdlàg}\}$$

and the set of relaxed paths is

$$\mathcal{M}_\varphi=\{\mu\,;\ \mu \text{ is a measure on } [0,T]\times\mathcal{Z} \text{ such that } \mu_{[0,T]}(dt)=dt \text{ and } \mu_t(dz)\in M_\varphi \text{ for a.e. } 0\le t\le T\}.$$

The set D_{M_φ} is endowed with its natural σ-field $\mathcal{A}=\sigma\left(\pi_{t,B}\,;\ 0\le t\le T, B\in\mathcal{B}(\mathcal{Z})\right)$ generated by the projections $\pi_{t,B}:\mu\mapsto\mu_t(B)$.

We introduce the space $\mathcal{C}_\varphi$ of test functions on $\mathcal{M}_\varphi$, consisting of all continuous functions $f\,:\,[0,T]\times\mathcal{Z}\longrightarrow I\!R$ such that $\sup\left\{\dfrac{|f(t,z)|}{\varphi(z)}\,;\ (t,z)\in[0,T]\times\mathcal{Z}\right\}<\infty$. Its algebraic dual space is denoted by $\mathcal{C}_\varphi^\sharp$. With the following dual bracket

$$\langle f,\mu\rangle=\int_{[0,T]\times\mathcal{Z}}f(t,z)\,\mu_t(dz)dt,\ \ f\in\mathcal{C}_\varphi,\mu\in\mathcal{M}_\varphi \tag{2.1}$$

one identifies $\mathcal{M}_\varphi$ as a subset of $\mathcal{C}_\varphi^\sharp$, so that

$$D_{M_\varphi}\subset\mathcal{M}_\varphi\subset\mathcal{C}_\varphi^\sharp.$$

These three sets are endowed with the weak-$*$ topologies $\sigma\left(D_{M_\varphi},\mathcal{C}_\varphi\right)$, $\sigma\left(\mathcal{M}_\varphi,\mathcal{C}_\varphi\right)$ and $\sigma\left(\mathcal{C}_\varphi^\sharp,\mathcal{C}_\varphi\right)$.

Remarks. $*$ It is interesting to introduce relaxed paths to obtain a larger class of compact subsets than with the Skorokhod topology, for which oscillations have to be controlled (see the proof of lemma 2.7 below). Of course, we get weaker large deviation results since closed and open subsets are less numerous in that topology. Nevertheless, the control of the oscillations may be performed following the proof of ([KOV], theorem 4.1).

$*$ If φ is a bounded function, M_φ is M_1 endowed with its usual weak topology. However, at section 3 we shall have to consider an unbounded function φ related to the kinetic energy of a particle (see (3.4)).

For any function F in the space $C_b(\mathcal{C}^\sharp_\varphi)$ of all real continuous bounded functions on $\mathcal{C}^\sharp_\varphi$, one defines

$$H(F) = \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E} e^{NF(\overline{X}^N)}$$

and for all μ in $\mathcal{C}^\sharp_\varphi$, let

$$L(\mu) = \sup\{F(\mu) - H(F)\,;\, F \in C_b(\mathcal{C}^\sharp_\varphi)\} \in [0, +\infty].$$

Let us state now the main results of this section.

Theorem 2.1. *Let us suppose that the function φ satisfies*

$$\forall a > 0, \exists b > 0, \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\sum_{i=1}^N \int_0^T \varphi(X_i^N(t))\, dt > Nb\right) \le -a \tag{2.2}$$

and that there exists an increasing sequence $(K_k)_{k\ge1}$ of compact subsets of $\mathcal{Z}$ such that

$$\lim_{k\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E} \exp\left(\gamma \sum_{i=1}^N \int_0^T \mathbb{1}_{\{X_i^N(t)\notin K_k\}}\, dt\right) = 0, \ \forall \gamma > 0. \tag{2.3}$$

Then, for any measurable subset A in $\mathcal{A}$, we have

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\overline{X}^N \in A) \le -\inf\{L(\mu)\,;\, \mu \in \overline{A}^{\mathcal{M}_\varphi}\}$$

where $\overline{A}^{\mathcal{M}_\varphi}$ stands for the closure of A in $\sigma\left(\mathcal{M}_\varphi, \mathcal{C}_\varphi\right)$.

Corollary 2.2. *Let us suppose that the conditions (2.2) and (2.3) are satisfied. Then, any $\mu \in \mathcal{C}^\sharp_\varphi$ such that $L(\mu)$ is finite belongs to $\mathcal{M}_\varphi$.*

If in addition, for any such μ one can find a version of $(\mu_t)_{0\le t\le T}$ such that $t \mapsto \int_{\mathcal{Z}} g\, d\mu_t$ is càdlàg for any $g \in C_\varphi$, then for any measurable subset A in $\mathcal{A}$, we have

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\overline{X}^N \in A) \le -\inf\{L(\mu)\,;\, \mu \in \overline{A}\}$$

where $\overline{A}$ stands for the closure of A in $\sigma\left(D_{M_\varphi}, \mathcal{C}_\varphi\right)$.

Proposition 2.3. *A sufficient condition for (2.2) to hold is that there exists $\alpha > 0$ such that*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E} \exp\left(\alpha \sum_{i=1}^N \int_0^T \varphi(X_i^N(t))\, dt\right) < +\infty \tag{2.4}$$

and a sufficient condition for (2.3) to hold is that there exist $\beta > 0$ and a measurable function $\psi : \mathcal{Z} \longrightarrow [0, +\infty[$ with compact level sets such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E} \exp\left(\beta \sum_{i=1}^{N} \int_0^T \psi(X_i^N(t))\, dt\right) < +\infty. \tag{2.5}$$

The remaining part of this section is devoted to the proof of these results.

Two standard results. Let X be a topological space and $(P_N)_{N\geq 1}$ be a sequence of probability measures built on its Borel σ-field. For any function F in the set $C_b(X)$ of all continuous bounded functions on X, let us define

$$H(F) = \limsup_{N\to\infty} \frac{1}{N} \log \int_X e^{NF}\, dP_N, \;\; F \in C_b(X).$$

Lemma 2.4. *Let us suppose that $(P_N)_{N\geq 1}$ is exponentially tight, then for any closed subset C of X, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N(C) \leq -\inf_{x\in C} \sup_{F\in C_b(X)} \{F(x) - H(F)\}.$$

Proof. See ([DZe], theorem 4.4.2). ■

Let Y be a topological space and X be a subset of Y endowed with its relative topology. Both X and Y are equipped with their respective Borel σ-fields: $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. Let P_N be a probability measure on $(X, \mathcal{B}(X))$. One defines its extension $\overline{P}_N$ on $(Y, \mathcal{B}(Y))$ by:

$$\overline{P}_N(A) = P_N(A \cap X), \;\; A \in \mathcal{B}(Y).$$

Lemma 2.5. *Let $(P_N)_{N\geq 1}$ be a sequence of probability measures on $(X, \mathcal{B}(X))$ and a rate function $I :\mapsto [0, +\infty]$ such that for any closed subset C of Y, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log \overline{P}_N(C) \leq -\inf\{I(y)\,;\, y \in C\}.$$

If the set $\{y \in Y\,;\, I(y) < +\infty\}$ is included in X, then for any closed subset C of X, we have

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N(C) \leq -\inf\{I(x)\,;\, x \in C\}.$$

Proof. See ([DZe], lemma 4.1.5). ■

The proof of theorem 2.1. The theorem 2.1 is a direct consequence of the lemmas 2.6, 2.7 and 2.8 which are proved below.

Let us begin with a simple measurability result. One denotes $h \otimes g(t, z) = h(t)g(z)$.

Lemma 2.6. *The Borel σ-field $\mathcal{B}(D_{M_\varphi})$ on D_{M_φ} is equal to the projection σ-field $\mathcal{A}$.*

Proof. Since $\mathcal{Z}$ is a metric space, $\mathcal{A}$ is equal to the σ-field generated by the projections: $\pi_{t,g} : \mu \mapsto \int_{\mathcal{Z}} g \, d\mu_t$, where t describes $[0,T]$ and g describes C_φ. Therefore, for any $f \in \mathcal{C}_\varphi$, $\mu \in D_{M_\varphi} \mapsto \langle f, \mu \rangle = \int_0^T \pi_{t,f(t,\cdot)}(\mu) \, dt$ is $\mathcal{A}$-measurable, so that: $\mathcal{B}(D_{M_\varphi}) \subset \mathcal{A}$.
Let θ be a nonnegative continuous function on $[0,1]$ such that $\int_0^1 \theta(s) \, ds = 1$. Consider the right-hand-side regular approximation $(\theta_n)_{n\geq 1}$ of δ_0, which is given by: $\theta_n(s) = n\theta(ns)$. For all $0 \leq t \leq T$, $g \in C_\varphi$ and $\mu \in D_{M_\varphi}$

$$\langle \theta_n(\cdot - t) \otimes g, \mu \rangle = \int_0^T \theta_n(s-t) \langle g, \mu_s \rangle \, ds \underset{n\to\infty}{\longrightarrow} \langle g, \mu_t \rangle = \pi_{t,g}(\mu)$$

since $s \mapsto \langle g, \mu_s \rangle$ is right continuous at t. Hence, $\pi_{t,g}$ is $\mathcal{B}(D_{M_\varphi})$-measurable and $\mathcal{A} \subset \mathcal{B}(D_{M_\varphi})$, which completes the proof of the lemma. ■

For any $N \geq 1$, let us define the probability measure $\overline{P}_N$ on the Borel σ-field $\mathcal{B}(\mathcal{C}_\varphi^\sharp)$ by

$$\overline{P}_N(A) = I\!P(\overline{X}^N \in A \cap D_{M_\varphi}), \; A \in \mathcal{B}(\mathcal{C}_\varphi^\sharp). \tag{2.6}$$

This definition is meaningful since: $A \in \mathcal{B}(\mathcal{C}_\varphi^\sharp) \Rightarrow A \cap D_{M_\varphi} \in \mathcal{A}$. Indeed, $\mathcal{B}(D_{M_\varphi})$ is the trace of $\mathcal{B}(\mathcal{C}_\varphi^\sharp)$ on D_{M_φ}, that is: $A \in \mathcal{B}(\mathcal{C}_\varphi^\sharp) \Rightarrow A \cap D_{M_\varphi} \in \mathcal{B}(D_{M_\varphi})$, and one concludes with lemma 2.6.

Lemma 2.7. *Let us suppose that φ satisfies the condition (2.2). Then, for any closed subset C of $\mathcal{C}_\varphi^\sharp$, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log \overline{P}_N(C) \leq -\inf\{L(\mu) \, ; \, \mu \in C\}.$$

Proof. We apply lemma 2.4 to $X = \mathcal{C}_\varphi^\sharp$ and to the sequence $(\overline{P}_N)_{N\geq 1}$ defined at (2.6). It remains to check the exponential tightness of $(\overline{P}_N)_{N\geq 1}$.
For any $b > 0$, let us set $K_b = \{\mu \in \mathcal{C}_\varphi^\sharp \, ; \, \mu \geq 0 \text{ and } \langle 1\!\!1 \otimes \varphi, \mu \rangle \leq b\}$, so that condition (2.2) can be expressed by

$$\forall a > 0, \exists b > 0, \limsup_{N\to\infty} \frac{1}{N} \log \overline{P}_N \left(\mathcal{C}_\varphi^\sharp \setminus K_b \right) \leq -a.$$

But K_b is compact in $\sigma\left(\mathcal{C}_\varphi^\sharp, \mathcal{C}_\varphi\right)$, by Tychonoff's theorem. ■

Lemma 2.8. *Let us suppose that φ satisfies condition (2.3). Then,*

$$\{\mu \in \mathcal{C}_\varphi^\sharp \, ; \, L(\mu) < +\infty\} \subset \mathcal{M}_\varphi.$$

Proof. A similar proof can be found in [DaG]. Let us first notice that $\mu \in \mathcal{C}_\varphi^\sharp$ belongs to $\mathcal{M}_\varphi$ if and only if

(a) $\forall f \in C_b([0,T] \times \mathcal{Z}), f \geq 0 \Rightarrow \langle f, \mu \rangle \geq 0$,
(b) $\forall g \in C_b([0,T]), \langle g \otimes 1\!\!1, \mu \rangle = \int_0^T g(t) \, dt$ and
(c) for any sequence $(f_p)_{p\geq 1}$ in $C_b([0,T] \times \mathcal{Z})$ which decreases pointwise to 0: $\lim_{p\to\infty} \langle f_p, \mu \rangle = 0$.

Indeed, thanks to (c) one can use Daniell's construction to extend the action of μ from $\mathcal{C}_\varphi$ to $B_b([0,T] \times \mathcal{Z})$ as a measure (see [Ne1], II.7). Property (a) implies that μ is a nonnegative measure, while (b) implies that $\mu(dt \times \mathcal{Z}) = dt$ and $\mu_t(\mathcal{Z}) = 1$ for almost every $0 \leq t \leq T$. Finally, since μ is in $\mathcal{C}^\sharp_\varphi$, taking the identification (2.1) into account, we obtain: $\langle \mathbb{1}\otimes\varphi, \mu\rangle = \int_{[0,T]\times\mathcal{Z}} \varphi(z)\,\mu_t(dz)dt < +\infty$.

Now, let $\mu \in \mathcal{C}^\sharp_\varphi$ be such that $L(\mu) < +\infty$. We are going to check that μ satisfies (a), (b) and (c). For any $f \in C_b([0,T]\times\mathcal{Z})$ and $n \geq 1$, let us define:

$$F_{f,n}(\mu) = (-n) \vee \langle f, \mu\rangle \wedge n,\ \ \mu \in \mathcal{C}^\sharp_\varphi$$

so that $F_{f,n}$ belongs to $C_b(\mathcal{C}^\sharp_\varphi)$.

Verification of (a). Let $f \in C_b([0,T]\times\mathcal{Z})$ be such that: $f \leq 0$. For all $n \geq 1$ and $\gamma \geq 0$, we have: $H(\gamma F_{f,n}) \leq 0$. Therefore, $\sup_{\gamma\geq 0} \gamma F_{f,n}(\mu) \leq \sup_{\gamma \geq 0}\{\gamma F_{f,n}(\mu) - H(\gamma F_{f,n})\} \leq L(\mu) < +\infty$. This implies that : $F_{f,n}(\mu) \leq 0, \forall n \geq 1$, that is: $\langle f, \mu\rangle \leq 0$.

Verification of (b). Let g stand in $C_b([0,T])$. For all $n \geq 1$ and $\gamma \in I\!R$

$$H(\gamma F_{g\otimes\mathbb{1},n}) = \limsup_{N\to\infty} \frac{1}{N}\log I\!E\exp\left(N\gamma\Big((-n)\vee\int_0^T g(t)\,dt \wedge n\Big)\right) = \gamma\Big((-n)\vee\int_0^T g(t)\,dt\wedge n\Big)$$

Therefore, $\sup_{\gamma\in I\!R} \gamma\left(F_{g\otimes\mathbb{1},n}(\mu) - \Big((-n)\vee \int_0^T g(t)\,dt\wedge n\Big)\right) \leq L(\mu) < +\infty$. This implies that for any $n \geq 1$, $(-n)\vee\langle g\otimes\mathbb{1}, \mu\rangle \wedge n = (-n)\vee\int_0^T g(t)\,dt\wedge n$, which yields (b).

Verification of (c). Since $\mathcal{Z}$ is a metric space, one can build a sequence $(l_k)_{k\geq 1}$ of continuous functions on $\mathcal{Z}$ such that: $0 \leq 1 - \mathbb{1}_{K_{k+1}} \leq l_k \leq 1 - \mathbb{1}_{K_k} \leq 1, \forall k \geq 1$. Let $(f_p)_{p\geq 1}$ be a sequence as in (c). It is enough to consider $(f_p)_{p\geq 1}$ such that $0 \leq f_p \leq 1, \forall p \geq 1$. Hence, for all $k \geq 1$, we have

$$0 \leq f_p \leq \mathbb{1}\otimes l_k + \mathbb{1}_{\{[0,T]\times K_{k+1}\}} f_p.$$

On the other hand, denoting $a_{k,p} = \sup\{f_p(t,z)\,;\,(t,z)\in[0,T]\times K_{k+1}\}$, we get for all $n \geq 1$ and $\gamma \geq 0$, $1 \leq I\!E e^{\gamma N F_{f_p,n}(\overline{X}^N)} \leq e^{\gamma N T a_{k,p}} I\!E \exp\big(\gamma N F_{\mathbb{1}\otimes l_k,n}(\overline{X}^N)\big)$ which gives

$$0 \leq H(\gamma F_{f_p,n}) \leq H(\gamma F_{\mathbb{1}\otimes l_k,n}) + \gamma T a_{k,p}.$$

An easy consequence of (2.3) is: $\lim_{k\to\infty} H(\gamma F_{\mathbb{1}\otimes l_k,n}) = 0$, and since any nonincreasing sequence of pointwise converging continuous functions on a compact set also converges uniformly, we get: $\lim_{p\to\infty} a_{k,p} = 0, \forall k \geq 1$. Therefore, for all $\gamma \geq 0$ and all $n \geq 1$, one gets $\lim_{p\to\infty} H(\gamma F_{f_p,n}) = 0$, and

$$0 \leq \gamma\big(\limsup_{p\to\infty}\langle f_p, \mu\rangle \wedge n\big) = \limsup_{p\to\infty}\{\gamma F_{f_p,n}(\mu) - H(\gamma F_{f_p,n})\} \leq L(\mu) < +\infty.$$

It follows that $\lim_{p\to\infty}\langle f_p, \mu\rangle = 0$. This completes the proof of the lemma. $\blacksquare$

The proof of corollary 2.2. The first part of corollary 2.2 is lemma 2.8. The additionnal assumption implies that $\{L < +\infty\} \subset D_{M_\varphi}$. One concludes with lemma 2.5 and theorem 2.1. $\blacksquare$

The proof of proposition 2.3. Let us begin with the implication: (2.4) $\Rightarrow$ (2.2). We set: $h(x) = \limsup_{N\to\infty}\frac{1}{N}\log I\!E\exp\left(x\sum_{i=1}^N\int_0^T \varphi(X_i^N(t))\,dt\right), x \in I\!R$. The assumption (2.4) states that there exists $\alpha > 0$ such that $h(\alpha) < +\infty$.

