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CHRISTIAN LÉONARD

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# Large deviations for long range interacting particle systems with jumps

by

**Christian LÉONARD**

Équipe de Modélisation Stochastique et Statistique,
U.R.A. C.N.R.S. D 0743, Université de Paris-Sud,
Département de Mathématiques, Bâtiment 425,
91405 Orsay Cedex, France.

ABSTRACT. – We prove large deviation principles for the empirical measure and the empirical process of large Markovian particle systems with a long range interaction.

*Key words:* Large deviations, interacting random process, jump processes, random measures.

RÉSUMÉ. – On prouve des principes de grandes déviations pour la mesure empirique et pour le processus empirique associés à des grands systèmes de particules en évolution markovienne subissant une interaction à longue portée.

## 1. INTRODUCTION

In this paper, we establish large deviation principles for a system of long range interacting particles in evolution, when the number of particles tends to infinity.

**a) The particle system.** Let us describe this particle system. The particle $i$ $(1 \leq i \leq N,\ N \geq 1)$ stands at site

$$s_i^N \in \mathcal{S} \subset \mathbb{R}^k$$

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and its random value at time $t$ is

$$X_i^N(t) \in \mathcal{Z} \subset \mathbb{R}^d,$$

so that the process $X_i^N$ takes its values in the set: $D([0, T], \mathcal{Z})$, of all càdlàg paths from $[0, T]$ $(T \geq 0)$ to $\mathcal{Z}$ and the system of $N$ particles: $X^N = (X_i^N)_{1 \leq i \leq N}$, is a $D([0, T], \mathcal{Z}^N) \cong D([0, T], \mathcal{Z})^N$-valued random variable. As a definition, we set that any path in $D([0, T], \mathcal{Z})$ is left continuous at $T$. Since the Lévy kernels we shall consider are absolutely continuous with respect to the Lebesgue measure $dt$, this corresponds to a negligeable modification of the processes.

The law of $X^N$ is a solution to the martingale problem on $D\left([0, T], \mathcal{Z}^N\right)$ associated with the Markov infinitesimal generator which is defined for some measurable functions $\Phi$ on $\mathcal{Z}^N$, by

$$\begin{aligned} A_N \Phi(z^N) = \sum_{i=1}^{N} \int_E \{\Phi(z^N + \Delta_{(i)}) - \Phi(z^N)\} \\ \times \mathcal{L}\left(z_i, s_i^N, \frac{1}{N}\sum_{j=1}^{N} \delta_{(z_j, s_j^N)}; d\Delta\right). \end{aligned} \tag{1.1}$$

We have denoted the set of all jumps for one particle

$$E = (\mathcal{Z} - \mathcal{Z}) \backslash \{0\}$$

and

$$z^N = (z_1, z_2, \ldots, z_N) \in \mathcal{Z}^N$$

and for any $\Delta \in E$, $1 \leq i \leq N$

$$\Delta_{(i)} = (0, \ldots, 0, \underset{\substack{\uparrow \\ i^{\text{th}}\text{-coordinate}}}{\Delta}, 0, \ldots, 0) \in \mathcal{Z}^N.$$

The family of generators $(A_N)_{N \geq 1}$ is determined by the Lévy kernel

$$\mathcal{L}: \quad (z, s, \xi) \in \mathcal{Z} \times \mathcal{S} \times M_1(\mathcal{Z} \times \mathcal{S}) \mapsto \mathcal{L}(z, s, \xi; \cdot) \in M_+(E)$$

where $M_+(E)$ is the set of all Radon nonnegative measures on $E$ and for any topological space $Y$, $M_1(Y)$ is the set of all probability measures on $Y$ endowed with its Borel $\sigma$-field $\mathcal{B}(Y)$.

The space $D([0, T], \mathcal{Z})$ is equipped with the Skorokhod topology, $\mathcal{B}(D([0, T], \mathcal{Z}))$ is the corresponding $\sigma$-field and $M_1(D([0, T], \mathcal{Z}) \times \mathcal{S})$ is the space of all probability measures on

$$(D([0, T], \mathcal{Z}) \times \mathcal{S}, \mathcal{B}(D([0, T], \mathcal{Z}) \times \mathcal{S})).$$

We are going to consider the random empirical measure on $D([0, T], \mathcal{Z}) \times \mathcal{S}$ defined by

$$\hat{X}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_i^N, s_i^N)} \in M_1(D([0, T], \mathcal{Z}) \times \mathcal{S})$$

and the $M_1(\mathcal{Z} \times \mathcal{S})$-valued càdlàg empirical process

$$\bar{X}^N: \quad t \in [0, T] \mapsto \bar{X}^N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_i^N(t), s_i^N)} \in M_1(\mathcal{Z} \times \mathcal{S}).$$

**b) An example.** Let us consider $N$ random spins located at the sites $\frac{i}{N}$ $(1 \leq i \leq N)$ on the circle $\mathcal{S} = \mathbb{R}/\mathbb{Z}$. Each spin takes its values in $\mathcal{Z} = \{-1, +1\}$. The statistical equilibrium of this system is described by the Gibbs probability measure on $\mathcal{Z}^N$:

$$(1.2) \quad \frac{1}{\Gamma_N} \exp\left[\sum_{i=1}^{N} z_i \frac{1}{N} \sum_{j=1}^{N} F\left(\frac{i}{N} - \frac{j}{N}\right) z_j\right] \bigotimes_{i=1}^{N} \frac{1}{2}(\delta_{-1} + \delta_{+1})(dz_i),$$

where $\Gamma_N$ is a normalizing constant, $F$ is a real symmetric function on $\mathcal{S}$ and $\delta_a$ stands for the Dirac measure at point $a$. Using the detailed balance conditions, it is easily checked that (1.2) is the unique invariant probability measure of the Markov process on $\mathcal{Z}^N$ associated with the generator defined for any function $\Phi$ on $\mathcal{Z}^N$ by

$$(1.3) \quad \sum_{i=1}^{N} \{\Phi(\widetilde{z^{N^i}}) - \Phi(z^N)\} \gamma \times \exp\left[\frac{4F(0)}{N} - \frac{2z_i}{N} \sum_{j=1}^{N} F\left(\frac{i}{N} - \frac{j}{N}\right) z_j\right]$$

where $z^N = (z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_N)$, $\widetilde{z^{N^i}} = (z_1, \ldots, z_{i-1}, -z_i$ $z_{i+1}, \ldots, z_N)$ and $\gamma$ is a positive constant.

The large deviations as $N$ tends to infinity for (1.2) have been studied by T. Eisele and R. S. Ellis in [EiE], while they have been studied for $(\bar{X}^N)_{N \geq 1}$ under (1.3) by F. Comets in [Com]. In the present paper, we extend the results of [Com] by means of an alternate proof to the more general interacting particle system (1.1) which includes (1.3).

*Remark.* – The generator (1.3) is of the form $A_N + O_{N\to\infty}(1)$ with

$$\mathcal{L}(z, s, \xi) = \exp\left(-2z\int_{\mathcal{Z}\times\mathcal{S}} F(s-s')\, z'\, \xi(dz'\, ds')\right) \times (\mathbf{1}_{\{z=+1\}}\,\delta_{-2} + \mathbf{1}_{\{z=-1\}}\,\delta_{+2}).$$

But our large deviation results are insensitive to perturbations of order $o_{N\to\infty}(N)$ so that the results of this paper hold for the long range Ising model given by (1.3).

**c) The law of large numbers.** Clearly, the evolution of $\bar{X}_t^N$ is specified by the values of $A_N\,\Phi_f$, where

$$\Phi_f(z^N) = \frac{1}{N}\sum_{i=1}^{N} f(z_i, s_i^N), \qquad z^N \in \mathcal{Z}^N,$$

when $f$ describes $C_c(\mathcal{Z}\times\mathcal{S})$.

From now on, $C_c(Y)$ will stand for the set of all continuous real functions with a compact support on a topological set $Y$. Denoting

$$\widehat{zs}^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{z_i, s_i^N}, \qquad z^N \in \mathcal{Z}^N,$$

we obtain for any $f$ in $C_c(\mathcal{Z}\times\mathcal{S})$

$$A_N\,\Phi_f(z^N) = \int_{\mathcal{Z}\times\mathcal{S}} A(\widehat{zs}^N)\, f(z, s)\, \widehat{zs}^N(dz\, ds)$$

where

$$(1.4)\quad \begin{cases} A(\xi)\, f(z, s) = \displaystyle\int_E \{f(z+\Delta, s) - f(z, s)\}\, \mathcal{L}(z, s, \xi, d\Delta), \\ \qquad (z, s) \in \mathcal{Z}\times\mathcal{S}, \qquad \xi \in M_1(\mathcal{Z}\times\mathcal{S}). \end{cases}$$

Hence, if a law of large numbers is satisfied, *i.e.*

$$(1.5a)\qquad \bar{X}^N \xrightarrow[N\to\infty]{\mathcal{L}} u \quad \text{in } D([0, T], M_1(\mathcal{Z}\times\mathcal{S})),$$

one expects the limit $u = (t \mapsto u_t)$ to be a solution to the weak non linear integrodifferential equation

$$(1.5b)\quad \left\langle f, \frac{\partial}{\partial t} u_t\right\rangle = \langle A(u_t)\, f, u_t\rangle, \quad \forall f \in C_c(\mathcal{Z}\times\mathcal{S}); \quad 0 \le t \le T.$$

Such laws of large numbers have been studied by many authors, mainly in the case of continuous diffusions where $A_N$ is given by

$$A_N \Phi(z^N) = \sum_{i=1}^{N} b\left(z_i, \frac{1}{N}\sum_{j=1}^{N}\delta_{z_j}\right) \cdot \Phi'(z_i) + \sum_{i=1}^{N} \frac{1}{2} \operatorname{trace}(a(z_i) \circ \Phi''(z_i)) \tag{1.6}$$

and $\mathcal{S}$ does not appear. On this topic, let us quote H. P. McKean [McK] and K. Oelschläger [Oel]. The law of large numbers for $\hat{X}^N$ with the evolution (1.6) has been obtained by A. S. Sznitman [Sz1] and extended by J. Gärtner in [Gär] and the author [Lé1]. A similar result for $\hat{X}^N$ with $A_N$ given by (1.1) would be

$$\hat{X}^N \xrightarrow[N\to\infty]{\mathcal{L}} Q \quad \text{in } M_1(D([0, T], \mathcal{Z}) \times \mathcal{S}) \tag{1.7}$$

where $Q$ is a solution to the nonlinear martingale problem (*see* [Sz1], for this notion) on $D([0, T], \mathcal{Z} \times \mathcal{S})$ associated with the generator (1.4).

It is easy to prove by means of the large deviations upper bounds of Theorems 2.1 and 3.1 below, that (1.5) and (1.7) hold under the assumptions of these theorems for the particle systems governed by (1.1). One even gets a strong law version of these results, using Borel-Cantelli lemma.

An important consequence of (1.7) in the special case where $\mathcal{S}$ does not appear (take $\mathcal{S} = \{\star\}$, for instance), is the following property of *propagation of chaos*. Suppose that at time $t = 0$, the particles $(X_i^N(0))_{1\le i\le N}$ are i.i.d. $(u_o)$, then for any $k \ge 1$ and any collection $(g_j)_{1\le j\le k}$ of continuous bounded functions on $D([0, T], \mathcal{Z})$, one gets

$$E\left(\prod_{j=1}^{k} g_j(X_j^N)\right) \xrightarrow[N\to\infty]{} \prod_{j=1}^{k} \langle g_j, Q\rangle$$

where $Q$ is given by (1.7) with $Q_0 = u_0$ (*see* for instance [Sz3]). In particular, for any collection $(g_j)_{1\le j\le k}$ of continuous bounded functions on $\mathcal{Z}$ and for any $0 \le t \le T$, one gets

$$E\left(\prod_{j=1}^{k} f_j(X_j^N(t))\right) \xrightarrow[N\to\infty]{} \prod_{j=1}^{k} \langle f_j, u_t\rangle$$

where $u$ is given by (1.5) with $u_0 = u_o$

**d) Some literature connected with this paper.** About the law of large numbers, we have already quoted [Gär], [McK], [Lé1], [Oel], [Sz1] and

[Sz3]. Several authors have studied the fluctuations associated with these laws of large numbers. Among others, let us mention G. Ben Arous and M. Brunaud in [BAB], D. A. Dawson in [Daw] and A. S. Sznitman in [Sz1] for particle systems governed by (1.6) and T. Shiga and H. Tanaka in [ShT] for particle systems governed by (1.1).

Large deviations for $\bar{X}^N$ have been investigated by F. Comets for (1.3) in [Com] and by D. A. Dawson and J. Gärtner in [DaG] for (1.6) under quite general assumptions on $b$ and $a$. In [Tan], H. Tanaka proved a large deviation principle for $\hat{X}^N$ under (1.6) with $a = \text{Id}$. Recently and independently, M. Sugiura [Sug] and S. Feng [Fen] gave alternate proofs for the large deviation principle for $\bar{X}^N$, under different hypotheses from ours. In any cases, these three approaches of the same problem are quite different. Let us also mention that special examples of systems governed by generators of the type (1.1), arising from epidemiology, are described in [Lé2].

Because of the contraction principle, if the rate function of the large deviation principle for $\hat{X}^N$ is $I$, the rate function $J$ of the large deviation principle for $\bar{X}^N$ is

$$\begin{aligned} J(\mu(\cdot)) = \inf \{ I(Q);\ & Q \in M_1(D([0, T], \mathcal{Z}) \times \mathcal{S}) \\ & \text{such that } Q_t = \mu(t), \forall\, 0 \le t \le T \}. \end{aligned}$$

An alternate way to obtain a concrete expression for $J(\mu(\cdot))$ is to solve this variational problem. This program has been performed in the case (1.6) by H. Föllmer [Fö2], with $a = \text{Id}$ and $b = 0$, and by M. Brunaud [Bru], with smooth coefficients $a$ and $b$. In a paper by P. Cattiaux and the author [CaL], this problem is solved for (1.6) with general coefficients. It is reasonable to conjecture similar results for jump Markov processes including (1.1), instead of continuous diffusions. In particular, if it is known that $J(\mu(\cdot))$ is finite, one obtains that the above infimum is attained for a time-nonhomogeneous Markov law which can be completely described in terms of the flow $t \mapsto \mu(t)$.

**e) Outline of the paper.** Section 2 of the present paper is devoted to the proof of the large deviation principle for $\hat{X}^N$. It is based on the Laplace-Varadhan principle (*see* [Var]). A large deviation principle for $\bar{X}^N$ is derived at section 3. Although we follow ([DaG], section 4) very closely, technicalities arising from the jumps of the processes force us to introduce Orlicz spaces related to the log-Laplace and the Cramér transforms of the centered Poisson law. This Orlicz space approach has already been used by the author in different contexts (*see* [Lé3]). At the end of the paper, one can find an appendix containing basic material about Orlicz spaces.

## 2. LARGE DEVIATIONS FOR THE EMPIRICAL MEASURES

The main result of this section is theorem 2.1 where a large deviation principle is stated for the empirical measures $\{\hat{X}^N; N \geq 1\}$. Its proof relies on the Laplace-Varadhan principle [Var].