For all b greater than the right-hand derivative of h at 0, we get

$$\limsup_{N\to\infty} \frac{1}{N} \log I\!P\left(\frac{1}{N}\sum_{i=1}^{N}\int_0^T \varphi(X_i^N(t))\,dt \geq b\right) \leq -\sup_{x\in I\!R}\{bx - h(x)\} := -h^*(b)$$

and for all $b \geq b^-$: the left-hand derivative of h at α (possibly choosing $\alpha/2$ instead of α, one can choose α such that $b^- < +\infty$), we get

$$h^*(b) \geq \sup_{x\leq\alpha}\{bx - h(x)\} = b\alpha - h(\alpha).$$

Therefore, taking $b = \max\left(b^-, \dfrac{a+h(\alpha)}{\alpha}\right)$, we have obtained (2.2).

Now, let us check: (2.5) $\Rightarrow$ (2.3). For all $k \geq 1$, let $K_k = \{\psi \leq k\}$ be a compact level set of ψ.

$$\begin{aligned}
I\!E\exp\left(\beta\sum_{i=1}^{N}\int_0^T \psi(X_i^N(t))\,dt\right) &\geq I\!E\exp\left(\frac{\beta k}{\gamma}\sum_{i=1}^{N}\gamma\int_0^T \mathbb{1}_{\{X_i^N(t)\notin K_k\}}\,dt\right)\\
&\geq \left(I\!E\exp\left(\sum_{i=1}^{N}\gamma\int_0^T \mathbb{1}_{\{X_i^N(t)\notin K_k\}}\,dt\right)\right)^{\frac{\beta k}{\gamma}}
\end{aligned}$$

as soon as $k \geq \gamma/\beta$, by Jensen's inequality. Taking (2.5) into account,

$$\begin{aligned}
0 &\leq \limsup_{N\to\infty}\frac{1}{N}\log I\!E\exp\left(\sum_{i=1}^{N}\gamma\int_0^T \mathbb{1}_{\{X_i^N(t)\notin K_k\}}\,dt\right)\\
&\leq \frac{\gamma}{\beta k}\limsup_{N\to\infty}\frac{1}{N}\log I\!E\exp\left(\beta\sum_{i=1}^{N}\int_0^T \psi(X_i^N(t))\,dt\right) \underset{k\to\infty}{\longrightarrow} 0
\end{aligned}$$

which is (2.3). ■

3. The upper bound for the Boltzmann-McKean system

In the present section, a large deviation upper bound is proved for the Boltzmann-McKean system: $\left(X_i^N\right)_{1\leq i\leq N}$, whose evolution is governed by the generator A_N described at (1.7). This result is stated in theorem 3.1. The starting point of its proof is theorem 2.1. In theorem 3.1, the rate function of this upper bound is obtained in a variational form. A non-variational expression for this rate function will be stated in theorem 7.1.

Let us assume that the initial data and the Lévy kernel $\mathcal{L}$ satisfy the following hypotheses.

The assumptions (A). There exists a C^1-function $\varphi : I\!R^d \longrightarrow [1, +\infty[$ such that

(3.1) $$\lim_{|z|\to\infty}\varphi(z) = +\infty$$

which satisfies the following requirements.

We assume that the initial configurations $\left(X_i^N(0)\right)_{1\leq i\leq N}$ are such that

$$(A_0) \qquad \limsup_{N\to\infty} \frac{1}{N} \log I\!E \exp\left(\alpha \sum_{i=1}^{N} \varphi\left(X_i^N(0)\right)\right) < +\infty \text{ for some } \alpha > 0.$$

Without restriction, we choose the collision kernel $\mathcal{L}$ such that

$$\mathcal{L}(z, z', d\Delta d\Delta') = \mathcal{L}(z', z, d\Delta' d\Delta), \ \forall z, z' \in I\!R^d, \forall d\Delta, d\Delta' \in \mathcal{B}(I\!R^d).$$

Let us denote $E = (I\!R^d)^2 \setminus \{(0,0)\}$: the set of the jumps of the biparticles. We assume that there exists a real number $\lambda \geq 0$ such that for all $z, z' \in I\!R^d$:

$$(A_1) \qquad \int_E \{\exp[\varphi \oplus \varphi(z+\Delta, z'+\Delta') - \varphi \oplus \varphi(z, z')] - 1\} \ \mathcal{L}(z, z', d\Delta d\Delta') \leq \lambda \ \varphi \oplus \varphi(z, z'),$$

$$(A_2) \qquad \int_E \left(1\!\!1_{\{|\Delta|+|\Delta'|<1\}} + (|\Delta| + |\Delta'|) 1\!\!1_{\{|\Delta|+|\Delta'|\geq 1\}}\right) \ \mathcal{L}(z, z', d\Delta d\Delta') \leq \lambda(1 + \varphi(z)\varphi(z')),$$

(A_3) for any continuous bounded function g on E,

$$(z, z') \in (I\!R^d)^2 \mapsto \int_E (|\Delta| + |\Delta'|) g(\Delta, \Delta') \, \mathcal{L}(z, z', d\Delta d\Delta') \in I\!R \text{ is continuous},$$

and one assumes that

(A_4) for any compact subset K of $(I\!R^d)^2$,

$$\sup_{(z,z')\in(I\!R^d)^2} \int_E 1\!\!1_{\{(z+\Delta, z'+\Delta')\in K\}} \left(1\!\!1_{\{|\Delta|+|\Delta'|<1\}} + (|\Delta| + |\Delta'|) 1\!\!1_{\{|\Delta|+|\Delta'|\geq 1\}}\right) \times \ \mathcal{L}(z, z', d\Delta d\Delta') < +\infty.$$

Some comments about the assumptions. We have introduced the control function φ to overcome the difficulty (1.9.iii).

For the initial condition, we may choose one of the following situations (3.2) or (3.3).

(3.2) *Deterministic initial condition:* There are a sequence $(z_i^o)_{i\geq 1}$ in $I\!R^d$ and a probability measure ρ_o in $M_1(I\!R^d)$ which satisfy:

$$\int_{I\!R^d} \varphi(z) \, \rho_o(dz) < +\infty, \quad \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i^o} \underset{N\to\infty}{\longrightarrow} \rho_o \text{ in } M_1(I\!R^d) \text{ and } \sup_{N\geq 1} \frac{1}{N} \sum_{i=1}^{N} \varphi(z_i^o) < +\infty,$$

such that for any $N \geq 2, \ X_i^N(0) = z_i^o, \ 1 \leq i \leq N,$ almost surely.

(3.3) *Chaotic initial condition:* For any $N \geq 2$ the law of $\left(X_i^N(0)\right)_{1\leq i\leq N}$ is equal to $\rho_o^{\otimes N}$ where $\rho_o \in M_1(I\!R^d)$ satisfies:

$$\int_{I\!R^d} \exp\left(\alpha\varphi(z)\right) \, \rho_o(dz) < +\infty, \quad \text{for some } \alpha > 0.$$

Let us see how the assumptions (A) are translated in the case of the spatially homogeneous Boltzmann equation (1.4). Because of the conservation of kinetic energy (1.3), we choose

$$(3.4) \qquad \varphi(z) = 1 + |z|^2$$

which immediately yields (A_1). Assumption (A_2) becomes

$$(3.5) \qquad \int_{S_2} \left(1\!\!1_{\{|(z-z')\cdot n|<1\}} + 1\!\!1_{\{|(z-z')\cdot n|\geq 1\}} |(z-z')\cdot n|\right) \, q(z, z', n) \, dn \leq \lambda \left(1 + |z|^2 |z'|^2\right),$$

for some $\lambda > 0$ and for any $z, z' \in I\!R^d$. Because of this, to get (A_3) it is enough that

$$(3.6) \qquad (z, z') \mapsto \int_{S_2} |(z - z') \cdot n| \, q(z, z', n) \, dn \text{ is continuous on } (I\!R^d)^2.$$

The conditions (3.5) and (3.6) are satisfied in the case of the hard spheres and of the Maxwellian molecules with cutoff, while (A_4) is a direct consequence of the conservation of kinetic energy: (1.3).

When $\mathcal{L}$ comes from (1.4), φ is given by (3.4) and the chaotic initial condition (3.3) is physically realistic since it includes the case of N independent particles at equilibrium. Indeed, in many cases the equilibrium distribution (as $t \to \infty$) is the Maxwell distribution which is a normal law. It is true for the hard spheres (see [Ark]) and for the Maxwellian molecules (see [Ta1]).

Assumption (A_2) implies

$$\int_E (|\Delta| + |\Delta'|) \, \mathcal{L}(z, z', d\Delta d\Delta') < +\infty, \ \forall \, z, z' \in \mathcal{Z}$$

which turns (1.7) into a meaningful definition of A_N. More generally, (A_2) also implies the local boundedness property

$$\sup \left\{ \int_E (|\Delta| + |\Delta'|) \, \mathcal{L}(z, z', d\Delta d\Delta') \, ; \, z, z' \in \mathcal{Z}, \ |z| + |z'| \leq k \right\} < +\infty, \ \forall k \geq 1.$$

Some notations. Let us denote

$$\begin{aligned} Df(t, z, \Delta) &= f(t, z + \Delta) - f(t, z) \\ Df(t, z, z', \Delta, \Delta') &= f(t, z + \Delta) - f(t, z) + f(t, z' + \Delta') - f(t, z'), \\ \Delta x_t &= x_t - x_{t^-}, \ 0 \leq t \leq T, x \in D([0, T], I\!R^d). \end{aligned}$$

The spaces $M_\varphi, \mathcal{M}_\varphi, D_{M_\varphi}, C_\varphi, \mathcal{C}_\varphi$ and $\mathcal{C}_\varphi^\sharp$ are defined as in section 2 with $\mathcal{Z} = I\!R^d$.

For any $N \geq 2$, the process X_N is built on the probability space $\left(\Omega_N, \mathcal{F}^N\right)$, where Ω_N stands for $D\left([0, T], (I\!R^d)^N\right)$ which is endowed with its canonical filtration $\left(\mathcal{F}_t^N\right)_{0 \leq t \leq T}$ and $\mathcal{F}^N = \mathcal{F}_T^N$. Its law Q_N on Ω_N is a solution to the martingale problem associated with the generator A_N on the domain $C_b^1((I\!R^d)^N)$ (see lemma 3.2 below). Although the only material needed for our purpose is $\{Q_N \, ; \, N \geq 2\}$, for the sake of clearer notations we introduce a strong representation $(\Omega, \mathcal{A}, I\!P)$ such that $I\!P \circ (X^N)^{-1} = Q_N$ for all $N \geq 2$.

(3.7) Remark. As a definition, the paths of $D\left([0, T], (I\!R^d)^N\right)$ are assumed to be left continuous at T. Since the Lévy kernel we consider are absolutely continuous with respect to the Lebesgue measure dt on $[0, T]$, this corresponds to a negligeable modification of the processes. The advantage is that both canonical projections at times $t = 0$ and $t = T$ are continuous.

Statement of the upper bound. The main result of this section is the theorem 3.1 below.

Let us mention that under a deterministic initial condition (3.2), the rate function for the initial condition is given by

$$I_o(\mu(0)) = \begin{cases} 0 & \text{if } \mu(0) = \rho_o \\ +\infty & \text{if } \mu(0) \neq \rho_o \end{cases}$$

and under a chaotic initial condition (3.3), we have

$$I_o(\mu(0)) = \begin{cases} \displaystyle\int_{I\!R^d} \log\left(\frac{d\mu(0)}{d\rho_o}(z)\right)\mu(0)(dz) & \text{if } \mu(0) \ll \rho_o \\ +\infty & \text{otherwise,} \end{cases}$$

i.e. $I_o(\mu(0))$ is the Kullback information of $\mu(0)$ with respect to ρ_o.

For all $\mu \in D_{M_\varphi}$ and all $f \in C_o^{1,1}([0,T] \times I\!R^d)$, let us denote

$$\int_0^T \langle f(t,\cdot), \dot{\mu}_t - A(\mu_t)^*\mu_t\rangle\, dt = \langle f(T,\cdot), \mu_T\rangle - \langle f(0,\cdot), \mu_0\rangle - \int_0^T \langle (\partial_t + A(\mu_t))f(t,\cdot), \mu_t\rangle\, dt.$$

We shall need the log-Laplace transform of the centered Poisson law with parameter 1, which is given by

$$\tau(u) = e^u - u - 1,\ \ u \in I\!R. \tag{3.8}$$

Theorem 3.1. (Upper Bound). *Let us suppose that the assumptions (A) are satisfied, then for any closed subset C of D_{M_φ}, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log I\!P(\overline{X}^N \in C) \leq -\inf_{\mu \in C} I(\mu)$$

where

$$I(\mu) = I_o(\mu(0)) + J(\mu)$$

and

$$J(\mu) = \sup_{f \in C_o^{1,1}([0,T]\times I\!R^d)} \left\{ \int_0^T \langle f(t,\cdot), \dot{\mu}_t - A(\mu_t)^*\mu_t\rangle\, dt \right. \\ \left. - \int_0^T dt \int_{(I\!R^d)^2\times E} \tau\left(Df(t,z,z',\Delta,\Delta')\right) \frac{1}{2}\mathcal{L}(z,z',d\Delta d\Delta')\, \mu_t^{\otimes 2}(dzdz') \right\}. \tag{3.9}$$

The remaining part of this section is devoted to the proof of this result, which is a consequence of the results of section 2.

The proof of theorem 3.1. Let us begin this proof by checking that the N-particle system under consideration exists.

Lemma 3.2. *Under the assumptions (A), for each $N \geq 2$, there exists a unique solution Q_N to the martingale problem associated with the generator A_N on the domain $C_b^1((I\!R^d)^N)$, (see (1.7)).*

Proof. Let us consider for each $n \geq 1$, the stopping times $T_n = \inf\left\{t \geq 0,\ \Phi\left(X_t^N\right) \geq n\right\}$ where $\Phi\left(z^N\right) = \sum_{i=1}^N \varphi(z_i)$. Because of (3.1), up to time T_n the process stands in a bounded set C_n such that $\bigcup_{n\geq 1} C_n = I\!R^{dN}$ so that $T_\infty := \sup_{n\geq 1} T_n$ is an explosion (stopping) time. Because of (A_2), for all $n \geq 1$: $\sup\limits_{z,z'\in C_n} \mathcal{L}(z,z',E) < \infty$ and one can build a stochastic integral with respect to a stationnary Poisson point process (the same one for each $n \geq 1$), which is a solution to the martingale problem associated with $\mathcal{L}$ and stopped at T_n (see [EKL] or the proof of ([IkW], theorem II.7.4)). Moreover, such a solution Q^n is unique (see [Sz1], Appendice, lemme 1), so that there exists a unique probability measure Q on $\mathcal{F}_{T_\infty^-}$ associated with A_N. By Itô's formula, with λ as in (A_1) (we drop N)

$$E^Q\left[e^{-\lambda(t\wedge T_n)}\Phi\left(X_{t\wedge T_n}\right)\right] = E^Q\Phi\left(X_0\right) + E^Q\int_0^{t\wedge T_n} e^{-\lambda s}\left(A_N\Phi\left(X_s\right) - \lambda\Phi\left(X_s\right)\right)ds$$
$$\leq E^Q\Phi\left(X_0\right), \quad \left(x \leq e^x - 1 \text{ and } (A_1)\right).$$

Therefore, $\forall\ 0 \leq t \leq T,\ n \geq 1,\ E^Q\Phi\left(X_{t\wedge T_n}\right) \leq e^{\lambda T}E^Q\Phi\left(X_0\right) < +\infty$ (by (A_0)). Letting n tend to infinity, by Fatou's lemma, we get $\sup_{0\leq t\leq T}\ E^Q\Phi\left(X_{t\wedge T_\infty}\right) < +\infty$, which implies that $Q\left(T_\infty \leq T\right) = 0$, thanks to (3.1). It remains to take for Q_N, the restriction of Q to $\mathcal{F}_T$. ■

Let us give now an elementary result of stochastic calculus which will be the starting point for the computation of the action functional.

Lemma 3.3. *Suppose that there exists a probability measure P on $D\left([0,T], I\!R^k\right)$, which is a solution to the martingale problem associated with the generator $(A_t\ ;\ 0 \leq t \leq T)$ on the domain $C_o^{1,2}\left([0,T]\times I\!R^k\right)$ given by*

$$A_t f(z) = b(t,z)\cdot\nabla f(z) + \int_{I\!R^k_*}\left\{f(z+\Delta) - f(z) - \Delta\cdot\nabla f(z)\right\}\mathcal{L}(t,z\ ;\ d\Delta)$$

where $b\ :\ [0,T]\times I\!R^k \to I\!R^k$ and $\mathcal{L}\ :\ [0,T]\times I\!R^k \to M_+\left(I\!R^k_*\right)$ are measurable and $\int_{I\!R^k_*}\left(|\Delta|^2\wedge|\Delta|\right)\mathcal{L}(t,z\ ;\ d\Delta) < +\infty$ for all (t,z). For any g in $C_o^{1,2}\left([0,T]\times I\!R^k\right)$, $0 \leq t \leq T$ and x in $D\left([0,T], I\!R^k\right)$, let us denote

$$Z_t^g(x) = \exp\left[g\left(t,x_t\right) - g\left(0,x_0\right) - \int_0^t(\partial_s + A_s)g\left(s,x_s\right)\,ds\right.$$
$$\left. - \int_0^t\left(\int_{I\!R^k_*}\tau\Big(g\left(s,x_s+\Delta\right) - g\left(s,x_s\right)\Big)\ \mathcal{L}\left(s,x_s,d\Delta\right)\right)ds\right]$$

where τ is given at (3.8).
Then, Z^g is a local nonnegative P-martingale, hence a P-supermartingale.

Proof. Let $(X_t)_{0\leq t\leq T}$ be the canonical process. To make the proof simpler, we shall view it as a formal solution of the stochastic differential equation
$dX_t = b(t,X_{t^-})\,dt + \Delta X_t - \left(\int_E \Delta\mathcal{L}(t,X_{t^-};d\Delta)\right)dt$ where $\Delta X_t - \left(\int_E \Delta\mathcal{L}(t,X_{t^-};d\Delta)\right)dt$ is the formal increment of a purely discontinuous local martingale with Lévy kernel: $\mathcal{L}(x;d\Delta)$, and set of jumps: E. For any regular function f, we have Itô's formula

$$df(t,X_t) = \partial_t f(t,X_{t^-})\,dt + \nabla_x f(t,X_{t^-})\cdot dX_t$$
$$+ f(t,X_{t^-}+\Delta X_t) - f(t,X_{t^-}) - \nabla_x f(t,X_{t^-})\cdot\Delta X_t.$$

Let us denote $Df(t,x,\Delta) = f(t,x+\Delta) - f(t,x)$ and $\frac{1}{2}D^2 f(t,x,\Delta) = f(t,x+\Delta) - f(t,x) - \nabla_x f(t,x)\cdot\Delta$. In the following, M and N will stand for any local martingales. As we have

$dX_t = b(t, X_{t^-})\,dt + dM_t$ and $\frac{1}{2}D^2f(t, X_{t^-}, \Delta X_t) = \left(\frac{1}{2}\int_E D^2f(t, X_{t^-}, \Delta)\,\mathcal{L}(t, X_{t^-}; d\Delta)\right)\,dt + dN_t^f$, thanks to Itô's formula:

$$
\begin{aligned}
df(t, X_t) = \partial_t f(t, X_{t^-})\,dt + \nabla_x f(t, X_{t^-}) \cdot (b(t, X_{t^-})\,dt + dM_t)& \\
+ \frac{1}{2}\left(\int_E D^2f(t, X_{t^-}, \Delta)\,\mathcal{L}(t, X_{t^-}; d\Delta)\right)\,dt + dN_t^f&.
\end{aligned}
$$

That is

$$
f(t, X_t) = f(0, X_0) + \int_0^t (\partial_s + A_s)f(s, X_s)\,ds + M_t^f \tag{3.10}
$$

where $(M_t^f)_{t\geq 0}$ is the local martingale formally defined by

$$
dM_t^f = Df(t, X_{t^-}, \Delta X_t) - [\int_E Df(t, X_{t^-}, \Delta)\,\mathcal{L}(t, X_{t^-}; d\Delta)]\,dt. \tag{3.11}
$$

Let $(Z_t)_{t\geq 0}$ be a process of the form $Z_t = \exp\left(f(t, X_t) - f(0, X_0) - \int_0^t U_s\,ds\right)$ where $(U_t)_{t\geq 0}$ is a previsible process. We want to specify $(U_t)_{t\geq 0}$ so that Z is a local martingale.
Let $Y_t = f(t, X_{t^-}) - f(0, X_0) - \int_0^t U_s\,ds$, so that $Z_t = e^{Y_t}$. Applying Itô's formula, we get

$$
dZ_t = Z_{t^-}\left(dY_t + \tau(\Delta Y_t)\right)
$$

where the function τ is given at (3.8). Thanks to (3.10), we have $dY_t = \Big((\partial_t + A_t)f(t, X_{t^-}) - U_t\Big)\,dt + dM_t^f$. We also get $\tau(\Delta Y_t) = \left(\int_E \tau(Df(t, X_{t^-}, \Delta))\,\mathcal{L}(t, X_{t^-}; d\Delta)\right)\,dt + dN_t^\tau$, since $\Delta Y_t = Df(t, X_{t^-}, \Delta X_t)$. Therefore,

$$
dZ_t = Z_{t^-}\left((\partial_t + A_t)f\,(t, X_{t^-}) + \int_E \tau(Df(t, X_{t^-}, \Delta))\,\mathcal{L}(t, X_{t^-}; d\Delta) - U_t\right)dt + Z_{t^-}(dM_t^f + dN_t^\tau)
$$

and $(Z_t)_{t\geq 0}$ is a local martingale if and only if U_t is such that the term multiplying $Z_{t^-}dt$ vanishes. Eventually, taking (3.10) and (3.11) into account

$$
\begin{aligned}
Z_t &= \exp\Bigg[f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + A_s)f\,(s, X_s)\,ds \\
&\qquad - \int_0^t \left(\int_E \tau\big(Df(s, X_s, \Delta)\big)\,\mathcal{L}(s, X_s; d\Delta)\right)\,ds\Bigg] \\
&= \exp\Bigg[\sum_{0\leq s\leq t} Df(s, X_{s^-}, \Delta X_s) - \int_0^t \left(\int_E Df(s, X_s, \Delta)\,\mathcal{L}(s, X_s; d\Delta)\right)\,ds \\
&\qquad - \int_0^t \left(\int_E \tau\big(Df(s, X_s, \Delta)\big)\,\mathcal{L}(s, X_s; d\Delta)\right)\,ds\Bigg] \\
&= \sum_{0\leq s\leq t} Z_{s^-}\left(e^{Df(s, X_{s^-}, \Delta X_s)} - 1\right) - \int_0^t \left(\int_E Z_s\left(e^{Df(s, X_s, \Delta)} - 1\right)\,\mathcal{L}(s, X_s; d\Delta)\right)\,ds
\end{aligned}
$$

is a local martingale. As a consequence of Fatou's lemma, any nonnegative local martingale is a supermartingale. ■

Let $\left(z_1^N, \ldots, z_N^N\right)$ be a N-tuple in $(I\!R^d)^N$. We have already introduced $\overline{z}^N = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{z_i^N} \in M_1(I\!R^d)$: its empirical measure. Let us also consider

$$\tilde{z}^N = \frac{1}{N(N-1)} \sum_{1\leq i\neq j\leq N} \delta_{(z_i^N, z_j^N)} = \frac{N}{(N-1)}\left((\overline{z}^N)^{\otimes 2} - \frac{1}{N^2}\sum_{i=1}^N \delta_{(z_i^N, z_i^N)}\right) \in M_1((I\!R^d)^2).$$

For any real function f on $I\!R^d$, we set $f \oplus f(z, z') = f(z) + f(z')$. It is immediate to see that

$$\frac{1}{(N-1)} \sum_{1\leq i<j\leq N} f \oplus f(z_i, z_j) = N\langle \frac{1}{2} f \oplus f, \tilde{z}^N\rangle = N\langle f, \overline{z}^N\rangle = \sum_{i=1}^N f(z_i). \tag{3.12}$$

Lemma 3.4. *Let us suppose that the assumptions (A) are satisfied, then there exists $\gamma > 0$ such that*

$$\limsup_{N\to\infty} \frac{1}{N} \log I\!\!E \exp\left(\gamma \sum_{i=1}^N \int_0^T \varphi(X_i^N(t))\, dt\right) < +\infty.$$

Proof. By Jensen's inequality, we get

$$\begin{aligned} I\!\!E \exp\left(\gamma \sum_{i=1}^N \int_0^T \varphi(X_i^N(t))\, dt\right) &= I\!\!E \exp\left(T\gamma \frac{1}{T}\int_0^T \sum_{i=1}^N \varphi(X_i^N(t))\, dt\right) \\ &\leq \frac{1}{T}\int_0^T I\!\!E \exp\left(T\gamma \sum_{i=1}^N \varphi(X_i^N(t))\right) dt \\ &\leq \sup_{0\leq t\leq T} I\!\!E \exp\left(T\gamma \sum_{i=1}^N \varphi(X_i^N(t))\right). \end{aligned} \tag{3.13}$$

Let α and λ be chosen as in (A_0) and (A_1). By lemma 3.3

$$\begin{aligned} Z_t = \exp\Bigg[\alpha e^{-\lambda t}\sum_{i=1}^N \varphi(X_i^N(t)) + \int_0^t e^{-\lambda s}\Big(\alpha\lambda \sum_{i=1}^N \varphi(X_i^N(s)) \\ - \frac{1}{N-1}\sum_{1\leq i<j\leq N}\int_E e^{\lambda s}\big\{\exp[\alpha e^{-\lambda s} D\varphi(X_i^N(s), X_j^N(s), \Delta, \Delta')] - 1\big\} \\ \mathcal{L}(X_i^N(s), X_j^N(s); d\Delta d\Delta')\Big)\, ds\Bigg] \end{aligned}$$

is a Q_N-supermartingale. And using (3.12), one obtains

$$\begin{aligned} Z_t = \exp\Bigg[N\Big\langle \alpha e^{-\lambda t}\varphi\oplus\varphi(x_t, x_t') + \int_0^t e^{-\lambda s}\Big(\alpha\lambda\varphi\oplus\varphi(x_s, x_s') \\ - \int_E e^{\lambda s}\{\exp[\alpha e^{-\lambda s} D\varphi(x_s, x_s', \Delta, \Delta')] - 1\} \\ \mathcal{L}(x_s, x_s', d\Delta d\Delta')\Big)\, ds, \frac{1}{2}\widetilde{X}^N(dx dx')\Big\rangle\Bigg]. \end{aligned}$$

But, $a(e^{\alpha D/a}-1)\le \alpha(e^D-1), \forall D\in I\!R, a\ge\alpha$. Therefore, choosing $\alpha\le 1$ in (A_0) (which is not a restriction), with $a=e^{\lambda s}$, we obtain: $e^{\lambda s}\left(\exp(\alpha e^{-\lambda s}D)-1\right)\le \alpha(e^D-1), \forall s\ge 0$ and

$$\begin{aligned}
&\alpha\lambda\varphi\oplus\varphi(x_s,x'_s)-\int_E e^{\lambda s}\{\exp(\alpha e^{-\lambda s}D\varphi(x_s,x'_s,\Delta,\Delta'))-1\}\,\mathcal{L}(x_s,x'_s,d\Delta d\Delta')\\
&\ge \alpha\left(\lambda\varphi\oplus\varphi(x_s,x'_s)-\int_E\{\exp(D\varphi(x_s,x'_s,\Delta,\Delta'))-1\}\,\mathcal{L}(x_s,x'_s,d\Delta d\Delta')\right)\\
&\ge 0.
\end{aligned}$$

The last inequality comes from (A_1). Hence,

$$Z_t\ge \exp\left(\alpha e^{-\lambda t}\sum_{i=1}^N\varphi(X_i^N(t))\right), \forall 0\le t\le T$$

and for all $0\le t\le T$

$$\begin{aligned}
I\!E\exp\left(\alpha e^{-\lambda T}\sum_{i=1}^N\varphi(X_i^N(t))\right)&\le I\!E\exp\left(\alpha e^{-\lambda t}\sum_{i=1}^N\varphi(X_i^N(t))\right)\\
&\le I\!E Z_t\le I\!E Z_0=I\!E\exp\left(\alpha\sum_{i=1}^N\varphi(X_i^N(0))\right).
\end{aligned}$$

Considering (3.13) and (A_0), we conclude with $\gamma=(\alpha\wedge 1)e^{-\lambda T}/T$. ■

The proof of theorem 3.1. Instead of $\overline{X}^N$, let us consider the process

$$t\in[0,T]\mapsto \widetilde{X}^N(t)=\frac{1}{N(N-1)}\sum_{1\le i\ne j\le N}\delta_{(X_i^N(t),X_j^N(t))}\in M_1((I\!R^d)^2).$$

Considering $\widetilde{X}^N$ instead of $\overline{X}^N$, $\mathcal{Z}^2$ instead of $\mathcal{Z}$ and $\varphi\oplus\varphi$ instead of φ, we define the spaces $\mathcal{M}_{\varphi\oplus\varphi}$, $D_{M_{\varphi\oplus\varphi}}$, $\mathcal{C}_{\varphi\oplus\varphi}$ and $\mathcal{C}^\sharp_{\varphi\oplus\varphi}$ similarly to $\mathcal{M}_\varphi$, D_{M_φ}, $\mathcal{C}_\varphi$ and $\mathcal{C}^\sharp_\varphi$ at section 2. It is immediate to see that the results of section 2 still hold for $\widetilde{X}^N$ with the corresponding spaces. In particular, by theorem 2.1 and proposition 2.3, if $\varphi\oplus\varphi$ has compact level sets in $\mathcal{Z}^2$ and if one can find $\gamma>0$ such that

$$\limsup_{N\to\infty}\frac{1}{N}\log I\!E\exp\left(\gamma N\int_0^T\langle\varphi\oplus\varphi,\widetilde{X}^N(t)\rangle\,dt\right)<+\infty,$$

then for all closed subset $\tilde{C}$ of $\mathcal{M}_{\varphi\oplus\varphi}$ (with respect to the topology $\sigma(\mathcal{M}_{\varphi\oplus\varphi},\mathcal{C}_{\varphi\oplus\varphi})$), we have

$$\text{(3.14)}\qquad \limsup_{N\to\infty}\frac{1}{N}\log I\!P(\widetilde{X}^N\in\tilde{C})\le-\inf_{\xi\in\tilde{C}}\sup\left\{\tilde{F}(\xi)-\tilde{H}(\tilde{F})\,;\,\tilde{F}\in C_b(\mathcal{C}^\sharp_{\varphi\oplus\varphi})\right\}$$

where $\tilde{H}(\tilde{F})=\limsup_{N\to\infty}\frac{1}{N}\log I\!E e^{N\tilde{F}(\widetilde{X}^N)}$.

As a consequence of lemma 3.4 and (3.12), (3.14) is satisfied under our assumptions.

For any $f \in C_o^{1,1}([0,T] \times I\!R^d)$, $g \in C_b(I\!R^d)$ and $\xi \in D_{M_{\varphi\oplus\varphi}}$, let us define

$$\begin{aligned}
\tilde{F}_f(\xi) &= \langle f \oplus f(T,\cdot), \xi_T\rangle - \langle f \oplus f(0,\cdot), \xi_0\rangle \\
&\quad - \int_{[0,T]\times(I\!R^d)^2} (f_t'(t,z) + f_t'(t,z'))\, \frac{1}{2}\xi(t; dzdz')dt \\
&\quad - \int_{[0,T]\times(I\!R^d)^2\times E} \big(\exp[Df(t,z,z',\Delta,\Delta')] - 1\big)\, \mathcal{L}(z,z',d\Delta d\Delta')\frac{1}{2}\xi(t;dzdz')dt \quad \text{and} \\
\tilde{F}_{f,g}(\xi) &= \langle \tfrac{1}{2}g \oplus g, \xi_0\rangle + \tilde{F}_f(\xi)
\end{aligned}$$

By (A_2), (A_3) and (A_4), $\tilde{F}_f$ is continuous and bounded on $D_{M_{\varphi\oplus\varphi}}$ (using remark (3.7)). Let us set

$$\begin{aligned}
Z_t^{N,f} = \exp\Bigg[N\Bigg(&\int_{I\!R^d} f(t,z)\,\overline{X}_t^N(dz) - \int_{I\!R^d} f(0,z)\,\overline{X}_0^N(dz) - \int_{[0,t]\times I\!R^d} \partial_s f(s,z)\,\overline{X}_s^N(dz)ds \\
&- \int_{[0,t]\times(I\!R^d)^2\times E} \Big(\exp[Df(t,z,z',\Delta,\Delta')] - 1\Big)\, \frac{1}{2}\mathcal{L}(z,z',d\Delta d\Delta')\widetilde{X}_s^N(dzdz')ds\Bigg)\Bigg].
\end{aligned}$$

Then, by lemma 3.3 and (3.12), $\exp(N\langle g, \overline{X}^N(0)\rangle)Z_t^{N,f}$ is a supermartingale, so that:

$$I\!E \exp(N\langle g, \overline{X}^N(0)\rangle)Z_T^{N,f} = I\!E \exp(N\tilde{F}_{f,g}(\widetilde{X}^N)) \leq I\!E \exp\left(\sum_{i=1}^N g(X_i^N(0))\right)$$

and $\tilde{H}(\tilde{F}_{f,g}) \leq H_o(g)$. This and (3.14) lead us to

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \log I\!P(\widetilde{X}^N \in \tilde{C}) &\leq - \inf_{\xi\in\tilde{C}} \sup\Big\{\tilde{F}_{f,g}(\xi) - H_o(g)\,;\, f \in C_o^{1,1}([0,T]\times I\!R^d), g \in C_b(I\!R^d)\Big\} \\
&= - \inf_{\xi\in\tilde{C}} \Bigg(\sup\Big\{\langle \frac{1}{2} g \oplus g, \xi_0\rangle - H_o(g)\,;\, g \in C_b(I\!R^d)\Big\} \\
&\qquad + \sup\Big\{\tilde{F}_f(\xi)\,;\, f \in C_o^{1,1}([0,T]\times I\!R^d)\Big\}\Bigg)
\end{aligned}$$

for all closed subset $\tilde{C}$ of $\mathcal{M}_{\varphi\oplus\varphi}$.
We get the upper bound for $\overline{X}^N$ in $\mathcal{M}_\varphi$, choosing $\tilde{C} = \{t \mapsto \mu_t \otimes \mu_t\,;\, \mu \in C\}$ with C a closed subset of $\mathcal{M}_\varphi$ and noticing that

$$\begin{aligned}
\tilde{F}_f(\mu^{\otimes 2}) &= \int_{[0,T]} \langle f(t,\cdot), \dot{\mu}_t - A(\mu_t)^*\mu_t\rangle\, dt \\
&\quad - \int_{[0,T]\times(I\!R^d)^2\times E} \tau\big(Df(t,z,z',\Delta,\Delta')\big)\, \frac{1}{2}\mathcal{L}(z,z',d\Delta d\Delta')\mu^{\otimes 2}(t;dzdz')dt.
\end{aligned}$$

Finally, it will be proved in proposition 7.2 that μ belongs to D_{M_φ} provided that $I(\mu) < +\infty$. We conclude by means of corollary 2.2 $\blacksquare$

4. How to study the rate function

The aim of the rest of the paper is to give a non-variational expression for the rate function $J(\mu)$ which was given in a variational form in theorem 3.1. This result will be stated in theorem 7.1. Such a result is well-known in the "Gaussian situation" arising from similar problems where the particles perform continuous diffusions (see [DaG] or [KiO], for instance). In contrast with the "Poissonian situation" which is studied here, in the Gaussian situation the exponential function τ is replaced by a quadratic one. When one wants to carry the proofs from the Gaussian situation to the Poissonian one, it is useful to introduce Orlicz spaces related to the function τ which will play the rôle of the L^2-spaces of the Gaussian situation. Nevertheless, the "Gaussian proof" heavily relies on the Riesz representation of the dual of a Hilbert space, and the proof breaks in the Poissonian situation where no natural Hilbert spaces arise. In the present section, we introduce the Orlicz spaces of interest, we begin the translation of the Gaussian proof into the Poissonian situation, we show where and why it breaks and we give a sketch of an alternate proof which will be developped in the remainding sections: 5, 6 and 7.

About Orlicz spaces. In this subsection, some basic notions about Orlicz spaces are recalled. For more details, see [KRu] or the appendix of [Ne2], for instance.