**a) Statement of theorem 2.1.** Let $\{s_i^N; 1 \leq i \leq N, N \geq 1\}$ be a triangular array in $\mathcal{S} \subset \mathbb{R}^k$ such that there exists $m \in M_1(\mathcal{Z})$ such that

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{s_i^N} \xrightarrow[N \to \infty]{} m \in M_1(\mathcal{Z}).$$

For sake of simplicity we shall assume that there exists $z_o \in \mathcal{Z}$ such that the initial conditions satisfy

$$(A.0) \qquad X_i^N(0) = z_o, \qquad \text{for any } 1 \leq i \leq N, \quad N \geq 1.$$

Let us consider the generators $A_N$ $(N \geq 1)$ defined at (1.1). We assume that the Lévy kernel $\mathcal{L}$ satisfies the following assumptions.

$$(A.1) \qquad \Lambda \text{ is a bounded measure on } E,$$

$$(A.2) \qquad 0 < \inf\left\{\frac{d\mathcal{L}(z, s, \xi)}{d\Lambda}(\Delta); (z, s, \xi, \Delta) \in \mathcal{Z} \times \mathcal{S} \times M_1(\mathcal{Z} \times \mathcal{S}) \times E\right\}$$
$$\leq \sup\left\{\frac{d\mathcal{L}(z, s, \xi)}{d\Lambda}(\Delta); (z, s, \xi, \Delta) \in \mathcal{Z} \times \mathcal{S} \times M_1(\mathcal{Z} \times \mathcal{S}) \times E\right\} < +\infty,$$

$$(A.3) \qquad (z, s, \xi, \Delta) \mapsto \frac{d\mathcal{L}(z, s, \xi)}{d\Lambda}(\Delta) \text{ is continuous on } \mathcal{Z} \times \mathcal{S} \times M_1(\mathcal{Z} \times \mathcal{S}) \times E$$

and

$$(A.4) \qquad \left\{\frac{d\mathcal{L}(\cdot, \cdot, \cdot)}{d\Lambda}(\Delta); \Delta \in E\right\} \text{ is a uniformly equicontinuous family of functions on } \mathcal{Z} \times \mathcal{S} \times M_1(\mathcal{Z} \times \mathcal{S}).$$

Since $(A.1)$ and $(A.2)$ imply $\sup \{\mathcal{L}(z, s, \xi; E); (z, s, \xi) \in \mathcal{Z} \times \mathcal{S} \times M_1(\mathcal{Z} \times \mathcal{S})\} < +\infty$, (1.1) makes sense for any bounded measurable function $\Phi$ on $\mathcal{Z}^N$ and the martingale problem associated with the generator $A_N$ on the domain $B_b(\mathcal{Z}^N)$ of all bounded measurable functions admits a unique solution

$$Q_N \in M_1(D([0, T], \mathcal{Z}^N)), \qquad N \geq 1$$

such that $\sum_{i=1}^{N} \int_0^T \mathcal{L}(X_i^N(t), s_i^N, \bar{X}^N(t); E)\, dt < \infty$, $Q_N$-a.s. (*see* [Sz2], Appendice, lemme 1).

In this section, we are interested in the law $\mathbb{P}_N$ of $\hat{X}^N$, that is the image of $Q_N$ on $M_1(D([0, T], \mathcal{Z}) \times \mathcal{S})$:

$$\mathbb{P}_N = Q_N \circ (\gamma_N)^{-1}$$

by the measurable application

$$\begin{aligned} \text{(2.1)} \qquad \gamma_N: \quad & (x_i)_{1 \leq i \leq N} \in D([0, T], \mathcal{Z}^N) \\ & \mapsto \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i, s_i^N)} \in M_1(D([0, T], \mathcal{Z}) \times \mathcal{S}). \end{aligned}$$

In order to state Theorem 2.1 let us give some notations.

Let $\tau$ be the log-Laplace transform of the centred Poisson law with parameter 1:

$$\text{(2.2)} \qquad \tau(u) = \exp(u) - u - 1, \qquad \forall\, u \in \mathbb{R}$$

and $\tau^*$ its Legendre conjugate

$$\text{(2.3)} \qquad \tau^*(u) = \begin{cases} (u+1)\log(u+1) - u & \text{if } u > -1 \\ 1 & \text{if } u = -1 \\ +\infty & \text{if } u < -1. \end{cases}$$

Their restrictions to $\mathbb{R}_+$ are Orlicz functions and the corresponding Orlicz spaces are denoted $L^\tau$ (*space, measure*) and $L^{\tau^*}$ (*space, measure*).

For all nonnegative measures $\mathcal{K}$ and $\mathcal{L}$ on a measurable space $\mathcal{R}$, one defines the Kullback information of $\mathcal{K}$ with respect to $\mathcal{L}$ by

$$I(\mathcal{K}|\mathcal{L}) = \begin{cases} \displaystyle\int_{\mathcal{R}} \tau^*\left(\frac{d\mathcal{K}}{d\mathcal{L}} - 1\right) d\mathcal{L} & \text{if } \mathcal{K} \ll \mathcal{L} \\ +\infty & \text{otherwise.} \end{cases}$$

In particular, if $\mu$ and $\rho$ are probability measures on $\mathcal{R}$, the Kullback information of $\mu$ with respect to $\rho$ is

$$I(\mu \mid \rho) = \begin{cases} \int_{\mathcal{R}} \log \left( \frac{d\mu}{d\rho} \right) d\mu & \text{if} \quad \mu \ll \rho \\ + \infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{R}$ and $\mathcal{S}$ be two measurable spaces, viewing any $\xi$ in $M_1(\mathcal{R} \times \mathcal{S})$ as the law of a $\mathcal{R} \times \mathcal{S}$-valued random vector $(Z, S)$, $\xi_S$ denotes the law of $S$ on $\mathcal{S}$ and $\xi^s$ denotes a regular version of the conditional law on $\mathcal{R}$ of $Z$ knowing that $S = s$.

The jump $x_t - x_{t^-}$ is denoted $\Delta x_t$ and $Q_t$ is the law of the random vector $(X_t, S)$ on $\mathcal{Z} \times \mathcal{S}$ where $Q$ is considered as the law of a random vector $(X, S)$ on $D([0, T], \mathcal{Z}) \times \mathcal{S}$.

Let $P_{z_o}$ be the unique solution [by $(A.1)$] to the martingale problem in $D([0, T], \mathcal{Z})$ associated with the generator

$$A^o \Phi(z) = \int_E \{\Phi(z + \Delta) - \Phi(z)\} \Lambda(d\Delta)$$

on the domain $B_b(\mathcal{Z})$ and the initial condition $\delta_{z_o}$.

For any $\mu$ in $D([0, T], M_1(\mathcal{Z} \times \mathcal{S}))$, let $R(\mu)$ be the probability measure on $D([0, T], \mathcal{Z}) \times \mathcal{S}$ defined by its density with respect to $P_{z_o} \otimes m$:

$$\begin{aligned} (2.4) \quad & \frac{dR(\mu)}{d(P_{z_o} \otimes m)}(x, s) \\ & = \exp \left( \sum_{0 \le t \le T} \mathbf{1}_{\{x_t \neq x_{t^-}\}} \log \left( \frac{d\mathcal{L}(x_{t^-}, s, \mu(t^-))}{d\Lambda}(\Delta x_t) \right) \right. \\ & \qquad - \int_0^T \int_E \left\{ \frac{d\mathcal{L}(x_t, s, \mu(t))}{d\Lambda}(\Delta) - 1 \right\} \\ & \qquad \left. \times \mathbf{1}_{\{x_t + \Delta \in \mathcal{Z}\}} \Lambda(d\Delta)\, dt \right). \end{aligned}$$

In other words, $R(\mu)$ is the unique solution to the martingale problem in $D([0, T], \mathcal{Z} \times \mathcal{S})$ associated with the Lévy kernel $\{\mathcal{L}(z, s, \mu_t) \otimes \delta_0;\ (z, s, t) \in \mathcal{Z} \times \mathcal{S} \times [0, T]\}$ ($\delta_0$ represents the absence of jumps for the process $S_t = S, \forall 0 \le t \le T$) with initial condition: $Law(X_0, S_0) = \delta_{z_o} \otimes m$ and which is absolutely continuous with respect to $P_{z_o} \otimes m$. The existence of $R(\mu)$ is a direct consequence of the uniform boundedness of $\mathcal{L}$ [by assumptions $(A.1)$ and $(A.2)$] and of Novikov's criterion.

For any $Q \in M_1(D([0, T], \mathcal{Z}) \times \mathcal{S})$, $R(Q)$ will stand for $R((Q_t)_{0 \le t \le T})$.

THEOREM 2.1. – *Under the assumptions* $(A.0)$-$(A.4)$, *the sequence of probability measures* $(\mathbb{P}_N)_{N\geq 1}$ *obeys a large deviation principle in* $M_1(D([0, T], \mathcal{Z}) \times \mathcal{S})$ *endowed with its usual weak topology. Its speed is* $N$ *and its rate function is*

$$I(Q) = \begin{cases} I(Q \mid R(Q)) = \int_S I(Q^s \mid R(Q)^s)\, m(ds) \\ \qquad if \quad Q_S = m \ \ and\, Q_0 = \delta_{z_o} \otimes m \\ +\infty \quad otherwise. \end{cases}$$

*Moreover, if* $Q \in M_1(D([0, T], \mathcal{Z}) \times \mathcal{S})$ *satisfies* $I(Q) < \infty$, *then* ($Q_S = m$ *and*) *one can find a Lévy kernel*

$$\{\mathcal{K}_Q(x, s, t);\ (x, s, t) \in D([0, T], \mathcal{Z}) \times \mathcal{S} \times [0, T]\} \subset M_+(E)$$

*such that:*

*for* $m$*-almost every* $s \in \mathcal{S}$, $Q^s$ *is the unique solution to the martingale problem associated with* $\mathcal{K}_Q(s)$ *on the domain* $B_b(\mathcal{Z})$, *which is absolutely continuous with respect to* $R(Q)^s$, $\mathcal{K}_Q(x, s, t) \ll \mathcal{L}(x_t, s, Q_t)$ *for* $Q(dx\, ds)\, dt$*-almost every* $(x, s, t)$ *and*

$$I(Q) = \int_0^T dt \int_{D([0, T], \mathcal{Z}) \times \mathcal{S}} I(\mathcal{K}_Q(x, s, t) \mid \mathcal{L}(x_t, s, Q_t))\, Q(dx\, ds).$$

*Remark.* – Technicalities of the proof of this result will lead us to a stronger result which is stated at Theorem 2.6.

The rest of this section is devoted to the proof of Theorem 2.1.

For more simplicity, this result will be proved without the variable $s \in \mathcal{S}$. The announced result is an easy extension of the $s$-independent one, its proof is left to the reader.

**b) The noninteracting case.** Let us begin to study the simple case when the Lévy kernel $\mathcal{L}$ is equal to $\Lambda$. The large deviation principle for the noninteracting system is a particular case of the following extension of Sanov's theorem.

Let $\mathcal{R}$ be a Polish space and $\varphi$ a nonnegative continuous function on $\mathcal{R}$. One denotes

$$C_\varphi(\mathcal{R}) = \{f;\ f : \mathcal{R} \to \mathbb{R},\ \text{continuous and } \|f\|_\varphi < +\infty\}$$

with

$$\|f\|_\varphi = \sup_{x \in \mathcal{R}} \frac{|f(x)|}{1 + \varphi(x)}$$

and

$$M_{\varphi}(\mathcal{R}) = \left\{\xi;\ \xi \in M_1(\mathcal{R}) \text{ such that } \int_{\mathcal{R}} \varphi(x)\,\xi(dx) < +\infty\right\}.$$

The set $M_{\varphi}(\mathcal{R})$ is endowed with the weak-$\star$ topology $\sigma(M_{\varphi}(\mathcal{R}), C_{\varphi}(\mathcal{R}))$.

PROPOSITION 2.2. – *Let $(Z_i)_{i\geq 1}$ be an i.i.d. sequence of random variables with common law $\rho$ on a Polish space $\mathcal{R}$ and let $\varphi$ be a nonnegative continuous real function on $\mathcal{R}$, such that*

$$for\ any\ \alpha > 0, \qquad \int_{\mathcal{R}} \exp(\alpha\varphi(x))\,d\rho(x) < +\infty.$$

*The sequence of the laws of the empirical measures*

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{Z_i}, \qquad N \geq 1$$

*obeys a large deviation principle in $M_{\varphi}(\mathcal{R})$ with speed N and rate function:*

$$\xi \mapsto J(\xi) = I(\xi \mid \rho).$$

*Moreover, any $\xi \in M_1(\mathcal{R})$ such that $J(\xi) < +\infty$ belongs to $M_{\varphi}(\mathcal{R})$.*

*Proof.* – This result is a simple modification of Sanov's theorem. ■

For technical reasons which will appear later at the next subsection, it is convenient to introduce the following space of paths

$$(2.5) \qquad \chi = \{x \in D([0, T], \mathcal{Z}),\ \sum_{0\leq t<T} \mathbf{1}_{\{x_t \neq x_{t-}\}} < +\infty\}.$$

We endow $\chi$ with a topology strong enough to turn the function

$$(2.6) \qquad \varphi:\quad x \in \chi \mapsto \sum_{0\leq t<T} \mathbf{1}_{\{x_t \neq x_{t-}\}} \in \mathbb{R}$$

into a continuous function. This topology is given by the metric $d$ on $\chi$ which is defined by:

$$d(x, y) = d_{\text{Sko}}(x, y) + \Big|\sum_{0\leq t<T} (\mathbf{1}_{\{x_t \neq x_{t-}\}} - \mathbf{1}_{\{y_t \neq y_{t-}\}})\Big|, \quad x, y \in \chi$$

where $d_{\text{Sko}}$ stands for a Skorokhod complete metric. The space $(\chi, d)$ is still Polish. For this topology, two paths are close to each other if they have the same number of jumps and if they are Skorokhod-close. Although it is rather technical, one can show by standard arguments (*see* [Bil]) that the

Borel $\sigma$-field of $\chi$ coincide with the $\sigma$-field generated by the elementary cylinders. From now on, $\chi$ will always be equipped with this topology. Let

$$\mathbb{P}_N^o = \bigotimes_{i=1}^{N} P_{z_o} \circ (\gamma_N)^{-1}$$

be the image on $M_1(D([0, T], \mathcal{Z}))$ by the application $\gamma_N$ [*see* (2.1)] of the law of the system of $N$ noninteracting particles with common law $P_{z_o}$, *i.e.* $P_{z_o}^{\otimes N}$. Clearly, under our assumptions, $\mathbb{P}_N^o$-almost every paths of the noninteracting system lie in $\chi$.

A direct application of Proposition 2.2 provides us with the following large deviation principle for $(\mathbb{P}_N^o)_{N \geq 1}$ in $M_\varphi(\chi)$.

LEMMA 2.3. – *The sequence of probability measures $(\mathbb{P}_N^o)_{N\geq 1}$ obeys a large deviation principle in $M_\varphi(\chi)$ with speed $N$ and rate function*

$$Q \mapsto J(Q) = I(Q \mid P_{z_o}).$$

*Moreover, any $Q \in M_1(D([0, T], \mathcal{Z}))$ such that $J(Q) < +\infty$ belongs to $M_\varphi(\chi)$.*

Now, let us give an alternate expression for $J(Q)$.

LEMMA 2.4. – *Let $Q$ in $M_\varphi(\chi)$ be such that $J(Q) < +\infty$. There exists a nonnegative function $L_Q$ which belongs to $L^{\tau^*}(\chi \times [0, T] \times E, \Lambda(d\Delta)\, Q(dx)\, dt)$ such that $Q$ is a solution to the martingale problem associated with the Lévy kernel $L_Q \cdot \Lambda$ on the domain $B_b(\mathcal{Z})$,*

$$J(Q) = \int_{\chi\times[0,T]\times E} \tau^*(L_Q(x, t, \Delta) - 1)\, \Lambda(d\Delta)\, Q(dx)\, dt$$

*and for all $0 \leq u \leq v \leq T$ and all $f \in B_b(\mathcal{Z})$,*

$$\int_{\mathcal{Z}} f\, d(Q_v - Q_u) = \int_u^v dt \int_{\mathcal{Z}\times E} \{f(z+\Delta) - f(z)\}\, l_Q(z, t, \Delta)\, \Lambda(d\Delta)\, Q_t(dz)$$

*where $l_Q$ is a version of the conditional expectation*

$$l_Q(z, t, \Delta) = \gamma\, E^{\Pi_Q}(L_Q(X, t, \Delta) \mid X_t = z)$$

*with $\Pi_Q(dx\, dt\, d\Delta) = \dfrac{1}{\gamma} Q(dx)\, dt\, \Lambda(d\Delta)$ and $\gamma = T\,\Lambda(E)$ (so that $\Pi_Q$ is a probability measure).*