Let X be a Polish space endowed with its Borel σ-field and a nonnegative bounded measure ρ. The space $L^p(X,\rho)$, when $1 < p < +\infty$ is the Orlicz space associated with the Orlicz function: $\theta_p(u) = u^p/p, u \geq 0$. As a definition, an Orlicz function is a convex function $\theta : [0,\infty[\to [0,\infty[$ such that $\theta(u) = 0 \iff u = 0$ and $\lim_{u\to\infty} \dfrac{\theta(u)}{u} = +\infty$. One proves that

$$\|h\|_\theta = \inf\left\{c > 0\,;\, \int_X \theta\left(\frac{|h|}{c}\right)\, d\rho \leq 1\right\}$$

defines a norm on the space $B(X)$ of all Borel measurable functions, where two ρ-almost everywhere equal functions are identified. The Orlicz space $L^\theta(X,\rho)$ is defined by

$$L^\theta(X,\rho) = \{h \in B(X)\,;\, \|h\|_\theta < +\infty\}.$$

Endowed with the norm $\|\cdot\|_\theta$, it is a Banach space. This space coincides with the set $\{h \in B(X)\,;\, \int_X \theta(|h|)\, d\rho < \infty\}$ if and only if the function θ satisfies the moderate growth condition:

$$\exists c > 0, u_o \geq 0, \text{such that } \theta(2u) \leq c\theta(u), \forall u \geq u_o, \tag{4.1}$$

as it is the case for $\theta_p(u) = u^p/p, 1 < p < \infty$.

Let us denote $E^\theta(X,\rho)$ the closure of the space of all bounded Borel functions in $L^\theta(X,\rho)$. The spaces $L^\theta(X,\rho)$ and $E^\theta(X,\rho)$ match if and only if θ satisfies (4.1).

Riesz theorem for Orlicz spaces is the following: one identifies the topological dual space of $E^\theta(X,\rho)$ with the Orlicz space $L^{\theta^*}(X,\rho)$ for the duality bracket:

$$\langle f, h\rangle = \int_X fh\, d\rho,\ \ f \in L^{\theta^*}(X,\rho), h \in E^\theta(X,\rho)$$

where θ^* is the Legendre transform of θ : $\theta^*(v) = \sup_{u\geq 0}\{uv - \theta(u)\},\ v \geq 0$, ($\theta^*$ is also an Orlicz function). The starting point for this identification is the Hölder inequality

$$\int_X |fh|\, d\rho \leq 2\|f\|_{\theta^*}\|h\|_\theta,\ \ f \in L^{\theta^*}(X,\rho), h \in L^\theta(X,\rho) \tag{4.2}$$

which one proves thanks to: $uv \le \theta(u) + \theta^*(v), \forall u, v \ge 0$. Notice that $\theta_p^* = \theta_q$ where $1/p + 1/q = 1$. The Legendre transform of the function τ given at (3.8) is

$$\tau^*(v) = \begin{cases} (v+1)\log(v+1) - v & \text{if } v > -1 \\ +1 & \text{if } v = -1 \\ +\infty & \text{if } v < -1. \end{cases} \tag{4.3}$$

The restrictions to $[0, \infty[$ of τ and τ^* are Orlicz functions; one denotes $L^\tau, E^\tau, L^{\tau^*}$ and E^{τ^*} the corresponding Orlicz spaces. Notice that although τ^* satisfies the condition (4.1), τ does not. Therefore, in general

$$E^\tau \underset{\neq}{\subset} L^\tau, \ (E^\tau)' = L^{\tau^*} = E^{\tau^*} \text{ and } (E^\tau)'' = (E^{\tau^*})' = L^\tau; \tag{4.4}$$

so that in most situations, the space E^τ is not reflexive.

A first approach. Let us proceed as in the Gaussian situation (see [DaG] or [KiO]). Pick up μ in D_{M_φ}. For any function $f \in C_o^{1,1}$ let us denote

$$\begin{aligned} l_\mu(f) &= \int_0^T \langle f(t,\cdot), \dot\mu_t - A(\mu_t)^*\mu_t\rangle \, dt \\ &= \langle f(T,\cdot), \mu_T\rangle - \langle f(0,\cdot), \mu_0\rangle - \int_0^T \langle (\partial_t + A(\mu_t)) f(t,\cdot), \mu_t\rangle \, dt, \end{aligned} \tag{4.5}$$

and define the measure on $[0,T] \times (IR^d)^2 \times E$

$$\Lambda_\mu(dt dz dz' d\Delta d\Delta') = \frac{1}{2}\mathcal{L}(z, z', d\Delta d\Delta') \, \mu_t^{\otimes 2}(dz dz') \, dt$$

which is bounded under the assumption (A_2).
Consider the Orlicz space $L^\tau([0,T] \times (IR^d)^2 \times E, \Lambda_\mu)$ endowed with its norm $\|\cdot\|_{\tau,\Lambda_\mu}$ and recall that for any $f \in C_o^{1,1}$, $Df(t, z, z', \Delta, \Delta') = f(t, z+\Delta) - f(t,z) + f(t, z'+\Delta') - f(t, z')$. Since

$$\tau(u) \le \tau(|u|), \ \forall u \in IR, \tag{4.6}$$

because of (3.9) and (4.5), we get:

$$\lambda l_\mu(f) - \int_{[0,T]\times(IR^d)^2\times E} \tau(|\lambda Df|) \, d\Lambda_\mu \le J(\mu), \ \forall \lambda \in IR, f \in C_o^{1,1}.$$

Therefore, choosing $\lambda = 1/\|Df\|_{\tau,\Lambda_\mu}$ and $\lambda = -1/\|Df\|_{\tau,\Lambda_\mu}$, we obtain

$$|l_\mu(f)| \le (1 + J(\mu))\|Df\|_{\tau,\Lambda_\mu}, \ \forall f \in C_o^{1,1}$$

and if μ is such that

$$J(\mu) < \infty,$$

passing to the factor space where f and g are identified whenever $Df = Dg$, we see that l_μ is a linear form on $DC_o^{1,1} = \{Df \, ; \, f \in C_o^{1,1}\}$ which is continuous with respect to the norm $\|\cdot\|_{\tau,\Lambda_\mu}$. Since $DC_o^{1,1}$

is included in $E^\tau([0,T]\times(I\!R^d)^2\times E,\Lambda_\mu)$, thanks to the Hahn-Banach extension theorem and to the Riesz representation for the Orlicz spaces, there exists a (in fact, infinitely many) measurable function K on $[0,T]\times(I\!R^d)^2\times E$ such that $K\in L^{\tau^*}([0,T]\times(I\!R^d)^2\times E,\Lambda_\mu)$ and

$$l_\mu(f)=\int_{[0,T]\times(I\!R^d)^2\times E} KDf\,d\Lambda_\mu,\ \forall f\in C_o^{1,1}. \tag{4.7}$$

Of course, one can also write (4.7):

$$\langle f(T,\cdot),\mu_T\rangle-\langle f(0,\cdot),\mu_0\rangle-\int_0^T\langle\partial_t f(t,\cdot),\mu_t\rangle\,dt=\int_{[0,T]\times(I\!R^d)^2\times E} Df\,d[(K+1)\cdot\Lambda_\mu]. \tag{4.8}$$

Because of the identity

$$uv-\tau(u)=\tau^*(v)-e^u\tau^*\left(\frac{v+1}{e^u}-1\right),\ \forall u\in I\!R, v\geq -1, \tag{4.9}$$

we obtain that if $K\geq -1$, then

$$\begin{aligned}J(\mu)&=\sup_{f\in C_o^{1,1}}\left\{l_\mu(f)-\int_{[0,T]\times(I\!R^d)^2\times E}\tau(Df)\,d\Lambda_\mu\right\}\\&=\int_{[0,T]\times(I\!R^d)^2\times E}\tau^*(K)\,d\Lambda_\mu-\inf_{f\in C_o^{1,1}}\int_{[0,T]\times(I\!R^d)^2\times E}\tau^*\left(\frac{K+1}{e^{Df}}-1\right)e^{Df}\,d\Lambda_\mu\\&=I\Big((K+1)\cdot\Lambda_\mu\mid\Lambda_\mu\Big)-\inf_{f\in C_o^{1,1}}I\Big((K+1)\cdot\Lambda_\mu\mid e^{Df}\cdot\Lambda_\mu\Big)\end{aligned} \tag{4.10}$$

where $I(\alpha\mid\beta)=\int\tau^*\left(\frac{d\alpha}{d\beta}-1\right)d\beta$ is the "Kullback information" of the nonnegative measure α with respect to the nonnegative measure β.

Hence, one has to solve the following problem: show that there exists a unique $K=K_\mu$ satisfying (4.8) such that

$$K_\mu\geq -1\quad\text{and} \tag{4.11.a}$$

$$\inf_{f\in C_o^{1,1}}I\Big((K_\mu+1)\cdot\Lambda_\mu\mid e^{Df}\cdot\Lambda_\mu\Big)=0. \tag{4.11.b}$$

Then for this priviliged K_μ, we would obtain that

$$J(\mu)=\int_{[0,T]\times(I\!R^d)^2\times E}\tau^*(K_\mu)\,d\Lambda_\mu. \tag{4.12}$$

In the Gaussian situation, the problem (4.11) does not appear: no analogue of (4.11.a) arises, because of the symmetry of $u\mapsto u^2/2$, and the analogue of (4.11.b) is simply solved by a usual orthogonal projection argument, while (4.11.b) has something to do with Csiszár projection ([Csi]). Notice that the inequality (4.6) prevents from proving (4.11.a) by means of the present method. In order to solve (4.11), we have to change our approach.

A second approach. Let $\mathcal{C}$ be a vector space, $\mathcal{C}^\sharp$ its algebraic dual space and Γ a nonnegative convex function defined on $\mathcal{C}$ such that $\Gamma(0)=0$. Its Legendre transform is defined by

$$\Gamma^* : l \in \mathcal{C}^\sharp \mapsto \sup_{f\in\mathcal{C}} \{\langle l, f\rangle - \Gamma(f)\} \in [0,\infty].$$

With $\mathcal{C} = D\mathcal{C}_o^{1,1}$, $l(Df) = l_\mu(f)$ given by (4.5) and $\Gamma = \Gamma_\mu$ given by

$$\Gamma_\mu(f) = \int_{[0,T]\times(I\!R^d)^2\times E} \tau(f)\, d\Lambda_\mu, \ \ f \in D\mathcal{C}_o^{1,1}, \tag{4.13}$$

one can write (3.9):

$$J(\mu) = \Gamma^*_\mu(l_\mu). \tag{4.14}$$

Our aim is to express $\Gamma^*(l)$ in a non-variational form. For this purpose, we shall use convex analysis. Suppose that Γ is Gâteaux-differentiable on $\mathcal{C}$ and denote $\Gamma'(f) \in \mathcal{C}^\sharp$ its Gâteaux-derivative. We shall recall in proposition 5.1 the classical result relating the derivative of Γ to Γ^*. If we apply this proposition with $(U,G) = (\mathcal{C},\Gamma)$ and $(V,G^*) = (\mathcal{C}^\sharp,\Gamma^*)$, we obtain that for all the l's of the form $l = \Gamma'(f_l)$ for some $f_l \in \mathcal{C}$:

$$\Gamma^*(l) = \langle l, f_l\rangle - \Gamma(f_l). \tag{4.15}$$

This would solve our problem if all the l's were of the form $\Gamma'(f)$. But, the effective domain of Γ^* is strictly larger than $\{\Gamma'(f)\,;\, f \in \mathcal{C}\}$ in the situation we are dealing with. It is the reason why we shall introduce the function $\overline{\Gamma}$ defined on the algebraic bidual $\mathcal{C}^{\sharp\sharp}$ of $\mathcal{C}$ by

$$\overline{\Gamma} : \xi \in \mathcal{C}^{\sharp\sharp} \mapsto \sup_{l\in\mathcal{C}^\sharp} \{\langle \xi, l\rangle - \Gamma^*(l)\} \in [0,\infty].$$

Indeed, since $\overline{\Gamma}$ is the Legendre transform of Γ^*, the analogue of (4.15) is $\overline{\Gamma}(\xi) = \langle \xi, l_\xi\rangle - \Gamma^*(l_\xi)$ where l_ξ is a solution of $\Gamma^{*\prime}(l_\xi) = \xi$.
It will be proved in proposition 5.3 that for all l, internal point of the effective domain of Γ^*, there exists $\xi_l \in \mathcal{C}^{\sharp\sharp}$ such that

$$\Gamma^{*\prime}(l) = \xi_l. \tag{4.16.a}$$

Consequently, for such l's, we obtain

$$\Gamma^*(l) = \langle \xi_l, l\rangle - \overline{\Gamma}(\xi_l), \tag{4.16.b}$$

which is the desired result. The main reason for the existence of ξ_l in $\mathcal{C}^{\sharp\sharp}$, is that ξ_l is a slope of the convex function Γ^* at the point l and the existence of an algebraic slope (without a priori continuity properties) is insured by the geometric form of Hahn-Banach theorem (and hence the axiom of choice).

Choosing $l = l_\mu$ and $\Gamma = \Gamma_\mu$ given by (4.5) and (4.13), (4.14) is $J(\mu) = \Gamma^*(l)$. Then, the identity (4.16.b) is (4.12) where the privileged K_μ is given by $K_\mu = \xi_{l_\mu}$: solution of (4.16.a).

Section 5 will be devoted to the proof of (4.16), while section 6 will be devoted to the computation of $\overline{\Gamma}_\mu$ (see the corollary 6.7). Finally, we shall use these results at section 7, to give a non-variational formulation of $J(\mu)$ (see (4.12)) in theorem 7.1.

5. About Legendre polarity

We begin recalling, in proposition 5.1, standard results about Legendre polarity. Then, the notion of Gâteaux-differentiation in the direction of a vector subspace is introduced. It will permit us, in proposition 5.2, to investigate relations between the slopes of a function and the domain of its Legendre transform, and to state in proposition 5.3, the desired relation (4.16) between Γ^* and $\overline{\Gamma}$. Finally, lemma 5.4 and proposition 5.5 are preliminary general results for the computation of $\overline{\Gamma}_\mu$.

A classical result. Let U and V be two vector spaces in duality with respect to the bracket: $(u, v) \in U \times V \mapsto \langle u, v\rangle \in I\!R$, a function $G : U \mapsto]-\infty, +\infty]$ and its Legendre transform G^* defined by

$$G^* : v \in V \mapsto \sup_{u \in U}\{\langle u, v\rangle - G(u)\} \in]-\infty, +\infty].$$

We suppose that there exists $u_o \in U$ such that $G(u_o) < +\infty$. This implies that $G^*(v) > -\infty, \forall v \in V$. If u belongs to the effective domain of G : *dom* $G = \{u \in U\, ;\, G(u) < +\infty\}$, the subdifferential of G at u is defined by:

$$\partial_V G(u) = \{v \in V\, ;\, G(u) + \langle h, v\rangle \leq G(u + h), \forall h \in U\}.$$

A vector $v \in V$ is called a slope of G if there exists $a \in I\!R$ such that $\langle u, v\rangle - a \leq G(u), \forall u \in U$ and the set of all the slopes of G is denoted by S_G. One defines the Legendre bipolar of G by

$$\overline{G} : u \in U \mapsto \sup_{v \in V}\{\langle u, v\rangle - G^*(v)\} \in]-\infty, +\infty].$$

We call convex $\sigma(U, V)$-lower semicontinuous regularized of the function G the largest convex $\sigma(U, V)$-lower semicontinuous function below G.

Proposition 5.1.

(a) $\displaystyle\bigcup_{u \in dom\ G} \partial_V G(u) \subset S_G = dom\ G^*$

(b) *for any* $u \in dom\ G$ *such that* $\partial_V G(u) \neq \emptyset$ *and any* $v \in V$, *we have:*
$v \in \partial_V G(u) \Longleftrightarrow G(u) + G^*(v) = \langle u, v\rangle$

(c) $v \in \partial_V G(u) \Longrightarrow u \in \partial_U G^*(v)$

(d) *if* G *is convex and* $\sigma(U, V)$*-lower semicontinuous, we have:*
$v \in \partial_V G(u) \Longleftrightarrow u \in \partial_U G^*(v)$

(e) *if* G *is convex and bounded below by an affine* $\sigma(U, V)$*-continuous function, then* $\overline{G}$ *is the convex* $\sigma(U, V)$*-lower semicontinuous regularized of* G.

Proof. (a) is a consequence of the definitions. For the other statements, see ([EkT], Proposition 5.1, Corollaire 5.2 and Proposition 3.3). ■

The slope-domain polarity. To study the relations between the domain of a convex function and the slopes of its Legendre transform, we are going to introduce the notions of continuity and Gâteaux-differentiability in the direction of a vector subspace. The properties stated below are simple modifications of basic properties of convex functions (see [EkT] or [Gil], for instance).

Geometric notions. Let Y be a vector space, C a vector subspace of Y and $\psi : Y \mapsto]-\infty, +\infty]$ a convex function. The effective domain of ψ is $dom\ \psi = \{y \in Y\,;\, \psi(y) < +\infty\}$. The C-interior of $dom\ \psi$ is the set of all $y \in dom\ \psi$ such that: $\forall h \in C, \exists \alpha > 0, [y, y+\alpha h] \subset dom\ \psi$, it is denoted by C-idom ψ (when $C = Y$, it is denoted idom ψ).
The function ψ is said to be Gâteaux-differentiable in the direction C (or C-Gâteaux-differentiable) at a point $y \in C$-idom ψ, if the limit $\lim_{t\to 0} \frac{\psi(y+th)-\psi(y)}{t}$ exists for all $h \in C$. We set

$$\langle \psi'(y), h\rangle = \lim_{t\to 0} \frac{\psi(y+th)-\psi(y)}{t}, y \in Y, h \in C.$$

Since ψ is convex, $\psi'(y)$ belongs to the algebraic dual space $\mathcal{C}^\sharp$ of C. If $C = Y$, the usual notion of Gâteaux-differentiability is recovered.
For any point y of $dom\ \psi$, one defines the $\mathcal{C}^\sharp$-subdifferential of ψ at y by

$$\partial_{\mathcal{C}^\sharp}\psi(y) = \{\xi \in \mathcal{C}^\sharp\,;\, \psi(y) + \langle \xi, h\rangle \le \psi(y+h), \forall h \in C\}.$$

Since ψ is convex, at any point y of C-idom ψ, the set $\partial_{\mathcal{C}^\sharp}\psi(y)$ is non-empty. Moreover, ψ is C-Gâteaux-differentiable at y if and only if $\partial_{\mathcal{C}^\sharp}\psi(y)$ is reduced to a unique point. In such a case: $\partial_{\mathcal{C}^\sharp}\psi(y) = \{\psi'(y)\}$.