*Proof.* – We follow [Föl]. Since $J(Q) < +\infty$, we have $Q \ll P_{z_o}$. By Girsanov's theorem (*see* [Jac], Theorems 12.17, 12.24, 12.48), there exists a (previsible) nonnegative function $L_Q$ on $\chi \times [0, T] \times E$ such that $Q$ is a solution to the martingale problem associated with the Lévy kernel $L_Q \cdot \Lambda$ on the domain $B_b(\mathcal{Z})$. With the stopping time

$$T_n = \inf\left\{t \geq 0; \int_{[0,T]\times E} \times \left(1 - \sqrt{L_Q(X, u, \Delta)}\right)^2 \Lambda(d\Lambda)\, du \geq n\right\} \wedge T, \qquad (n \geq 1)$$

the density process of $Q$ with respect to $P_{z_o}$ is

$$Z_t = \begin{cases} \exp\left(\int_0^T (L_Q(t, \cdot) - 1)\, dM_t - \sum_{0 \leq t < T} \mathbf{1}_{\{x_t \neq x_{t-}\}}\, \tau(\log L_Q(t, \Delta X_t))\right) & \text{on } \bigcup_{n\geq 1} [\![0, T_n]\!] \\ 0 & \text{on } \bigcap_{n \geq 1} ]\!]T_n, T]\!] \end{cases}$$

where the compensated sum of jumps $M_t = X_t - z_0 - t \int_E \Delta\Lambda(d\Delta)$ is a $P_{z_o}$-martingale and $\int_0^t Y_s\, dM_s$ is the stochastic integral of $Y$ with respect to $M$. [The expression of $M_t$ is formal since $\int_E \Delta\Lambda(d\Delta)$ may be undefined; we shall keep this integral notation for all compensated sum of jumps.]

We also know that $N_t = M_t - \int_0^t \int_E (L_Q(\cdot, \Delta) - 1)\, \Delta\Lambda(d\Delta)\, du$ is a local $Q$-martingale which is localized by the sequence $(T_n)_{n\geq 1}$. Let $Q_n = Z_{T_n} \cdot P_{z_o}$. By Novikov's criterion ([Jac], th. 8.25), $Q_n$ is a probability measure, $Q \ll Q_n$ and these two measures are equal in restriction to the $\sigma$-field $\mathcal{F}_{T_n}$, so that

$$\begin{aligned} I(Q_n \mid P_{z_o}) &= E^{Q_n}\left[\sum_{0\leq t \leq T_n} \mathbf{1}_{\{x_t \neq x_{t-}\}} \log(L_Q(t, \Delta x_t)) - \int_{[0,T_n]\times E} \{L_Q(t, \Delta) - 1\}\, \mathbf{1}_{\{x_t + \Delta \in \mathcal{Z}\}}\, \Lambda(d\Delta)\, dt\right] \\ &= E^{Q_n}\left[\int_0^{T_n} \log L_Q(t)\, dN_t + \int_{[0,T_n]\times E} \tau^*(L_Q(t, \Delta) - 1)\, \Lambda(d\Delta)\, dt\right] \\ &= E^Q\left[\int_{[0,T_n]\times E} \tau^*(L_Q(t, \Delta) - 1)\, \Lambda(d\Delta)\, dt\right]. \end{aligned}$$

On the other hand, $I(Q_n \mid P_{z_o}) = E^{P_{z_o}}[Z_{T_n} \log Z_{T_n}]$. Since the function $x \log x$ is bounded below, one can apply Fatou's lemma and deduce from $T_n \xrightarrow[n\to\infty]{} T$, $Q$-a.s. that

$$I(Q \mid P_{z_o}) \leq E^Q \int_{[0,T]\times E} \tau^* (L_Q(t, \Delta) - 1) \Lambda(d\Delta)\, dt$$

(notice that $\liminf_{n\to\infty} Z_{T_n} \log Z_{T_n} = Z_T \log Z_T$, $P_{z_o}$-a.s. since $Z_s = 0$ if $s > T_n$, $\forall\, n \geq 1$). We also have

$$I(Q \mid P_{z_o}) = I(Q \mid Q_n) + E^Q[\log Z_{T_n}] \geq E^Q[\log Z_{T_n}] = I(Q_n \mid P_{z_o})$$

then, taking the $\sup_{n\geq 1}$ in the right hand side, one deduces the converse inequality and the announced expression for $J(Q)$.

By Itô's formula, for all $0 \leq u \leq T$, all $n \geq 1$ and all $f \in B_b(\mathcal{Z})$

$$f(X_{v\wedge T_n}) - f(X_{u\wedge T_n}) - \int_{u\wedge T_n}^{v\wedge T_n} dt \int_E \{f(X_t + \Delta) - f(X_t)\} L_Q(X_t, t, \Delta) \Lambda(d\Delta)$$

$(u \leq v \leq T)$ is a $Q$-martingale. Since $T_n \xrightarrow[n\to\infty]{} T$, $Q$-a.s., $f$ is bounded and $L_Q$ belongs to $L^1(\chi \times [0, T] \times E, \Lambda(d\Delta)\, Q(dx)\, dt)$, taking the $Q$-expectation of this martingale and letting $n$ tend to infinity, one can apply the bounded convergence theorem to obtain the second part of the lemma. ■

**c) The proof of Theorem 2.1.** Our proof is based on the Laplace-Varadhan principle which is stated at the next proposition.

PROPOSITION 2.5. – *Let $\{\rho^N; N \geq 1\}$ be a sequence of probability measures which obeys a large deviation principle in a topological space $\mathcal{T}$, with a rate function $J$ and speed $N$. Let $h$ be a real function on $\mathcal{T}$ which is continuous on $\{J < +\infty\}$ and such that*

$$(2.7) \quad \textit{for any } \alpha > 0, \limsup_{N\to\infty} \frac{1}{N} \log \int_{\mathcal{T}} \exp(N\alpha\, |h|)\, d\rho^N < +\infty.$$

*Then,* $\frac{1}{N} \log \int_{\mathcal{T}} \exp(N h)\, d\rho^N \xrightarrow[N\to\infty]{} \sup_{x\in\mathcal{T}} \{h(x) - J(x)\} < +\infty$ *and the sequence of probability measures*

$$\left\{ \frac{\exp(N h)}{\int_{\mathcal{T}} \exp(N h)\, d\rho^N} \cdot \rho^N;\ N \geq 1 \right\}$$

*obeys a large deviation principle in* $\mathcal{T}$ *with speed* $N$ *and rate function* $x \mapsto J(x) - h(x) - \inf_{y\in\mathcal{T}} \{J(y) - h(y)\}$.

*Proof.* – In [Var], it is only required that

$$\lim_{A\to\infty} \limsup_{N\to\infty} \frac{1}{N} \int_{\{h>A\}} \exp(N h)\, d\rho^N = -\infty.$$

It is easy to check that this requirement is satisfied under our assumptions. ■

*The proof of Theorem* 2.1. – Thanks to assumption $(A.1)$, $P_{z_o}^{\otimes N}$ is the unique solution of its corresponding martingale problem, so that by Girsanov's formula (*see* [Jac], théorème 12.48*a*) one obtains

$$(2.8) \qquad d\mathbf{P}_N(Q) = \exp(N h(Q)) \cdot d\mathbf{P}_N^o(Q)$$

where for any $Q$ in $M_1(D([0, T], \mathcal{Z}))$

$$(2.9) \quad h(Q) = \int_{D([0,T],\mathcal{Z})} \left[ \sum_{0\leq t<T} \mathbf{1}_{\{x_t \neq x_{t-}\}} \log \left( \frac{d\mathcal{L}(x_{t-}, Q_{t-})}{d\Lambda} (\Delta x_t) \right) - \int_0^T \int_E \left\{ \frac{d\mathcal{L}(x_t, Q_t)}{d\Lambda}(\Delta) - 1 \right\} \times \mathbf{1}_{\{x_t+\Delta\in\mathcal{Z}\}}\, \Lambda(d\Delta)\, dt \right] Q(dx).$$

One desires to apply Proposition 2.5 to the sequence $(\mathbf{P}_N)_{N\geq 1}$ with $\mathcal{T} = M_1\, D([0, T], \mathcal{Z}))$ and $\rho^N = \mathbf{P}_N^o$. Unfortunately, the term $\int_{D([0,T],\mathcal{Z})} \left[ \sum_{0\leq t<T} \mathbf{1}_{\{x_t\neq x_{t-}\}} \log \left( \frac{d\mathcal{L}(x_{t-}, Q_{t-})}{d\Lambda} (\Delta x_t) \right) \right] Q(dx)$ in $h$

is not defined everywhere on $M_1(D[0, T], \mathcal{Z}))$ and it is neither bounded nor continuous on the subset where it is defined. In order to remedy this situation, we have introduced the topological space $\chi$ at (2.5) which turns the function $\varphi$ [*see* (2.6)] into a continuous function. Note that $\mathbb{P}_N^o$-almost every paths (of the noninteracting system) lie in $\chi$.

We shall show at (2.15) that one can find $C \geq 0$ such that for all $N \geq 1$ and $\mathbb{P}_N^o$-almost every $Q$ in $M_\varphi(\chi)$, $|h(Q)| \leq C\left(1 + \int_\chi \varphi(x)\, Q(dx)\right)$. Hence, the restriction of $h$ to the topological space

$$M_\varphi(\chi) = \left\{Q \in M_\varphi(\chi);\ \int_\chi \varphi(x)\, Q(dx) < +\infty\right\}$$

endowed with the weak-$\star$ topology of pointwise convergence on the space

$$C_\varphi(\chi) = \{f;\ f:\ \chi \to \mathbb{R},\ \text{continuous and } \|f\|_\varphi < +\infty\}$$

with

$$\|f\|_\varphi = \sup_{x\in\chi} \frac{|f(x)|}{1+\varphi(x)},$$

should not be far from being continuous. Indeed, we prove at Lemma 2.9 that $h$ is continuous on $M_\varphi(\chi)$ at any $Q$ such that $J(Q) < +\infty$ where $J$ is given at Lemma 2.3.

In view of the equality (2.8), to apply Proposition 2.5 to $\rho^N = \mathbb{P}_N^o$ in $\mathcal{T} = M_\varphi(\chi)$ with $h$ defined by (2.9) and $J$ given at Lemma 2.3, it remains to check the condition (2.7). This is done at Lemma 2.10 below.

This leads to a large deviation principle for $\{\mathbb{P}_N;\ N \geq 1\}$ in $M_\varphi(\chi)$ with speed $N$ and rate function

$$I(Q) = I(Q\,|\,P_{z_o}) - h(Q) - \inf_{Q'\in M_\varphi(\chi)} \{I(Q\,|\,P_{z_o}) - h(Q')\}.$$

But, $I(Q\,|\,P_{z_o}) = \int_\chi \log\left(\frac{dQ}{dP_{z_o}}\right) dQ$ and $h(Q) = \int_\chi \log\left(\frac{dR(Q)}{dP_{z_o}}\right) dQ$ [*see* (2.4)], so that

$$I(Q\,|\,P_{z_o}) - h(Q) = \int_\chi \log\left(\frac{dQ}{dR(Q)}\right) dQ = I(Q\,|\,R(Q)).$$

Since $\int_{M_\varphi(\chi)} \exp(N\,h)\, d\mathbb{P}_N^o = 1,\ \forall N \geq 1$, a consequence of the proof of Proposition 2.5 (*see* [Var]) is: $\inf_{Q'\in M_\varphi(\chi)} \{I(Q'\,|\,P_{z_o}) - h(Q')\} = 0$. This gives the expression of $I(Q)$.

The second part of the theorem is given by Lemma 2.4 with $R(Q)$, $\mathcal{L}(Q)$ and $\dfrac{d\mathcal{K}_Q}{d\mathcal{L}_Q}$ instead of $P_{z_o}$, $\Lambda$ and $L_Q$ respectively. ■

In fact we have proved the following result which is a little stronger than theorem 2.1.

THEOREM 2.6. – *The sequence of probability measures $(\mathbb{P}_N)_{N\geq 1}$ obeys a large deviation principle in $M_\varphi(\chi \times \mathcal{S})$ endowed with the weak-$\star$ topology of pointwise convergence on the set $C_\varphi(\chi \times \mathcal{S})$. Its speed is $N$ and its rate function is $I(\cdot)$.*

It remains to prove Lemmas 2.9 and 2.10. In order to prove Lemma 2.9 we shall need preliminary results which are stated at the next Lemmas 2.7 and 2.8.

LEMMA 2.7. – *a) For every real function $f$ on $\mathcal{Z} \times [0, T]$ which is bounded on any compact set and such that $\forall\, 0 \leq t \leq T$, $z \mapsto f(z, t)$ is continuous on $\mathcal{Z}$, the function $x \mapsto \int_0^T f(x_t, t)\, dt$ is continuous on $\chi$.*

*b) For any real continuous function $g$ on $\mathcal{Z} \times [0, T] \times E$,*

$$x \mapsto \sum_{0\leq t<T} \mathbf{1}_{\{x_t \neq x_{t-}\}}\, g(x_{t-}, t, \Delta x_t)$$

*is a continuous function on $\chi$.*

*Proof.* – The part *a*) is clear and the topology of $\chi$ is built in order to turn functions of the type of $x \mapsto \sum_{0\leq t<T} \mathbf{1}_{\{x_t \neq x_{t-}\}}\, g(x_{t-}, t, \Delta x_t)$ in b) into continuous functions. ■

LEMMA 2.8. – *Let $M_1(D[0, T], \mathcal{Z}))$ be endowed with its usual weak topology and $D([0, T], M_1(\mathcal{Z}))$ be equipped with the metric*

$$\rho(\mu, \nu) = \sup_{0\leq t\leq T} \delta(\mu_t, \nu_t), \qquad \mu, \nu \in D([0, T], M_1(\mathcal{Z})), \tag{2.10}$$

*where $\delta(\alpha, \beta)$, $\alpha, \beta \in M_1(\mathcal{Z})$ is a metric on $M_1(\mathcal{Z})$ which generates the weak topology $\sigma(M_1(\mathcal{Z}), C_b(\mathcal{Z}))$. The projection*

$$\begin{aligned} \Pi: \quad Q \in M_1(D([0, T], \mathcal{Z})) &\mapsto \Pi(Q) \\ &\doteq (Q_t)_{0\leq t\leq T} \in D([0, T], M_1(\mathcal{Z})) \end{aligned} \tag{2.11}$$

*is continuous at each $Q \in M_1(D([0, T], \mathcal{Z}))$ such that $J(Q) < +\infty$.*

*Remark.* – As a consequence of the following proof, it can be noticed that this projecion is not continuous as these $Q$'s which admit fixed discontinuity times.

*Proof.* – Since $M_1(D([0, T], \mathcal{Z}))$ and $D([0, T], M_1(\mathcal{Z}))$ are metric spaces, it is enough to check that when $J(Q) < +\infty$ and $Q^n \xrightarrow[n\to\infty]{} Q$ in $M_1(D([0, T], \mathcal{Z}))$, then $\Pi(Q^n) \xrightarrow[n\to\infty]{} \Pi(Q)$ in $D([0, T], M_1(\mathcal{Z}))$.

By the Skorokhod representation theorem, there exist a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $D([0, T], \mathcal{Z})$-valued random variables $X$ and $(X^n)_{n\geq 1}$ such that $\mathbb{P} \circ X^{-1} = Q$, $\mathbb{P} \circ (X^n)^{-1} = Q^n$, $\forall\, n \geq 1$ and $X^n \xrightarrow[n\to\infty]{} X$, $\mathbb{P}$-almost surely.