Topological notions. Suppose now that Y is a topological vector space and that C is endowed with its relative topology. Denote C' the topological dual space of C. One says that ψ is continuous in the direction C (or C-continuous) at $y \in C$-idom ψ if there exists a neighbourhood V_o of 0 in C such that $y + V_o \subset dom\ \psi$ and if the function $h \in V_o \mapsto \psi(y+h) \in I\!R$ is continuous at 0. As ψ is convex, if it is C-continuous at $y \in C$-idom ψ, then $\partial_{\mathcal{C}^\sharp}\psi(y) \subset C'$.
In the particular case where $C = Y$, we have: ψ is continuous on idom ψ if and only if ψ is continuous at one point of idom ψ (or equivalently: ψ is bounded above on a neighbourhood of this point). Then, idom ψ coincide with the topological interior of $dom\ \psi$. This result does not extend to the case where C is a strict vector subspace of Y.

The slope-domain polarity. After these general considerations, let us explore some of the relations between the domain of a convex function and the slopes of its Legendre transform.

Let C be a topological vector space, C' and C'' its topological dual and bidual spaces. We consider a function $G : C' \mapsto]-\infty, +\infty]$ and its Legendre transform G^* on $C'' : G^*(y) = \sup_{x\in C'}\{\langle x, y\rangle - G(x)\}$, $y \in C''$. We perform the canonical embedding $C \subset C''$.

Proposition 5.2. *Suppose that G is a proper convex $\sigma(C', C'')$-lower semicontinuous function which is bounded below by a $\sigma(C', C'')$-continuous affine function.*

(i) If

(a)	*G is Gâteaux-differentiable,*
(b)	idom $G = dom\ G$ *and*
(c)	*G^* is C-continuous on C-*idom G^*,

then

$$C\text{-idom } G^* \subset \{G'(x)\,;\, x \in dom\ G\} \subset dom\ G^*.$$

(ii) If

(d)	*G^* is C-Gâteaux-differentiable,*
(e)	C-idom $G^* = dom\ G^*$ *and*
(f)	*G is continuous on* idom G,

then

$$\text{idom } G \subset \{G^{*\prime}(y)\,;\, y \in dom\ G^*\} \subset dom\ G.$$

<u>Proof</u>. The duality which we consider is (C', C''). We shall write $\partial G(x)$ for $\partial_{C''} G(x)$ and $\partial G^*(y)$ for $\partial_{C'} G^*(y)$. Since G is convex and $\sigma(C', C'')$-lower semicontinuous, thanks to 5.1.e (part (e) of the proposition 5.1), we get $\overline{G} = G$.

Let us show (i). By 5.1.b, we have

$$\bigcup_{x \in dom\ G} \partial G(x) \subset dom\ G^*.$$

Together with (a) and (b), this yields: $\{G'(x)\,;\, x \in dom\ G\} \subset dom\ G^*$.
Let $y \in C$-idom G^*, as by (c): G^* is C-continuous, there exists $x_y \in C'$ such that $x_y \in \partial G^*(y)$. But, being the upper envelope of affine continuous functions, G^* is convex and $\sigma(C'', C')$-lower semicontinuous, so that 5.1.d leads us to: $y \in \partial\overline{G}(x_y) = \partial G(x_y)$. Therefore:

$$C\text{-idom } G^* \subset \bigcup_{x \in C'} \partial G(x).$$

Thanks to (a) and (b), one concludes that: C-idom $G^* \subset \{G'(x)\,;\, x \in dom\ G\}$.

Let us show (ii). By 5.1.b, we have

$$\bigcup_{y \in dom\ G^*} \partial G^*(y) \subset dom\ \overline{G} = dom\ G.$$

Together with (d) and (e), this yields: $\{G^{*\prime}(y)\,;\, y \in dom\ G^*\} \subset dom\ G$.
Under the assumption (f), for any $x \in$ idom G, there exists $y_x \in C''$ such that $y_x \in \partial G(x)$. Thanks to 5.1.d, since G is convex and $\sigma(C', C'')$-lower semicontinuous, we get: $x \in \partial G^*(y_x)$, therefore:

$$\text{idom } G \subset \bigcup_{y \in dom\ G^*} \partial G^*(y).$$

One concludes with (d) and (e) that: idom $G \subset \{G^{*\prime}\,;\, y \in dom\ G^*\}$. ■

Study of Γ^*. Let $\mathcal{C}$ be a vector space, $\mathcal{C}^\sharp$ and $\mathcal{C}^{\sharp\sharp}$ are its algebraic dual and bidual spaces. We perform the embedding $\mathcal{C} \subset \mathcal{C}^{\sharp\sharp}$. Let $\Gamma : \mathcal{C} \mapsto [0, +\infty]$ be a convex function such that $\Gamma(0) = 0$. Let us define the Legendre transforms:

$$\Gamma^* : l \in \mathcal{C}^\sharp \mapsto \sup_{f \in \mathcal{C}} \{\langle l, f\rangle - \Gamma(f)\} \in [0, +\infty]$$
$$\overline{\Gamma} : \xi \in \mathcal{C}^{\sharp\sharp} \mapsto \sup_{l \in \mathcal{C}^\sharp} \{\langle \xi, l\rangle - \Gamma^*(l)\} \in [0, +\infty].$$

Let us recall that if $\overline{\Gamma}$ is $\mathcal{C}$-Gâteaux-differentiable at $\xi \in \mathcal{C}^{\sharp\sharp}$, then $\overline{\Gamma}'(\xi)$ which is defined for all $f \in \mathcal{C}$, by: $\langle \overline{\Gamma}'(\xi), f\rangle = \lim_{t \to 0} \dfrac{\overline{\Gamma}(\xi + tf) - \overline{\Gamma}(\xi)}{t}$, belongs to $\mathcal{C}^\sharp$.
We are now ready to state the following results.

Proposition 5.3. *For any l in* idom Γ^*, *there exists $\xi_l \in \mathcal{C}^{\sharp\sharp}$ satisfying*

$$\xi_l \in \partial_{\mathcal{C}^{\sharp\sharp}} \Gamma^*(l), \quad l \in \partial_{\mathcal{C}^\sharp} \overline{\Gamma}(\xi_l) \quad \textit{and} \quad \Gamma^*(l) = \langle \xi_l, l\rangle - \overline{\Gamma}(\xi_l).$$

*In particular, if $\mathcal{C}$-*idom $\overline{\Gamma}$ = *dom $\overline{\Gamma}$ and if $\overline{\Gamma}$ is Gâteaux-differentiable in the direction $\mathcal{C}$ on its whole domain, we have*

$$l = \overline{\Gamma}'(\xi_l) \quad \textit{and} \tag{5.1}$$
$$\Gamma^*(l) = \langle \overline{\Gamma}'(\xi_l), \xi_l\rangle - \overline{\Gamma}(\xi_l). \tag{5.2}$$

If in addition $\overline{\Gamma}$ is strictly convex, then for all l in idom Γ^*, *there exists a unique $\xi_l \in \mathcal{C}^{\sharp\sharp}$ satisfying the equality (5.1).*

Proof. Apply the proposition 5.1 to $(U, G) = (\mathcal{C}^\sharp, \Gamma^*)$ and $(V, G^*) = (\mathcal{C}^{\sharp\sharp}, \overline{\Gamma})$. Since Γ^* is convex, thanks to the geometric form of the Hahn-Banah theorem, for all $l \in$ idom Γ^*, there exists at least one linear form $\xi_l \in \mathcal{C}^{\sharp\sharp}$ such that: $\xi_l \in \partial_{\mathcal{C}^{\sharp\sharp}} \Gamma^*(l)$. More, Γ^* is $\sigma(\mathcal{C}^\sharp, \mathcal{C})$-lower semicontinuous and a fortiori $\sigma(\mathcal{C}^\sharp, \mathcal{C}^{\sharp\sharp})$-lower semicontinuous; therefore, one is allowed to apply parts (b) and (d) of proposition 5.1, which give the first assertion of the proposition 5.3.

Then, one gets (5.1) and (5.2), noticing that when $\overline{\Gamma}$ is Gâteaux-differentiable in the direction $\mathcal{C}$ at ξ, $\partial_{\mathcal{C}^\sharp} \overline{\Gamma}(\xi) = \{\overline{\Gamma}'(\xi)\}$.

Now, let us check the last assertion of the proposition. Suppose that there exist distinct ξ and ξ' such that $\xi, \xi' \in \partial_{\mathcal{C}^{\sharp\sharp}} \Gamma^*(l)$. Then, (proposition 5.1.(d)) $l \in \partial_{\mathcal{C}^\sharp} \overline{\Gamma}(\xi)$ and $l \in \partial_{\mathcal{C}^\sharp} \overline{\Gamma}(\xi')$. Since by proposition 5.1.(e), $\overline{\Gamma}$ is the upper envelope of its affine lower bounds, it is affine on the segment $[\xi, \xi']$ (if not, Hahn-Banach theorem would lead to a contradiction). Hence, it is not strictly convex. ■

What we want now, is to compute $\overline{\Gamma}_\mu$. This computation relies on the following lemma.

Lemma 5.4. *Suppose that there exists a norm $\|\cdot\|$ on $\mathcal{C}$ such that:*

$$\textit{there exists } r > 0 \textit{ such that } \sup\{\Gamma(f)\,;\, \|f\| \le r\} \le 1. \tag{5.3.a}$$

Then,

$$\textit{dom}\ \Gamma^* \subset \mathcal{C}',$$

and Γ^* *is* $\|\cdot\|^*$*-continuous on* idom Γ^*, *where* $\mathcal{C}'$ *is the topological dual of* $(\mathcal{C}, \|\cdot\|)$, *endowed with the uniform norm* $\|\cdot\|^*$.

If in addition,

$$\text{(5.3.}b\text{)} \qquad \textit{there exists } t_o > 0, \textit{such that } \inf\{\Gamma(f)\,;\, \|f\| = t_o\} > 0,$$

then

$$\textit{dom}\ \overline{\Gamma} \subset \mathcal{C}'',$$

where $\mathcal{C}''$ *is the topological bidual of* $(\mathcal{C}, \|\cdot\|)$.

Remarks. $*$ In the next section, we shall apply this lemma choosing for $(\mathcal{C}, \|\cdot\|)$ some Orlicz spaces related to the functions τ and τ^* given by (3.8) and (4.3).

$*$ The condition (5.3.a) implies that Γ is $\|\cdot\|$-continuous on the $\|\cdot\|$-interior of *dom* Γ.

Proof. Let $\|\cdot\|$ satisfying (5.3.a) be given. Then, there exists $r > 0$ such that $\sup\{\Gamma(f)\,;\, \|f\| \le r\} \le 1$. Let l be an element of $\mathcal{C}^\sharp$, then for all $f \in \mathcal{C}$ and all $a \neq 0$

$$\langle l, f/a\rangle \le \Gamma(f/a) + \Gamma^*(l)$$

and choosing $a = \|f\|/r$ and $a = -\|f\|/r$ if $f \neq 0$, one obtains

$$|\langle l, f\rangle| \le \frac{1 + \Gamma^*(l)}{r}\|f\|, \ \forall f \in \mathcal{C}$$

so that if $\Gamma^*(l) < \infty$, then l belongs to $\mathcal{C}'$.

Suppose now that (5.3.b) is satisfied. Let l be in *dom* $\Gamma^* \subset \mathcal{C}'$. Denote $\psi(t) = \inf\{\Gamma(f)\,;\, \|f\| = t\}, t \ge 0$ and ψ^* its Legendre transform. Then,

$$\begin{aligned}
\Gamma^*(l) &= \sup_{f \in \mathcal{C}}\{\langle l, f\rangle - \Gamma(f)\} \\
&\le \sup_{f \in \mathcal{C}}\{\|l\|^*\|f\| - \Gamma(f)\} \\
&= \sup_{t \ge 0}\{t\|l\|^* - \psi(t)\} \\
&= \psi^*(\|l\|^*).
\end{aligned}$$

But (5.3.b) implies that $\psi(t_o)/t_o > 0$ and that $[0, \psi(t_o)/t_o]$ is included in the domain of ψ^*. Hence, denoting $\beta = \psi^*(\psi(t_o)/t_o)$ and $U = \{l \in \mathcal{C}'\,;\, \|l\|^* < \psi(t_o)/t_o\}$, we get

$$\text{(5.4)} \qquad \sup_{l \in U} \Gamma^*(l) \le \beta < \infty.$$

A consequence of (5.4) is that the convex function Γ^* is $\|\cdot\|^*$-continuous on idom Γ^*.

Denote $\delta(l \mid U) = \begin{cases} 0 & \text{if } l \in U \\ +\infty & \text{otherwise} \end{cases}$. Taking the inclusion: *dom* $\Gamma^* \subset \mathcal{C}'$ into account, thanks to (5.4), for all $l \in \mathcal{C}^\sharp$, $\Gamma^*(l) \leq \beta + \delta(l \mid U)$. It follows that for all $\xi \in \mathcal{C}^{\sharp\sharp}$,

$$\begin{aligned} \overline{\Gamma}(\xi) &\geq \sup_{l \in \mathcal{C}^\sharp} \{\langle \xi, l\rangle - \delta(l \mid U) - \beta\} \\ &= \sup_{l \in U} \langle \xi, l\rangle - \beta, \end{aligned} \tag{5.5}$$

which completes the proof of the lemma. ■

Let a norm $\|\cdot\|$ satisfying the conditions (5.3) of lemma 5.4, be given. Since, thanks to this lemma: *dom* $\overline{\Gamma} \subset \mathcal{C}''$, the study of $\overline{\Gamma}$ reduces to the study of its restriction to $\mathcal{C}''$. Denote $\widehat{\Gamma}$ the convex $\sigma(\mathcal{C}, \mathcal{C}^\sharp)$-lower semicontinuous regularized of Γ.

Proposition 5.5. *Under the conditions (5.3), $\widehat{\Gamma}$ is the convex $\|\cdot\|$-lower semicontinuous regularized of Γ and the restriction of $\overline{\Gamma}$ to $\mathcal{C}''$ is the unique extension of $\widehat{\Gamma}$ to $\mathcal{C}''$ which is $\sigma(\mathcal{C}'', \mathcal{C}')$-lower semicontinuous.*

Proof. Because of proposition 5.1.e, $\overline{\Gamma}$ is the convex $\sigma(\mathcal{C}^{\sharp\sharp}, \mathcal{C}^\sharp)$-lower semicontinuous regularized of the function $\xi \in \mathcal{C}^{\sharp\sharp} \mapsto \begin{cases} \Gamma(\xi) & \text{if } \xi \in \mathcal{C} \\ +\infty & \text{if } \xi \in \mathcal{C}^{\sharp\sharp} \setminus \mathcal{C} \end{cases}$ and $\widehat{\Gamma}$ is the restriction of $\overline{\Gamma}$ to $\mathcal{C}$. Since: *dom* $\Gamma^* \subset \mathcal{C}'$ (lemma 5.4), it is clear that $\widehat{\Gamma}$ and $\overline{\Gamma}$ are respectively $\sigma(\mathcal{C}, \mathcal{C}')$ and $\sigma(\mathcal{C}'', \mathcal{C}')$-lower semicontinuous. But, a convex function is $\sigma(\mathcal{C}, \mathcal{C}')$-lower semicontinuous if and only if it is $\|\cdot\|$-lower semicontinuous; therefore $\widehat{\Gamma}$ is the convex $\|\cdot\|$-lower semicontinuous regularized of Γ.

Recall that the epigraph of the lower semicontinuous convex regularized $\overline{G}$ of a function G is the closed convex hull of the epigraph of G. Since the epigraph of $\overline{\Gamma}$ is the $\sigma(\mathcal{C}'', \mathcal{C}')$-closure of the epigraph of $\xi \in \mathcal{C}'' \mapsto \begin{cases} \widehat{\Gamma}(\xi) & \text{if } \xi \in \mathcal{C} \\ +\infty & \text{if } \xi \in \mathcal{C}'' \setminus \mathcal{C} \end{cases}$, the uniqueness of the extension follows with Goldstine theorem ([Bré], lemme III.4) (see (6.7)) which states that $\mathcal{C}$ is $\sigma(\mathcal{C}'', \mathcal{C}')$-dense in $\mathcal{C}''$. ■

We are now ready for the computation of the extension $\overline{\Gamma}_\mu$ of the function Γ_μ given at (4.13).

6. Computation of $\overline{\Gamma}_\mu$.

In this section, μ is fixed. If one chooses $\|\cdot\| = \|\cdot\|_{\tau, \Lambda_\mu}$: the norm of $L^\tau([0,T] \times (IR^d)^2 \times E, \Lambda_\mu)$, the condition (5.3.a) is satisfied for Γ_μ, and it follows that (see (4.7))

$$dom\ \Gamma_\mu^* \subset L^{\tau^*}([0,T] \times (IR^d)^2 \times E, \Lambda_\mu), \tag{6.1}$$

but (5.3.b) is not satisfied, on account of the asymmetry of the function τ; once again, we are faced with the problem of the inequality (4.6) which is too coarse too solve (4.11). Hence, it will be convenient to treat the positive and negative contributions separately. For this purpose, it will be comfortable to consider in a first step the extension of Γ_μ to the space C_b of all bounded continuous functions on $[0,T] \times (IR^d)^2 \times E$, which is given by

$$\begin{aligned} \Gamma(f) &= \int_{[0,T]\times(IR^d)^2\times E} \tau(f)\, d\Lambda_\mu, \ f \in \mathcal{C}, \quad \text{with} \\ \mathcal{C} &= C_b. \end{aligned}$$

The function τ is given at (3.8). Denoting $f_+ = f \vee 0$ and $f_- = (-f) \vee 0$: the nonnegative and nonpositive parts of the function f, we get

$$\Gamma(f) = \Gamma(f_+ - f_-) = \Gamma_+(f_+) + \Gamma_-(f_-), f \in C_b \tag{6.2}$$

where for any $f \in C_b$,

$$\begin{aligned} \Gamma_+(f) &= \int_{[0,T]\times(IR^d)^2\times E} \tau(|f|)\, d\Lambda_\mu, \\ \Gamma_-(f) &= \int_{[0,T]\times(IR^d)^2\times E} \gamma(|f|)\, d\Lambda_\mu, \text{ with} \\ \gamma(u) = \tau(-u) &= u + e^{-u} - 1, \; u \geq 0. \end{aligned}$$

The normed space $(\mathcal{C}, \|\cdot\|) = (C_b, \|\cdot\|_{\tau,\Lambda_\mu})$ satisfies the conditions (5.3) for the function Γ_+, and the normed space $(\mathcal{C}, \|\cdot\|) = (C_b, \|\cdot\|_{1,\Lambda_\mu})$, where $\|\cdot\|_{1,\Lambda_\mu}$ stands for the norm of $L^1([0,T] \times (IR^d)^2 \times E, \Lambda_\mu)$, satisfies the conditions (5.3) for the function Γ_-. Let us notice that for this last function, the norm $\|f\|_{\gamma,\Lambda_\mu} = \inf\left\{c > 0\,;\, \int \gamma\left(\frac{f}{c}\right) d\Lambda_\mu \leq 1\right\}$ is equivalent to $\|\cdot\|_{1,\Lambda_\mu}$ since Λ_μ is a bounded measure. The lemma 5.4, (4.4) and the inclusion $(L^1)' \subset L^\infty$, allow us to state that

$$\begin{array}{lclcl} dom\ \Gamma_+^* \subset L^{\tau^*} & \quad\text{and}\quad & dom\ \overline{\Gamma}_+ \subset L^\tau, \\ dom\ \Gamma_-^* \subset L^\infty & \quad\text{and}\quad & dom\ \overline{\Gamma}_- \subset (L^\infty)'. \end{array} \tag{6.3}$$

Because of (6.1), which is still satisfied by the function Γ^*, for all $\xi, \xi' \in C_b^{\sharp\sharp}$, we have:

$$\xi_{|_{L^{\tau^*}}} = \xi'_{|_{L^{\tau^*}}} \Longrightarrow \overline{\Gamma}(\xi) = \overline{\Gamma}(\xi')$$

where $\xi_{|_{L^{\tau^*}}}$ is the restriction of ξ to L^{τ^*}. This allows us to denote $\overline{\Gamma}(\xi) = \overline{\Gamma}(\xi_{|_{L^{\tau^*}}})$ and to introduce the notion of nonnegative element of $C_b^{\sharp\sharp}$:

$$\xi \in C_{b,+}^{\sharp\sharp} \Longleftrightarrow \langle \xi, l \rangle \geq 0, \forall l \in L_+^{\tau^*} = \{l \in L^{\tau^*}\,;\, l \geq 0\}.$$

Now, our aim is to show that the relation (6.2) extends to $C_b^{\sharp\sharp}$. This result is stated in proposition 6.3. Let us begin proving in proposition 6.1 and lemma 6.2, preliminary results for the proof of proposition 6.3.

In proposition 6.1, the expression of Γ^* on $C_b^\sharp$ will be given. It could seem that our work to obtain Γ_μ^* is almost done, but it is not true. Indeed, we wish to compute $\sup_{f \in \mathcal{C}}\{\langle l_\mu, f\rangle - \Gamma_\mu(f)\}$ where $\mathcal{C} = DC_o^{1,1}$ is a vector subspace of C_b which is not dense in L^{τ^*}. Because of this, there exist infinitely many functions K associated with l_μ by (4.7) and

$$\Gamma_\mu^*(l_\mu) \leq \inf\left\{\int \tau^*(K)\, d\Lambda_\mu\,;\, K \text{ associated with } l_\mu \text{ by (4.7)}\right\}.$$

The function τ^* is given at (4.3). We are going to show that this inequality is in fact an equality. Then, we shall solve the associated minimization problem using proposition 5.3; this requires the computation of $\overline{\Gamma}_\mu$.

Proposition 6.1. *The function Γ^* is defined for all l in $C_b^\sharp$ by*

$$\Gamma^*(l) = \begin{cases} \int \tau^*(l)\, d\Lambda_\mu & \text{if } l \in L^{\tau^*} \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Thanks to (4.6), the property 5.3.a is satisfied for $\|\cdot\|_\tau$. It follows that, *dom* $\Gamma^* \subset L^{\tau^*}$ and that Γ^* is $\|\cdot\|_{\tau^*}$-continuous on idom Γ^* (lemma 5.4).

Let $l \in L^{\tau^*}$, we have:

$$\begin{aligned} \Gamma^*(l) &= \sup_{f \in C_b} \int (lf - \tau(f))\, d\Lambda_\mu \\ &\leq \int \sup_{u \in IR} \{l(x)u - \tau(u)\}\, \Lambda_\mu(dx) \\ &= \int \tau^*(l)\, d\Lambda_\mu \in [0, \infty]. \end{aligned} \tag{6.4}$$

If $l \in L^{\tau^*}$ and if $\Lambda_\mu(l < -1) > 0$, then $\Gamma^*(l) = +\infty$. Indeed, there exists an open set U such that $\Lambda_\mu(U) > 0$ on which $l < -1$, Λ_μ-almost everywhere. Taking advantage of the inner regularity of Λ_μ and of Lusin theorem (Lusin theorem is valid in a metric space for a regular measure, hence in a Polish space), one obtains the existence of a compact set $\kappa \subset U$ such that $\Lambda_\mu(\kappa) > 0$ on which l is continuous; in particular: $\sup_{x \in \kappa} l(x) := \alpha < -1$. Let $g \in C_b$ be such that $0 \leq g \leq 1$, $g(x) = 0$, if $x \notin U$ and $g(x) = 1$ if $x \in \kappa$. Notice that: $v < -1, u \leq 0 \Rightarrow uv - \tau(u) = u(v+1) + 1 - e^u \geq 0$ so for any $n \geq 1$,

$$\begin{aligned} \Gamma^*(l) &\geq \int_U (-nlg - \tau(-ng))\, d\Lambda_\mu \\ &\geq \int_\kappa (-nl - \tau(-n))\, d\Lambda_\mu \\ &\geq (-n(\alpha+1) + 1 - e^{-n})\Lambda_\mu(\kappa), \end{aligned}$$

and letting n tend to infinity: $\Gamma^*(l) = +\infty$.
It follows that if $\Gamma^*(l) < +\infty$, then $\int \tau^*(l)\, d\Lambda_\mu < +\infty$. Indeed, if $\Gamma^*(l) < +\infty$, then $\int_{\{l \geq 0\}} \tau^*(l)\, d\Lambda_\mu < +\infty$ since $l \in L^{\tau^*}$ and $\int_{\{l \leq 0\}} \tau^*(l)\, d\Lambda_\mu < +\infty$ since Λ_μ is a bounded measure and we have just seen that $l \geq -1$ Λ_μ-almost everywhere.

The converse is clear, thanks to (6.4), and we have

$$\textit{dom}\ \Gamma^* = \left\{ l \in C_b^\sharp\, ;\, l \in L^{\tau^*} \text{and } \int \tau^*(l)\, d\Lambda_\mu < +\infty \right\}. \tag{6.5}$$

Let l be a step function: $l = \sum_i \lambda_i \mathbb{1}_{A_i}$, where $\lambda_i > -1$ and the A_i's are Borel subsets. For any $\varepsilon > 0$, one approximates $\mathbb{1}_{A_i}$ by $\varphi_i^\varepsilon \in C_b$, as follows. Since Λ_μ is a regular measure, there exists a compact set K_i^ε and an open set U_i^ε such that $K_i^\varepsilon \subset A_i \subset U_i^\varepsilon$, $\Lambda_\mu(U_i^\varepsilon \setminus A_i) \leq \varepsilon/2$ and $\Lambda_\mu(A_i \setminus K_i^\varepsilon) \leq \varepsilon/2$.

One builds $\varphi_i^\varepsilon \in C_b$ such that $0 \le \varphi_i^\varepsilon \le 1$, $\varphi_i^\varepsilon(x) = 1, \forall x \in K_i^\varepsilon$ and $\varphi_i^\varepsilon(x) = 0, \forall x \notin U_i^\varepsilon$. Defining $f^\varepsilon = \sum_i \log(\lambda_i + 1)\varphi_i^\varepsilon$:

$$\begin{aligned}
\Gamma^*(l) &\ge \int (lf^\varepsilon - \tau(f^\varepsilon))\, d\Lambda_\mu \\
&\ge \sum_i [\lambda_i \log(\lambda_i+1) - \tau(\log(\lambda_i+1))]\Lambda_\mu(K_i^\varepsilon) - \sum_i \int_{U_i^\varepsilon \setminus K_i^\varepsilon} |lf^\varepsilon - \tau(f^\varepsilon)|\, d\Lambda_\mu \\
&\ge \int \tau^*(\sum_i \lambda_i \mathbb{1}_{K_i^\varepsilon})\, d\Lambda_\mu - \max_i(\lambda_i \log(\lambda_i+1)) \sum_i \Lambda_\mu(U_i^\varepsilon \setminus K_i^\varepsilon) \\
&\ge \int \tau^*(\sum_i \lambda_i \mathbb{1}_{K_i^\varepsilon})\, d\Lambda_\mu - C\varepsilon \\
&\ge \int \tau^*(\sum_i \lambda_i \mathbb{1}_{A_i})\, d\Lambda_\mu - (\max_i \tau^*(\lambda_i)) \sum_i \Lambda_\mu(A_i \setminus K_i^\varepsilon) - C\varepsilon \\
&\ge \int \tau^*(l)\, d\Lambda_\mu - 2C\varepsilon.
\end{aligned}$$

As ε is arbitrary, it follows that: $\Gamma^*(l) \ge \int \tau^*(l)\, d\Lambda_\mu$ and taking (6.4) into account, we obtain that for any step function l such that $l > -1$:

$$\Gamma^*(l) = \int \tau^*(l)\, d\Lambda_\mu.$$

Let $l \in L^{\tau^*}$ be such that $\int \tau^*(l)\, d\Lambda_\mu < +\infty$. Tracking the traditional proof for the everywhere density of the step functions in L^p, one shows that there exists a sequence $(l_n)_{n\ge1}$ of $]-1, +\infty[$-valued step functions such that $\lim_{n\to\infty} l_n = l$ in L^{τ^*}, $0 \le l_n \le l$ on $\{l \ge 0\}$, Λ_μ-almost everywhere and $-1 \le l \le l_n \le 0$ on $\{l \le 0\}$, Λ_μ-almost everywhere. Choose l in idom Γ^* in such a way that Γ^* is $\|\cdot\|_{\tau^*}$-continuous at l (lemma 5.4), this provides us with: $\Gamma^*(l) = \lim_{n\to\infty} \Gamma^*(l_n) = \lim_{n\to\infty} \int \tau^*(l_n)\, d\Lambda_\mu = \int \tau^*(l)\, d\Lambda_\mu$ (Beppo-Levi theorem).

Now, let l be a boundary point of *dom* Γ^*. Since Γ^* is $\sigma(L^{\tau^*}, C_b)$-lower semicontinuous, it is $\|\cdot\|_{\tau^*}$-lower semicontinuous and $t \in [0,1] \mapsto \Gamma^*(tl)$ is convex and lower semicontinuous. But, $\Gamma^*(tl) < +\infty, \forall t \in [0,1[$, hence:

$$\Gamma^*(l) = \lim_{t\uparrow1} \Gamma^*(tl) = \lim_{t\uparrow1} \int \tau^*(tl)\, d\Lambda_\mu = \int \tau^*\, d\Lambda_\mu \in [0,\infty]$$

where the last equality is a consequence of Beppo-Levi theorem. Therefore, for all $l \in$ *dom* Γ^*, we get: $\Gamma^*(l) = \int \tau^*(l)\, d\Lambda_\mu$. Together with (6.5), this completes the proof of the proposition. ■

Lemma 6.2. *For any ξ, ξ' in $C_{b,+}^{\sharp\sharp}$, we have*

$$\begin{aligned}
\xi_{|L^{\tau^*}} = \xi'_{|L^{\tau^*}} &\implies \overline{\Gamma}(\xi) = \overline{\Gamma}(\xi') \\
\xi_{|L^\infty} = \xi'_{|L^\infty} &\implies \overline{\Gamma}(-\xi) = \overline{\Gamma}(-\xi')
\end{aligned}$$

and for any $\xi \in C_{b,+}^{\sharp\sharp}$,

$$\begin{aligned}
\overline{\Gamma}(\xi) &= \overline{\Gamma}_+(\xi) = \sup_{l \in L_+^{\tau^*}} \{\langle \xi, l\rangle - \Gamma_+^*(l)\} \quad \text{and} \\
\overline{\Gamma}(-\xi) &= \overline{\Gamma}_-(\xi) = \sup_{l \in L_+^{\infty}} \{\langle \xi, l\rangle - \Gamma_-^*(l)\}.
\end{aligned}$$

Proof. Denoting $l = l_+ - l_-$, for any $\xi \in C^{\sharp\sharp}_{b,+}$, one gets

$$\begin{aligned}
\overline{\Gamma}(\xi) &= \sup\{\langle \xi, l_+\rangle - \langle \xi, l-\rangle - \Gamma^*(l_+) - \Gamma^*(l_-)\,;\, l \in L^{\tau^*}\} \\
&\leq \sup\{\langle \xi, l_+\rangle - \Gamma^*(l_+)\,;\, l \in L^{\tau^*}\} \\
&= \sup\{\langle \xi, l\rangle - \Gamma^*(l)\,;\, l \in L^{\tau^*}_+\} \\
&\leq \sup\{\langle \xi, l\rangle - \Gamma^*(l)\,;\, l \in L^{\tau^*}\} \\
&= \overline{\Gamma}(\xi),
\end{aligned}$$

which yields

$$\overline{\Gamma}(\xi) = \sup\{\langle \xi, l\rangle - \Gamma^*(l)\,;\, l \in L^{\tau^*}_+\}.$$

One also gets,

$$\overline{\Gamma}_+(\xi) = \sup\{\langle \xi, l\rangle - \Gamma^*_+(l)\,;\, l \in L^{\tau^*}_+\}.$$

As in proposition 6.1, one shows that: $\Gamma^*_+(l) = \int \tau^*(|l|)\, d\Lambda_\mu$, so that

$$\overline{\Gamma}(\xi) = \overline{\Gamma}_+(\xi) = \sup\{\langle \xi, l\rangle - \Gamma^*_+(l)\,;\, l \in L^{\tau^*}_+\},\ \forall \xi \in C^{\sharp\sharp}_{b,+}.$$

Similarly, one shows that

$$\overline{\Gamma}(-\xi) = \overline{\Gamma}_-(\xi) = \sup\{\langle \xi, l\rangle - \Gamma^*_-(l)\,;\, l \in L^{\infty}_+\},\ \forall \xi \in C^{\sharp\sharp}_{b,+}.$$

One concludes thanks to (6.3). ∎

We are now ready to prove the following proposition.

Proposition 6.3. *For any ξ in dom $\overline{\Gamma}$, there exists at least a couple*

$$(\zeta_+, \zeta_-) \in (C^{\sharp\sharp}_{b,+} \cap L^\tau, C^{\sharp\sharp}_{b,+} \cap (L^\infty)')$$

such that $\xi_{|L^\infty} = \zeta_{+|L^\infty} - \zeta_-$ and $\overline{\Gamma}(\xi) = \overline{\Gamma}_+(\zeta_+) + \overline{\Gamma}_-(\zeta_-)$.

Proof. Let us begin showing that for any $\xi \in C^{\sharp\sharp}_b$,

(6.6) $$(\exists \xi_+, \xi_- \in C^{\sharp\sharp}_{b,+} \text{ such that } \xi = \xi_+ - \xi_-) \Rightarrow \overline{\Gamma}(\xi) \leq \overline{\Gamma}(\xi_{+|L^\infty}) + \overline{\Gamma}(-\xi_{-|L^\infty}),$$

where it has been emphasized that $\overline{\Gamma}(\xi_{|L^\infty}) = \overline{\Gamma}(\xi), \overline{\Gamma}(-\xi_{|L^\infty}) = \overline{\Gamma}(-\xi)$ when $\xi \in C^{\sharp\sharp}_{b,+}$. Indeed, by lemma 6.2, for any $\xi \in C^{\sharp\sharp}_{b,+}$, $\overline{\Gamma}(\xi) = \overline{\Gamma}(\xi_{|L^{\tau^*}})$ and $\overline{\Gamma}(-\xi) = \overline{\Gamma}(-\xi_{|L^\infty})$. But $L^{\tau^*} \cap L^\infty = L^\infty$ is dense in $E^{\tau^*} = L^{\tau^*}$.

Let (u_α) and (v_α) be two filters in C_b, convergent with respect to the topology $\sigma(C^{\sharp\sharp}_b, C^\sharp_b)$ with

$$\begin{array}{lll} u_\alpha \underset{\alpha}{\longrightarrow} \xi_+ & \text{and} & \lim_\alpha \Gamma(u_\alpha) = \overline{\Gamma}(\xi_+) \\ v_\alpha \underset{\alpha}{\longrightarrow} \xi_- & \text{and} & \lim_\alpha \Gamma(-v_\alpha) = \overline{\Gamma}(-\xi_-) \end{array} \quad \text{for } \sigma(C^{\sharp\sharp}_b, C^\sharp_b).$$

We have $u_\alpha - v_\alpha \underset{\alpha}{\longrightarrow} \xi_+ - \xi_- = \xi$ and

$$\begin{aligned}
\overline{\Gamma}(\xi_+) + \overline{\Gamma}(-\xi_-) &= \lim_\alpha(\Gamma(u_\alpha) + \Gamma(-v_\alpha)) \\
&\geq \liminf_\alpha \Gamma(u_\alpha - v_\alpha) \\
&\geq \overline{\Gamma}(\xi),
\end{aligned}$$

which is (6.6).

Let $\xi \in dom\,\overline{\Gamma}$, then one can find a $\sigma(C_b^{\sharp\sharp}, C_b^{\sharp})$-convergent filter in $C_b : (f_\alpha)$, such that $f_\alpha \underset{\alpha}{\longrightarrow} \xi$ and $\lim_\alpha \Gamma(f_\alpha) = \overline{\Gamma}(\xi)$. But, $f_\alpha = f_\alpha^+ - f_\alpha^-$ for some $f_\alpha^+, f_\alpha^- \in C_{b,+}$ such that $\Gamma(f_\alpha) = \Gamma(f_\alpha^+) + \Gamma(-f_\alpha^-)$. Possibly refining the filter (f_α^+), (f_α^+) can be chosen in such a way that $\liminf_\alpha \Gamma(f_\alpha^+) = \lim_\alpha \Gamma(f_\alpha^+)$. Since $\Gamma(f_\alpha) = \Gamma(f_\alpha^+) + \Gamma(-f_\alpha^-)$ is a convergent filter, one also gets $\liminf_\alpha \Gamma(f_\alpha^-) = \lim_\alpha \Gamma(f_\alpha^-)$.