Making use of Wasserstein's metric (or Dudley's metric as well) (*see* [Rac], corollary 10.2.1), it is immediate to see that it remains to show that

$$\sup_{0\leq t\leq T} \mathbb{E}(|X_t^n - X_t| \wedge 1) \xrightarrow[n\to\infty]{} 0. \tag{2.12}$$

Let $x \in D([0, T], \mathcal{Z})$, $\delta > 0$ and $S \subset [0, T]$. As in [Bil], we denote

$$w_x(S) = \sup\{|x_s - x_t|;\ s, t \in S\}$$
$$w'_x(\delta) = \inf_{\{t_i\}} \max_{1\leq i\leq r} w_x([t_{i-1}, t_i[)$$

where the infimum extends over the finite sets $\{t_i\}$ of points such that

$$\begin{cases} 0 = t_0 < t_1 < \cdots < t_r = T \\ t_i - t_{i-1} > \delta, \qquad i = 1, 2, \ldots, r. \end{cases}$$

Let $\varepsilon > 0$, choose $\{t_i\}$ as above such that

$$\max_{1\leq i\leq r} w_x([t_{i-1}, t_i[) < w'_x(\delta) + \varepsilon.$$

Let $s \leq u$ be such that $|s - u| \leq \delta$, then

$$t_{i-1} \leq s \leq u \leq t_i \qquad \textit{and} \qquad |x_s - x_u| \leq w_x([t_{i-1}, t_i[) \leq w'_x(\delta) + \varepsilon,$$

or

$$t_{i-1} \leq s \leq t_i \leq u \leq t_{i+1}$$

and

$$\begin{aligned} |x_s - x_u| &\leq w_x([t_{i-1}, t_i[) + w_x([t_i, t_{i+1}[) + |\Delta x_{t_i}| \\ &\leq 2\, w'_x(\delta) + \sup_{s\leq\tau\leq u} |\Delta x_\tau| + 2\,\varepsilon. \end{aligned}$$

From which we deduce that

$$w_x([a, b]) \le 2\, w'_x(b-a) + \sup_{a \le s \le b} |\Delta x_s|, \quad \forall\, 0 \le a \le b \le T. \tag{2.13}$$

Let $\alpha > 0$, $0 \le t \le T$ and $x, y \in D([0, T], \mathcal{Z})$ be such that $d_{\text{Sko}}(x, y) \le \alpha$, then for any $\varepsilon > 0$, one can find a time change $\lambda$ such that $\sup_{0 \le s < T} |\lambda(s) - s| \le \alpha + \varepsilon$ and $\sup_{0 \le s < T} |y_s - x_{\lambda(s)}| \le \alpha + \varepsilon$. Since

$$|x_t - y_t| \le |y_t - x_{\lambda(t)}| + |x_{\lambda(t)} - x_t|,$$

we get:

$$d_{\text{Sko}}(x, y) \le \alpha \Rightarrow |x_t - y_t| \le \alpha + w_x([t-\alpha, t+\alpha]).$$

In addition with (2.13), this leads us for all $\alpha > 0$, to

$$\begin{aligned} \mathbb{E}(|X_t^n - X_t| \wedge 1) &\le \mathbb{P}(d_{\text{Sko}}(X^n, X) > \alpha) \\ &\quad + \mathbb{E}(d_{\text{Sko}}(X^n, X) \le \alpha;\ |X_t^n - X_t| \wedge 1) \\ &\le \mathbb{P}(d_{\text{Sko}}(X^n, X) > \alpha) + \alpha \\ &\quad + \mathbb{E}(w_X([t-\alpha, t+\alpha]) \wedge 1) \\ &\le \mathbb{P}(d_{\text{Sko}}(X^n, X) > \alpha) + \alpha \\ &\quad + 2\, \mathbb{E}(w'_X(2\alpha) \wedge 1) \\ &\quad + \mathbb{E}(\sup_{s \in [t-\alpha, t+\alpha]} |\Delta X_s| \wedge 1). \end{aligned}$$

As $X^n \xrightarrow[n \to \infty]{} X$, $\mathbb{P}$-almost surely and for all $x \in D([0, T], \mathcal{Z})$, $w'_x(\delta)$ tends to zero as $\delta$ tends to zero ([Bil], (14.8)), to get (2.12), it remains to check that

$$\sup_{0 \le t \le T} \mathbb{E}(\sup_{s \in [t-\alpha, t+\alpha]} |\Delta X_s| \wedge 1) \xrightarrow[\alpha \downarrow 0]{} 0. \tag{2.14}$$

Clearly,

$$\begin{aligned} &\mathbb{E}(\sup_{s \in [t-\alpha, t+\alpha]} |\Delta X_s| \wedge 1) \\ &= \int_{D([0, T], \mathcal{Z})} (\sup_{s \in [t-\alpha, t+\alpha]} |\Delta x_s| \wedge 1)\, Q(dx) \\ &\le Q(\{x;\ \exists\, s \in [t-\alpha, t+\alpha],\ \Delta x_s \ne 0\}) \\ &= \int_{\mathcal{X} \times [0, T] \times E} \mathbf{1}_{[t-\alpha, t+\alpha]}(s)\, L_Q(x, s, \Delta)\, \Lambda(d\Delta)\, Q(dx)\, ds \\ &\le \| L_Q \|_{\tau^*, \Gamma_Q} \| \mathbf{1}_{[t-\alpha, t+\alpha]} \|_{\tau, \Gamma_Q} \end{aligned}$$

where $L_Q$ appears in lemma 2.4, $\|\cdot\|_{\tau^*, \Gamma_Q}$ and $\|\cdot\|_{\tau, \Gamma_Q}$ are the Orlicz norms with respect to the measure $\Gamma_Q(d\Delta\, dx\, ds) := \Lambda(d\Delta)\, Q(dx)\, ds$ on $\chi \times [0, T] \times E$ and the functions $\tau$ and $\tau^*$ are given at (2.2) and (2.3). Last inequality is Hölder's inequality in Orlicz spaces.

Since $J(Q) < +\infty$, by lemma 2.4, we have $\| L_Q \|_{\tau^*, \Gamma_Q} < +\infty$. Moreover,

$$\begin{aligned}\| \mathbf{1}_{[t-\alpha, t+\alpha]} \|_{\tau, \Gamma_Q} &= 1/\tau^{-1}\left(1/\Gamma_Q(\chi \times [t-\alpha, t+\alpha] \times E)\right) \\ &\leq 1/\tau^{-1}\left(\frac{1}{2\Lambda(E)\alpha}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_{0 \leq t \leq T} \mathbb{E}\left(\sup_{s \in [t-\alpha, t+\alpha]} |\Delta X_s| \wedge 1\right) \\ &\leq \| L_Q \|_{\tau^*, \Gamma_Q} / \tau^{-1}\left(\frac{1}{2\Lambda(E)\alpha}\right) \xrightarrow[\alpha \downarrow 0]{} 0\end{aligned}$$

which is (2.14). This completes the proof of the lemma. ■

We are now ready to prove Lemma 2.9.

LEMMA 2.9. – *The function $h : M_\varphi(\chi) \to \mathbb{R}$ given at* (2.9) *is continuous at any $Q$ such that $J(Q) < +\infty$.*

*Proof.* – Let us show that with

$$\theta_Q(x) = \sum_{0 \leq t < T} \mathbf{1}_{\{x_t \neq x_{t^-}\}} \log\left(\frac{d\mathcal{L}(x_{t^-}, Q(t^-))}{d\Lambda}(\Delta x_t)\right)$$

and

$$\gamma_Q(x) = \int_{[0, T] \times E} \left\{\frac{d\mathcal{L}(x_t, Q(t))}{d\Lambda}(\Delta) - 1\right\} \mathbf{1}_{\{x_t + \Delta \in \mathcal{Z}\}}\, \Lambda(d\Delta)\, dt,$$

the functions $Q \mapsto \int_\chi \theta_Q\, dQ$ and $Q \mapsto \int_\chi \gamma_Q\, dQ$ are continuous at any $Q$ such that $J(Q) < +\infty$. By assumption $(A.2)$ we have for some $C \geq 0$

$$\begin{aligned}|\theta_Q(x)| &\leq \sup_{z, \xi, \Delta}\left(\left|\log\left(\frac{d\mathcal{L}(z, \xi)}{d\Lambda}(\Delta)\right)\right|\right)\varphi(x) \\ &\leq C(1 + \varphi(x)), \qquad \forall x \in \chi.\end{aligned}$$

Similarly, by assumptions $(A.1)$ and $(A.2)$ we have for some $C \geq 0$

$$|\gamma_Q(x)| \leq \sup_{z,\,\xi} \left| \int_{[0,\,T]\times E} \left\{ \frac{d\mathcal{L}(z,\,\xi)}{d\Lambda}(\Delta) - 1 \right\} \times \mathbf{1}_{\{z+\Delta\in\mathcal{Z}\}}\, \Lambda(d\Delta)\, dt \right| < +\infty, \qquad \forall\, x \in \chi.$$