Thanks to (5.5), the filter $(f^+_{\alpha\,|L^{\tau^*}})$ is bounded in L^τ and the filter $(f^-_{\alpha\,|L^\infty})$ is bounded in $(L^\infty)'$. Refining the filter $(f^+_{\alpha\,|L^{\tau^*}}, f^-_{\alpha\,|L^\infty})$ one obtains the convergent filters:

$$\begin{aligned} f^+_{\alpha\,|L^{\tau^*}} &\underset{\alpha}{\longrightarrow} \zeta_+ \in L^\tau \quad \text{for } \sigma(L^\tau, L^{\tau^*}) \\ f^-_{\alpha\,|L^\infty} &\underset{\alpha}{\longrightarrow} \zeta_- \in (L^\infty)' \quad \text{for } \sigma((L^\infty)', L^\infty). \end{aligned}$$

Any continuous linear form on L^{τ^*} being determined by its restriction to L^∞, one obtains $\overline{\Gamma}(\zeta_+) = \overline{\Gamma}(\zeta_{+|L^\infty})$ and $\overline{\Gamma}(-\zeta_-) = \overline{\Gamma}(-\zeta_{-|L^\infty})$. From the semicontinuity inequalities:

$$\begin{aligned} \lim_\alpha \Gamma(f_\alpha^+) &= \liminf_\alpha \Gamma(f_\alpha^+) \geq \overline{\Gamma}(\zeta_+) \\ \lim_\alpha \Gamma(-f_\alpha^-) &= \liminf_\alpha \Gamma(-f_\alpha^-) \geq \overline{\Gamma}(-\zeta_-), \end{aligned}$$

it comes out that

$$\begin{aligned} \overline{\Gamma}(\xi) &= \lim_\alpha (\Gamma(f_\alpha^+) + \Gamma(-f_\alpha^-)) \\ &\geq \overline{\Gamma}(\zeta_+) + \overline{\Gamma}(-\zeta_-) \\ &= \overline{\Gamma}(\zeta_{+|L^\infty}) + \overline{\Gamma}(-\zeta_{-|L^\infty}) \\ &\geq \overline{\Gamma}(\xi). \end{aligned}$$

For the last equality, notice that $\xi_{|L^\infty} = \zeta_{+|L^\infty} - \zeta_{-|L^\infty}$ and use (6.6). We have just obtained the existence of $(\zeta_+, \zeta_-) \in (C_{b,+}^{\sharp\sharp} \cap L^\tau) \times (C_{b,+}^{\sharp\sharp} \cap (L^\infty)')$ such that $\xi_{|L^\infty} = \zeta_{+|L^\infty} - \zeta_{-|L^\infty}$ and $\overline{\Gamma}(\xi) = \overline{\Gamma}(\zeta_{+|L^\infty}) + \overline{\Gamma}(-\zeta_{-|L^\infty})$. One concludes by means of lemma 6.2. ■

Since $(L^\infty)'$ is not a nice space, we still have to work: it remains to show that the nonpositive part of an element of $dom\,\overline{\Gamma}$ is a function. During the proof of this result which is stated in lemma 6.4, we shall invoke the following classical results of convex analysis.

(6.7) <u>Goldstine theorem</u>. *For a normed vector space E, the canonical embedding $E \subset E''$ is $\sigma(E'', E')$-dense.*

(6.8) *A linear form on a topological vector space E is continuous if and only if it is $\sigma(E, E')$-continuous.*

For (6.7), see ([Bré], lemme III.4) and for (6.8), see ([Bré], théorème III.9).

Lemma 6.4. *For any $\xi \in (L^\infty)'$, the restriction $\xi_{|C_c}$ of ξ to C_c : the space of all continuous functions with compact support in $[0,T] \times (IR^d)^2 \times E$, belongs to L^1.*

Denoting $\widetilde{\Gamma}_-$ for the restriction of $\overline{\Gamma}_-$ to the space $L^1 \subset (L^\infty)'$, we have

$$\overline{\Gamma}_- : \xi \in C_b^{\sharp\sharp} \mapsto \begin{cases} \widetilde{\Gamma}_-(\xi_{|C_c}) & \textit{if } \xi \in (L^\infty)' \\ +\infty & \textit{otherwise.} \end{cases}$$

More, if $\xi, \xi' \in L^\tau$ are such that $\xi_{|C_c} = \xi'_{|C_c}$, then $\overline{\Gamma}_+(\xi) = \overline{\Gamma}_+(\xi')$.

Proof. The first statement is a consequence of F. Riesz representation of the dual of C_c and of Radon-Nykodym theorem.

It has already been noticed at (6.3) that $dom\, \overline{\Gamma}_- \subset (L^\infty)'$.

Denote E^∞ the space C_c endowed with the norm of L^∞. Denoting $A \xrightarrow{\ '\ } B$ to mean that B is the topological dual of A, we have

$$E^\infty \xrightarrow{\ '\ } L^1 \xrightarrow{\ '\ } L^\infty \xrightarrow{\ '\ } (L^\infty)',$$

with the duality bracket $\langle f, h\rangle = \int fh\, d\Lambda_\mu$, for the two first arrows.

Of course, we have

$$\widetilde{\Gamma}_-(\theta) = \sup_{l \in L^\infty} \{\langle \theta, l\rangle - \Gamma_-^*(l)\}, \quad \theta \in Ł^1.$$

But, C_c is $\sigma(L^\infty, L^1)$-dense in L^∞ (by (6.7)), and for any $\theta \in L^1$, the function $l \in L^\infty \mapsto \langle \theta, l\rangle - \Gamma_-^*(l)$ is $\sigma(L^\infty, L^1)$-upper semicontinuous. Therefore, for any $\theta \in L^1$,

$$\widetilde{\Gamma}_-(\theta) = \sup_{l \in C_c} \{\langle \theta, l\rangle - \Gamma_-^*(l)\}.$$

It follows that $\widetilde{\Gamma}_-$ is $\sigma(L^1, C_c)$-lower semicontinuous and convex.

On the other hand, $\xi \in (L^\infty)' \mapsto \xi_{|C_c} \in L^1$ is linear and $\sigma((L^\infty)', L^\infty)$-$\sigma(L^1, C_c)$-continuous. It follows that the function $\xi \in (L^\infty)' \mapsto \widetilde{\Gamma}_-(\xi_{|C_c})$ is convex and $\sigma((L^\infty)', L^\infty)$-lower semicontinuous.

Thanks to (6.8), the elements of $(L^\infty)'$ are linear forms on L^∞ which are $\sigma(L^\infty, (L^\infty)')$-continuous. The subspace L^1 of $(L^\infty)'$ is the space of all linear forms on L^∞ which are $\sigma(L^\infty, Ł^1)$-continuous. Hence, both functions $\xi \in (L^\infty)' \mapsto \overline{\Gamma}_-(\xi)$ and $\xi \in (L^\infty)' \mapsto \widetilde{\Gamma}_-(\xi_{|C_c})$ match on L^1. On the other hand, their epigraphs being $\sigma((L^\infty)', L^\infty)$-closed, they are equal since L^1 is $\sigma((L^\infty)', L^\infty)$-dense in $(L^\infty)'$. One deduces that $\overline{\Gamma}_-$ has the desired form.

Finally, the statement about $\overline{\Gamma}_+$ holds since C_c is dense in L^{τ^*}. ■

Proposition 6.5. *Let us denote $C_c \cdot \Lambda_\mu$: the space of all the $l \in C_b^\sharp$ of the form $l = h \cdot \Lambda_\mu$ with $h \in C_c$, and also $\xi_{|C_c\cdot\Lambda_\mu}$: the restriction of $\xi \in C_b^{\sharp\sharp}$ to $C_c \cdot \Lambda_\mu$. Then, for any $\xi, \xi' \in C_b^{\sharp\sharp}$,*

$$\xi_{|C_c\cdot\Lambda_\mu} = \xi'_{|C_c\cdot\Lambda_\mu} \Longrightarrow \overline{\Gamma}(\xi) = \overline{\Gamma}(\xi').$$

Denoting $L_{\tau,1}$ the set of all measurable functions f such that $f_+ \in L^\tau$ and $f_- \in L^1$, we have

$$dom\, \overline{\Gamma} \subset \{\xi \in C_b^{\sharp\sharp}\, ; \xi_{|C_c\cdot\Lambda_\mu} \in L_{\tau,1}\}.$$

Proof. It is a straightforward consequence of proposition 6.3 and lemma 6.4. ■

Theorem 6.6. *Let the function Γ on C_b be defined by*

$$\Gamma : f \in C_b \mapsto \int \tau(f)\, d\Lambda_\mu \in [0, \infty[.$$

The upper bound: $\overline{\Gamma}$, of the convex and $\sigma(C_b^{\sharp\sharp}, C_b^{\sharp})$-lower semicontinuous extensions of Γ to $C_b^{\sharp\sharp}$ is given, for all $\xi \in C_b^{\sharp\sharp}$, by

$$\overline{\Gamma}(\xi) = \widetilde{\Gamma}(\xi_{|C_c\cdot\Lambda_\mu}), \quad \text{where for any } \theta \in (C_c \cdot \Lambda_\mu)^\sharp,$$

$$\widetilde{\Gamma}(\theta) = \begin{cases} \int \tau(\theta)\, d\Lambda_\mu & \text{if } \theta \in L_{\tau,1} \text{ and } \int \tau^*(e^\theta - 1)\, d\Lambda_\mu < +\infty \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. For any measurable function h on $[0,T] \times (IR^d)^2 \times E$, let us define the following $[0,+\infty]$-valued functions

$$\psi_+(h) = \int \tau^*(|h|)\, d\Lambda_\mu \quad \text{and} \quad \psi_-(h) = \int \gamma^*(|h|)\, d\Lambda_\mu,$$

with $\gamma^*(u) = \tau^*(-u),\ u \geq 0$, so that

$$\psi(h) := \int \tau^*(h)\, d\Lambda_\mu = \psi_+(h_+) + \psi_-(h_-).$$

For any function $f \in L_{\tau,1}$, one defines

$$\psi^*(f) = \sup_{h \in C_c} \{\langle h, f\rangle - \int \tau^*(h)\, d\Lambda_\mu\}.$$

As in the proof of lemma 6.2, one obtains

$$\begin{aligned} \psi^*(f) &= \psi_+^*(f_+) + \psi_-^*(f_-), \quad \text{where} \\ \psi_+^*(f) &= \sup\{\langle h, f\rangle - \psi_+(h)\,;\, h \in C_c, h \geq 0\}, \quad \text{and} \\ \psi_-^*(f) &= \sup\{\langle h, f\rangle - \psi_-(h)\,;\, h \in C_c, h \geq 0\}. \end{aligned} \tag{6.9}$$

But, since ψ_+ is $\|\cdot\|_{\tau^*}$-continuous on L^{τ^*} and since C_c is dense in L^{τ^*}, one gets

$$\psi_+^*(f) = \sup_{h \in L_+^{\tau^*}} \{\langle h, f\rangle - \psi_+(h)\},\ f \in L^\tau. \tag{6.10}$$

Similarly, since ψ_- is $\|\cdot\|_\infty$-continuous on C_c and convex, it is $\sigma(C_c, L^1)$-lower semicontinuous. But, thanks to Goldstine theorem (6.7), C_c is $\sigma(L^\infty, L^1)$-dense in L^∞. Therefore, taking (6.8) into account,

$$\psi_-^*(f) = \sup_{h \in L_+^\infty} \{\langle h, f\rangle - \psi_+(h)\},\ f \in L^1.$$

On the other hand, for any function $h \in L_+^{\tau^*}$: $hf - \tau^*(h) \leq \tau(f)$, Λ_μ-almost everywhere. Hence, if $f \in L_+^\tau$, one gets: $\forall h \in L_+^{\tau^*}, \langle h, f\rangle - \psi_+(h) \leq \int \tau(f)\, d\Lambda_\mu$. Together with (6.10), this yields

$$\psi_+^*(f) = \sup_{h \in L_+^{\tau^*}} \{\langle h, f\rangle - \psi_+(h)\} \leq \int \tau(f)\, d\Lambda_\mu. \tag{6.11}$$

More, $\tau(f) = h_f f - \tau^*(h_f)$ with $h_f = e^f - 1$. Therefore, if $f \in L_+^\tau$, one gets $h_f \geq 0$ and if $h_f \in L^{\tau^*}$:

$$\begin{aligned} \int \tau(f)\, d\Lambda_\mu &= \langle h_f, f\rangle - \psi_+(h_f) \\ &\leq \sup_{h \in L_+^{\tau^*}} \{\langle h, f\rangle - \psi_+(h)\} \\ &= \psi_+^*(f). \end{aligned}$$

With (6.11), it follows that if $f \in L^\tau_+$ and $e^f - 1 \in L^{\tau^*}$, then $\psi^*_+(f) = \int \tau(f)\, d\Lambda_\mu$.
One shows similarly that, if $f \in L^1_+$ (hence $1 - e^{-f} \in L^\infty$) then $\psi^*_-(f) = \int \gamma(f)\, d\Lambda_\mu$. Thanks to (6.9), one obtains that for any function $f \in L_{\tau,1}$ satisfying $\displaystyle\int \tau^*(e^f - 1)\, d\Lambda_\mu < \infty$, one gets

$$\psi^*(f) = \sup_{h \in C_c} \left\{ \langle h, f\rangle - \int \tau^*(h)\, d\Lambda_\mu \right\} = \int \tau(f)\, d\Lambda_\mu. \tag{6.12}$$

Therefore, the function ψ^* is an extension of Γ. Moreover, it is convex and $\sigma(L_{\tau,1}, C_c)$-lower semicontinuous, and a fortiori $\sigma(L_{\tau,1}, L^{\tau^*})$-lower semicontinuous. Together with the proposition 6.5 and (6.7), it follows that for any $\xi \in C_b^{\sharp\sharp}$,

$$\overline{\Gamma}(\xi) = \begin{cases} \psi^*(\xi_{|C_c \cdot \Lambda_\mu}) & \text{if } \xi_{|C_c \cdot \Lambda_\mu} \in L_{\tau,1} \\ +\infty & \text{otherwise.} \end{cases}$$

Because of (6.12), it remains to show that

$$\mathit{dom}\ \psi^* = \{ f \in L_{\tau,1}\,;\, e^f - 1 \in L^{\tau^*}\}. \tag{6.13}$$

Thanks to lemma 5.4, one shows that ψ^*_+ is $\|\cdot\|_\tau$-continuous in the direction C_c. Hence, one is allowed to apply to it the proposition 5.2.(i) to obtain: C_c-idom $\psi^*_+ \subset \{\psi'_+(h)\,;\, h \in L^{\tau^*}\} \subset \mathit{dom}\ \psi^*_+$. One shows similarly that C_c-idom $\psi^*_- \subset \{\psi'_-(h)\,;\, \|h\|_\infty < 1\} \subset \mathit{dom}\ \psi^*_-$. Together with (6.9), this yields

$$C_c\text{-idom}\ \psi^* \subset \{\psi'(h)\,;\, h_+ \in L^{\tau^*}, h_- < 1\} \subset \mathit{dom}\ \psi^*.$$

For any $A \subset L_{\tau,1}$, denote C_c-icor $A = \{x \in A\,; \forall h \in C_c, \exists t > 0, [x, x+th] \subset A\}$. Since $\{\psi'(h)\,;\, h_+ \in L^{\tau^*}, h_- < 1\} = \{\log(h+1)\,;\, h \in L^{\tau^*}, h > -1\} = \{g \in L_{\tau,1}\,;\, e^g - 1 \in L^{\tau^*}\}$ and $S := \{f \in L_{\tau,1}\,;\, e^f - 1 \in L^{\tau^*}\}$ is equal to C_c-icor S, and $(L_{\tau,1}) \setminus S$ is equal to C_c-icor $\Big((L_{\tau,1}) \setminus S\Big)$, one deduces (6.13). This completes the proof of the theorem. ■

Let us consider now, the extension of the function Γ which we are interested in, whose domain is a subspace $\mathcal{C}$ of C_b.

Denote N^τ : the closure in L^τ of $\mathcal{C}$ for the topology $\sigma(L^\tau, L^{\tau^*})$, and N^1 : the closure in L^1 of $\mathcal{C}$. Denote $N_{\tau,1}$ the set of all functions f whose nonnegative part f_+ is in N^τ and whose nonpositive part f_- is in N^1.

Corollary 6.7. *Let the function $\Gamma_{\mathcal{C}}$ on $\mathcal{C} \subset C_b$ be defined by*

$$\Gamma_{\mathcal{C}} : f \in \mathcal{C} \mapsto \int \tau(f)\, d\Lambda_\mu \in [0, \infty[.$$

The upper bound: $\overline{\Gamma}_{\mathcal{C}}$, of all the convex and $\sigma(\mathcal{C}^{\sharp\sharp}, \mathcal{C}^\sharp)$-lower semicontinuous extensions of $\Gamma_{\mathcal{C}}$ to $\mathcal{C}^{\sharp\sharp}$ is given, for any $\xi \in \mathcal{C}^{\sharp\sharp}$, by

$$\overline{\Gamma}_{\mathcal{C}}(\xi) = \widetilde{\Gamma}_{\mathcal{C}}(\xi_{|\mathcal{C}\cdot\Lambda_\mu}), \quad \textit{where for any } \theta \in (\mathcal{C}\cdot\Lambda_\mu)^\sharp,$$
$$\widetilde{\Gamma}_{\mathcal{C}}(\theta) = \begin{cases} \displaystyle\int \tau(\theta)\, d\Lambda_\mu & \textit{if } \theta \in N_{\tau,1} \textit{ and } \int \tau^*(e^\theta - 1)\, d\Lambda_\mu < +\infty, \\ +\infty & \textit{otherwise.} \end{cases}$$

Remark. The result still holds with $(C_c \cap \mathcal{C}) \cdot \Lambda_\mu$ instead of $\mathcal{C} \cdot \Lambda_\mu$.

Proof. Let us fix some notations. One chooses a space $\mathcal{E}$ supplementary to $\mathcal{C}$ in C_b. One has the following decompositions and identifications:

$$\begin{array}{rclcrcl} C_b & = & \mathcal{C} \oplus \mathcal{E}, & & & & \\ C_b^\sharp & = & \mathcal{C}^\perp \oplus \mathcal{E}^\perp, & & \mathcal{C}^\sharp & \simeq & \mathcal{E}^\perp, \\ C_b^{\sharp\sharp} & = & \mathcal{C}^{\perp\perp} \oplus \mathcal{E}^{\perp\perp}, & & \mathcal{C}^{\sharp\sharp} & \simeq & \mathcal{C}^{\perp\perp}. \end{array}$$

It will be of practical use to consider the function $\Gamma_o : f \in C_b \mapsto \begin{cases} \Gamma(f) & \text{if } f \in \mathcal{C} \\ +\infty & \text{otherwise} \end{cases}$. So that for any $l \in C_b^\sharp$:

$$\begin{aligned} \Gamma_o^*(l) = \sup_{f \in C_b} \{\langle l, f\rangle - \Gamma_o(f)\} &= \sup_{f \in \mathcal{C}} \{\langle l, f\rangle - \Gamma_{\mathcal{C}}(f)\} = \Gamma_{\mathcal{C}}^*(l_{|\mathcal{C}}) \\ &= \sup_{f \in C_b} \{\langle \Pi_{\mathcal{E}^\perp}(l), f\rangle - \Gamma(f)\} = \Gamma^*(\Pi_{\mathcal{E}^\perp}(l)), \end{aligned}$$

where $\Pi_{\mathcal{E}^\perp}$ is the projection in $C_b^\sharp$ on $\mathcal{E}^\perp$ parallel to $\mathcal{C}^\perp$. Hence, for any l in $C_b^\sharp$,

$$\Gamma_o^*(l) = \Gamma_{\mathcal{C}}^*(l_{|\mathcal{C}}) = \Gamma^*(\Pi_{\mathcal{E}^\perp}(l)). \tag{6.14}$$

For any $\xi \in C_b^{\sharp\sharp}$, we get

$$\begin{aligned} \overline{\Gamma}_o(\xi) &= \sup_{l \in C_b^\sharp} \{\langle \xi, l\rangle - \Gamma_o^*(l)\} \\ &= \sup_{l \in C_b^\sharp} \{\langle \xi, l\rangle - \Gamma^*(\Pi_{\mathcal{E}^\perp}(l))\} \qquad \text{(with (6.14))} \\ &= \sup_{l_1 \in \mathcal{E}^\perp, l_2 \in \mathcal{C}^\perp} \{\langle \xi, l_2\rangle + \langle \xi, l_1\rangle - \Gamma^*(l_1)\} \\ &= \begin{cases} \sup_{l \in \mathcal{E}^\perp} \{\langle \xi, l\rangle - \Gamma^*(l)\} & \text{if } \xi \in \mathcal{C}^{\perp\perp} \simeq \mathcal{C}^{\sharp\sharp} \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

But, by (6.14), we have

$$\begin{aligned} \sup_{l \in \mathcal{E}^\perp} \{\langle \xi, l\rangle - \Gamma^*(l)\} &= \sup_{l \in \mathcal{E}^\perp} \{\langle \xi_{|\mathcal{C}^\sharp}, l_{|\mathcal{C}}\rangle - \Gamma_{\mathcal{C}}^*(l_{|\mathcal{C}})\} \\ &= \sup_{\tilde{l} \in \mathcal{C}^\sharp} \{\langle \xi_{|\mathcal{C}^\sharp}, \tilde{l}\rangle - \Gamma_{\mathcal{C}}^*(\tilde{l})\} \\ &= \overline{\Gamma}_{\mathcal{C}}(\xi_{|\mathcal{C}^\sharp}). \end{aligned}$$

On the other hand, if $\xi \in \mathcal{C}^{\sharp\sharp} \simeq \mathcal{C}^{\perp\perp}$ and if ξ is geometrically interior to *dom* $\overline{\Gamma}_o$,

$$\begin{aligned} \overline{\Gamma}_o(\xi) &= \sup_{l \in C_b^\sharp} \{\langle \xi, l\rangle - \Gamma_o^*(l)\} \\ &= \sup_{l \in C_b^\sharp} \{\langle \xi, l\rangle - \Gamma^*(\Pi_{\mathcal{E}^\perp}(l))\} \\ &= \sup_{l \in \mathcal{C}^\sharp} \{\langle \xi, l\rangle - \Gamma^*(l)\} \\ &= \sup_{l \in C_b^\sharp} \{\langle \xi, l\rangle - \Gamma^*(l)\} \\ &= \overline{\Gamma}(\xi) \end{aligned}$$

since, thanks to proposition 5.1 : $\xi \in \partial_{\mathcal{C}^{\sharp\sharp}} \Gamma_o^*(l_\xi) \Longleftrightarrow l_\xi \in \partial_{\mathcal{C}^\sharp} \overline{\Gamma}_o(\xi) \Longrightarrow l_\xi \in \mathcal{C}^\sharp$.

Therefore, $\overline{\Gamma}_{\mathcal{C}}$ is a restriction of $\overline{\Gamma}$.

Hence it is $\sigma(\mathcal{C}^{\sharp\sharp}, L_+^{\tau^*} - L_+^\infty)$-lower semicontinuous and $dom\ \overline{\Gamma}_{\mathcal{C}} \subset (\mathcal{C}^{\sharp\sharp} \cap dom\ \overline{\Gamma}) \subset (\mathcal{C}^{\sharp\sharp} \cap L_{\tau,1})$. By Goldstine theorem ((6.7)), it is the unique extension of $\Gamma_{\mathcal{C}}$ to $\mathcal{C}^{\sharp\sharp} \cap L_{\tau,1}$ which is $\sigma(\mathcal{C}^{\sharp\sharp} \cap L_{\tau,1}, \mathcal{C}^\sharp \cap (L_+^{\tau^*} - L_+^\infty))$-lower semicontinuous and convex. It follows that $\overline{\Gamma}_{\mathcal{C}}$ is the restriction of $\overline{\Gamma}$ to $\mathcal{C}^{\sharp\sharp}$ and that its domain is included in $N_{\tau,1}$ (notice that the closure of $\mathcal{C}$ with respect to the topology $\sigma(L^1, L^\infty)$ is also its closure with respect to $\|\cdot\|_1$). Indeed, in restriction to L^τ (resp. L^1) the biorthogonal $\mathcal{C}^{\perp\perp}$ of $\mathcal{C}$ coincide with the bipolar (hence the closed convex hull) of $\mathcal{C}$ for the topology $\sigma(L^\tau, L^{\tau^*})$ (resp. $\sigma(L^1, L^\infty)$). One completes the proof of the corollary with theorem 6.6.
■

7. Non-variational formulation of the rate function

Before stating, in theorem 7.1, the non-variational formulation of $J(\mu)$ (see (3.9)), let us recall some notations. The function τ^* is defined by $\tau^*(v) = \begin{cases} (v+1)\log(v+1) - v & \text{if } v > -1 \\ +1 & \text{if } v = -1 \\ +\infty & \text{if } v < -1, \end{cases}$ the "Kullback information" of the nonnegative measure α with respect to the nonnegative measure β is defined by: $I(\alpha \mid \beta) = \int \tau^*\left(\frac{d\alpha}{d\beta} - 1\right) d\beta$. For any $\mu \in D_{M_\varphi}$, we have defined the nonnegative measure on $[0,T] \times (IR^d)^2 \times E$: $\Lambda_\mu(dtdzdz'd\Delta d\Delta') = \frac{1}{2}\mathcal{L}(z,z',d\Delta d\Delta')\,\mu_t^{\otimes 2}(dzdz')\,dt$ where E is the set of all the jumps of the biparticles and for any $f \in C_o^{1,1}$: $Df(t,z,z',\Delta,\Delta') = f(t,z+\Delta) - f(t,z) + f(t,z'+\Delta') - f(t,z')$.

Notice that for any measurable function K : $\int \tau^*(K)\,d\Lambda_\mu = I\Big((K+1)\cdot\Lambda_\mu \mid \Lambda_\mu\Big)$.

Theorem 7.1. (Non-variational formulation of J). *A path $\mu \in D_{M_\varphi}$ is such that $J(\mu) < +\infty$ if and only if there exists a measurable function $K : [0,T] \times (IR^d)^2 \times E \longrightarrow IR$ such that μ is a solution to the Boltzmann weak equation (see (1.6))*

$$\Big\langle \big(\partial_t + A_{K,t}(\mu_t)\big) f, \mu_t \Big\rangle = 0,\ \forall f \in C_o^{1,1}([0,T] \times IR^d) \tag{7.1}$$

with

$$A_{K,t}(\mu_t)f(z) = \int_{(\Delta)} \{f(z+\Delta) - f(z)\} \left(\int_{(z',\Delta')} \mathcal{L}_K(t,z,z',\cdot \times d\Delta')\,\mu_t(dz') \right)(d\Delta),$$

$$\mathcal{L}_K(t,z,z',d\Delta d\Delta') = \Big(K(t,z,z',\Delta,\Delta') + 1 \Big)\,\mathcal{L}(z,z',d\Delta d\Delta')$$

and

$$\int_{[0,T]\times(IR^d)^2\times E} \tau^*(K)\,d\Lambda_\mu < +\infty, \tag{7.2}$$

(hence $K \geq -1$, Λ_μ-almost everywhere).

In such a case, there exists a unique (up to Λ_μ-almost everywhere equality) measurable function $K = K_\mu$ which satisfies both (7.1) and (7.2), and such that

$$\inf_{f \in C_o^{1,1}} I\Big((K_\mu + 1)\cdot\Lambda_\mu \mid e^{Df}\cdot\Lambda_\mu\Big) = 0. \tag{7.3}$$

Moreover,

$$J(\mu) = \int \tau^*(K_\mu)\, d\Lambda_\mu \tag{7.4}$$
$$= \inf \left\{ \int \tau^*(K)\, d\Lambda_\mu \,;\, K \textit{ measurable function satisfying } (7.1) \right\}.$$

<u>Remarks</u>. $*$ In particular, if $\mathcal{L}$ is the Lévy kernel of equation (1.4) and if $\mu \in D_{M_\varphi}$ is such that $J(\mu)$ is finite, since $\mathcal{L}_{K_\mu}$ is absolutely continuous with respect to $\mathcal{L}$, the conservation equations (1.2) and (1.3) still hold for μ.

$*$ Notice that the nonnegative function $K_\mu + 1$ may vanish on some measurable subset of $[0,T] \times (I\!R^d)^2 \times E$.

<u>Proof</u>. We have already seen in section 4, that with l_μ and Γ_μ given by (4.5) and (4.13), one writes: $J(\mu) = \Gamma_\mu^*(l_\mu)$.

Let $\mu \in D_{M_\varphi}$ be such that $J(\mu) < \infty$. Denote $\mathcal{C} = I\!DC_o^{1,1}$ and $\Gamma = \Gamma_\mu$. Applying the corollary 6.7, one obtains that the upper bound: $\overline{\Gamma}$, of the extensions of Γ to $\mathcal{C}^{\sharp\sharp}$ which are convex and $\sigma(\mathcal{C}^{\sharp\sharp}, \mathcal{C}^\sharp)$-lower semicontinuous is given by $\overline{\Gamma}(\xi) = \begin{cases} \int \tau(\xi)\, d\Lambda_\mu & \text{if } \xi \in N_{\tau,1} \text{ and } \int \tau^*(e^\xi - 1)\, d\Lambda_\mu < +\infty \\ +\infty & \text{otherwise,} \end{cases}$

where ξ and $\xi' \in \mathcal{C}^{\sharp\sharp}$ are identified whenever $\xi_{|\mathcal{C}\cdot\Lambda_\mu} = \xi'_{|\mathcal{C}\cdot\Lambda_\mu}$. With this identification, $\overline{\Gamma}$ is strictly convex. More, $\mathcal{C}$-idom $\overline{\Gamma} = \textit{dom}\, \overline{\Gamma}$ (a priori $\mathcal{C}$-idom $\overline{\Gamma} \subset \textit{dom}\, \overline{\Gamma}$), $\overline{\Gamma}$ is Gâteaux-differentiable in the direction $\mathcal{C}$ on $\textit{dom}\, \overline{\Gamma}$ and for any $\xi \in \textit{dom}\, \overline{\Gamma}$:

$$\overline{\Gamma}'(\xi) = (e^\xi - 1) \cdot \Lambda_\mu.$$

<u>Interior case</u>. Suppose that $l_\mu \in \text{idom}\, \Gamma^*$.

Applying the proposition 5.3, one gets the existence of a unique $\xi_\mu \in \textit{dom}\, \overline{\Gamma}$ satisfying (5.1), that is: $l_\mu(f) = \int Df\, (e^{\xi_\mu} - 1)\, d\Lambda_\mu, \forall f \in C_o^{1,1}$. More, (5.2) is: $\Gamma^*(l_\mu) = \int \Big((e^{\xi_\mu} - 1)\xi_\mu - \tau(\xi_\mu)\Big)\, d\Lambda_\mu$ $= \int \tau^*(e^{\xi_\mu} - 1)\, d\Lambda_\mu$. Taking (4.5) into account and defining $K_\mu = e^{\xi_\mu} - 1$, we obtain (7.1) and (7.2) with $K = K_\mu$, as well as the first equality in (7.4).

<u>Boundary case</u>. Suppose that $l_\mu \in \textit{dom}\, \Gamma^* \setminus \text{idom}\, \Gamma^*$.

Then, there exists $l_1 \in \text{idom}\, \Gamma^*$ such that $[l_1, l_\mu] \subset \textit{dom}\, \Gamma^*$. Let us set $l_n = \frac{1}{n} l_1 + \frac{n-1}{n} l_\mu,\ n \geq 1$. Since Γ^* is convex and lower semicontinuous, we have

$$\lim_{n\to\infty} \Gamma^*(l_n) = \Gamma^*(l_\mu). \tag{7.5}$$

For all $n \geq 1$, l_n is interior and we have just seen that there exists K_n such that for any $f \in C_o^{1,1}$, $l_n(f) = \int Df\, K_n\, d\Lambda_\mu$, and

$$\Gamma^*(l_n) = \int \tau^*(K_n)\, d\Lambda_\mu. \tag{7.6}$$

As $l_\mu = 2l_2 - l_1$, with $K_\mu = 2K_2 - K_1$, we obtain that for any $n \geq 1$, $K_n = 2\dfrac{n-1}{n} K_2 - \dfrac{n-2}{n} K_1$ and: $l_\mu(f) = \int Df\, K_\mu\, d\Lambda_\mu,\ \forall f \in C_o^{1,1}$, which is (7.1) with $K = K_\mu$.

Considering (4.7), K_1 and K_2 belong to $L^{\tau^*}([0,T] \times (IR^d)^2 \times E, \Lambda_\mu)$ so that $|K_n| \le 2|K_2| + |K_1| \in L^{\tau^*}([0,T] \times (IR^d)^2 \times E, \Lambda_\mu)$. This allows us to apply Lebesgue bounded convergence theorem, to get: $\lim_{n\to\infty} \int \tau^*(K_n)\, d\Lambda_\mu = \int \tau^*(K_\mu)\, d\Lambda_\mu$. It comes out that with (7.5) and (7.6), one gets $\Gamma^*(l_\mu) = \int \tau^*(K_\mu)\, d\Lambda_\mu$, (which is (7.2) with $K = K_\mu$) as well as the first equality in (7.4).

Now, let us look at the converse part of the statement. Let $\mu \in D_{M_\varphi}$. Thanks to (4.9), one shows as in (4.10), that for any measurable function $K \ge -1$ associated with μ by (7.1), one obtains: $J(\mu) \le \int \tau^*(K)\, d\Lambda_\mu$. Therefore

$$J(\mu) \le \inf \left\{ \int \tau^*(K)\, d\Lambda_\mu \,;\, K \text{ satisfies } (7.1) \right\} \tag{7.7}$$

and if there exists K satisfying (7.1) and (7.2), we have $J(\mu) < +\infty$. In such a case, it is already proved that there exists K_μ satisfying (7.1) such that $J(\mu) = \int \tau^*(K_\mu)\, d\Lambda_\mu$. Together with (7.7), this provides us with the second equality in (7.4).

Finally, (7.3) is a consequence of (4.10) and the uniqueness of K_μ comes from the strict convexity of the function $K \mapsto \int \tau^*(K)\, d\Lambda_\mu$ which is minimized on the convex set $\{K\,;\, K \text{ satisfies } (7.1)\}$. This completes the proof of the theorem. ∎

Let us recall a notion introduced by D.A. Dawson and J. Gärtner in [DaG]. Let $\mathcal{D}$ denote the Schwartz space of test functions $IR^d \to IR$ with its usual inductive topology and $\mathcal{D}'$ the corresponding space of real distributions. For each compact set $\kappa \subset IR^d$ let $\mathcal{D}_\kappa$ be the subspace of $\mathcal{D}$ consisting of all test functions with their support in κ.

(7.8) **Definition.** ([DaG]). *A map $\mu(\cdot) : [0,T] \to \mathcal{D}'$ is called* **absolutely continuous** *if for each compact set $\kappa \subset IR^d$ there exist a neighbourhood U_κ of 0 in $\mathcal{D}_\kappa$ and an absolutely continuous function $H_\kappa : [0,T] \to IR$ such that*

$$|\langle g, \mu(u)\rangle - \langle g, \mu(v)\rangle| \le |H_\kappa(u) - H_\kappa(v)| \quad \forall 0 \le u, v \le T,\ \forall g \in U_\kappa.$$

It has been proved in ([DaG], lemma 4.2) that for any absolutely continuous map $\mu(\cdot) : [0,T] \to \mathcal{D}'$ and any g in $\mathcal{D}$, $\langle g, \mu(\cdot)\rangle$ is absolutely continuous and that the derivative in the distribution sense

$$\dot{\mu}(t) = \lim_{h\to 0} \frac{1}{h}[\mu(t+h) - \mu(t)]$$

exists for almost all $0 \le t \le T$.

Proposition 7.2. *If a relaxed path $\mu \in \mathcal{M}_\varphi$ is such that $J(\mu) < +\infty$, then it is absolutely continuous in the sense of (7.8). In particular, it belongs to D_{M_φ}.*

Proof. Notice that for all $0 \le s \le u \le T$, and $f \in C_o^{1,1}$ such that $f(u,\cdot) = f(s,\cdot) \equiv 0$, as in the proof of theorem 7.1, one gets

$$\int_{]s,u[} \langle f(t,\cdot), \dot{\mu}_t\rangle\, dt = \int \mathbb{1}_{]s,u[}(K_\mu + 1) Df\, d\Lambda_\mu$$

(a priori K_μ should depend on s and u, but one can check that it does not, see ([DaG], (4.29))). The absolute continuity follows from Hölder's inequality (see (4.2)):

$$\left| \int 1\!\!1_{]s,u[}(K_\mu + 1) Df \, d\Lambda_\mu \right| \leq 2 \left\| 1\!\!1_{]s,u[} \sqrt{|Df|} \right\|_{\tau, \Lambda_\mu} \left\| (K_\mu + 1) \sqrt{|Df|} \right\|_{\tau^*, \Lambda_\mu} .$$

Notice that K_μ belongs to L^{τ^*} and that Df has a compact support in $[0, T] \times (I\!R^d)^2 \times E$, so that $(K_\mu + 1)\sqrt{|Df|}$ belongs to L^{τ^*} and $1\!\!1_{]s,u[}\sqrt{|Df|}$ belongs to L^τ. This completes the proof of the proposition. ∎

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