Therefore,

$$|h(Q)| \leq C\left(1 + \int_\chi \varphi(x)\, Q(dx)\right), \qquad \forall\, Q \in M_\varphi(\chi). \tag{2.15}$$

We have

$$|\langle \theta_Q,\, Q\rangle - \langle \theta_{Q'},\, Q'\rangle| \leq |\langle \theta_Q,\, Q - Q'\rangle| + |\langle \theta_Q - \theta_{Q'},\, Q'\rangle|.$$

To control $|\langle \theta_Q,\, Q - Q'\rangle|$ it is enough to nocite that $\theta_Q$ belongs to $C_\varphi(\chi)$ provided that $J(Q) < +\infty$. Indeed, by Lemma 2.7*b*, Lemma 2.8 and assumption $(A.3)$ it is continuous and we have just seen that $\theta_Q \leq C(1+\varphi)$. On the other hand,

$$|\langle \theta_Q - \theta_{Q'},\, Q'\rangle| \leq \sup_{z,\,t,\,\Delta} \left| \log\left(\frac{d\mathcal{L}(z,\,Q_t)}{d\Lambda}(\Delta)\right) - \log\left(\frac{d\mathcal{L}(z,\,Q'_t)}{d\Lambda}(\Delta)\right) \right| \int_\chi \varphi\, dQ'$$

which is controlled for any $Q'$ in a neighbourhood of $Q$ in $M_\varphi(\chi)$, provided that $J(Q) < +\infty$, thanks to Lemma 2.8 and assumption $(A.4)$.

Similarly, one proves that $Q \mapsto \int_\chi \gamma_Q\, dQ$ is continuous on $M_\varphi(\chi)$ [with Lemma 2.7*a* and assumption $(A.4)$ instead of Lemma 2.7*b* and assumption $(A.3)$]. This completes the proof of the lemma. ■

It remains to show that $h$ and $\mathbb{P}^o_N$, $N \geq 1$ satisfy (2.7).

LEMMA 2.10. – *For any nonnegative real* $\alpha$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \int_{M_\varphi(\chi)} \exp(N\,\alpha\,|h|)\, d\mathbb{P}^o_N < +\infty.$$

*Proof.* – Let us show that

$$\limsup_{N\to\infty} \frac{1}{N} \log \int_{M_\varphi(\chi)} \exp\left(N\,\alpha \int_\chi \varphi(x)\, Q(dx)\right) d\mathbb{P}^o_N(Q) < +\infty.$$

We have

$$\int_{M_\varphi(\chi)} \exp\left(N\alpha \int_\chi \varphi(x)\, Q(dx)\right) d\mathbb{P}_N^o(Q) = \int_{\chi^N} \exp\left(\alpha \sum_{i=1}^N \varphi(x_i)\right) P_{z_o}^{\otimes N}(dx_1 \dots dx_N)$$

for which we deduce that

$$\limsup_{N\to\infty} \frac{1}{N} \log \int_{M_\varphi(\chi)} \exp(N\alpha\varphi)\, d\mathbb{P}_N^o = \int_\chi \exp(\alpha\varphi(x)\, P_{z_o}(dx)) = \exp(T\Lambda(E)(e^\alpha - 1)) < +\infty,$$

since $\exp(\alpha \sum_{0\le u\le t} \mathbf{1}_{\{x_u \ne x_{u-}\}} - t\Lambda(E)(e^\alpha - 1))$ is a $P_{z_o}$-martingale with expectation 1. One completes the proof with (2.15). ■

## 3. LARGE DEVIATION FOR THE EMPIRICAL PROCESSES

At Theorem 3.1 below, we state a large deviation principle for the sequence $(P_N)_{N\ge 1}$ of the laws of the $\bar{X}^N$'s. Clearly, for any $N \ge 1$, $P_N$ is the image law

$$P_N = \mathbb{P}_N \circ \theta_N^{-1}$$

of $\mathbb{P}_N$ on $D([0,\, T],\, M_1(\mathcal{Z} \times \mathcal{S}))$ by the application

$$\hat{\theta}_N: \quad \hat{x}^N = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i,\, s_i^N)} \in M_1(D([0,\, T],\, \mathcal{Z}) \times \mathcal{S})$$
$$\mapsto \bar{x}^N = \left(t \mapsto \bar{x}^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i(t),\, s_i^N)}\right) \in D([0,\, T],\, M_1(\mathcal{Z} \times \mathcal{S})).$$

This large deviation principle is similar to ([DaG], Therorem 4.5). When its proof is similar to ([DaG], Theorem 4.5)'s one, we simply refer to similar statements in [DaG], otherwise we give some details on the differences.

Before setting the rate function, we need some notions which have been introduced by D. A. Dawson and J. Gärtner.

**a) About measure-valued functions.** ([DaG], section 4.1). Let us recall a notion introduced by D. A. Dawson and J. Gärtner in [DaG]. Let $\mathcal{D}$ denote the Schwartz space of test functions $\mathbb{R}^d \to \mathbb{R}$ with its usual inductive topology and $\mathcal{D}'$ the corresponding space of real distributions. For each compact set $K \subset \mathbb{R}^d$ let $\mathcal{D}_K$ be the subspace of $\mathcal{D}$ consisting of all test functions with their support in $K$.

3.1. DEFINITION. – *A map* $\nu(\cdot) : [0, T] \to \mathcal{D}'$ *is called* absolutely continuous *if for each compact set* $K \subset \mathbb{R}^d$ *there exist a neighbourhood* $U_K$ *of* 0 *in* $\mathcal{D}_K$ *and an absolutely continuous function* $H_K : [0, T] \to \mathbb{R}$ *such that*

$$|\langle f, \nu(u)\rangle - \langle f, \nu(v)\rangle| \leq |H_K(u) - H_K(v)|$$
$$\forall\, 0 \leq u, v \leq T, \quad \forall\, f \in U_K.$$

It has been proved in ([DaG], Lemma 4.2) that for any absolutely continuous map $\nu(\cdot) : [0, T] \to \mathcal{D}'$ and any $f$ in $\mathcal{D}$, $\langle f, \nu(\cdot)\rangle$ is absolutely continuous and that the derivative in the distribution sense

$$\dot{\nu}(t) = \lim_{h \to 0} \frac{1}{h}[\nu(t+h) - \nu(t)]$$

exists for almost all $0 \leq t \leq T$ and the following integration by part formula holds

$$\begin{aligned}(3.2) \qquad &\langle \nu(v), f(v, \cdot)\rangle - \langle \nu(u), f(u, \cdot)\rangle \\ &= \int_u^v \langle \dot{\nu}(t), f(t, \cdot)\rangle\, dt + \int_u^v \langle \nu(t), \dot{f}(t, \cdot)\rangle\, dt.\end{aligned}$$

for each $0 \leq u \leq v \leq T$ and each $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that $f(t, \cdot) \in \mathcal{D}$, $\forall\, 0 \leq t \leq T$ and $f(\cdot, z) \in C^1([0, T])$, $\forall\, z \in \mathbb{R}^d$.

**b) The large deviation principle.** For any $\theta \in M(\mathbb{R}^d)$, $0 \leq t \leq T$, $s \in \mathcal{S}$ and $\mu \in D([0, T]), M_1(\mathcal{Z} \times \mathcal{S}))$, let us define

$$|||\theta|||_{\mu, t, s} = \sup_{f \in C}\left\{\langle \theta, f\rangle - \int_{\mathcal{Z} \times E} \tau(Df(z, \Delta))\,\mathcal{L}(z, s, \mu_t; d\Delta)\,\mu_t^s(dz)\right\}$$

where the function $\tau$ is defined at (2.2) and

$$Df(z, \Delta) = f(z + \Delta) - f(z)$$

and

$$\mathcal{C} = C_c\,(]0,\, T[) \times \mathcal{Z}).$$

Notice that $|||\ \cdot\ |||_{\mu,\,t,\,s}$ is not a norm and that $f\,(0,\,\cdot\,) = f\,(T,\,\cdot\,) \equiv 0$ for all $f \in \mathcal{C}$.

For any $\xi \in M_1\,(\mathcal{Z} \times \mathcal{S})$ and $s \in \mathcal{S}$, let $A_{\xi,\,s}$ be the infinitesimal generator defined for any $f$ in $B_b\,(\mathcal{Z})$ by:

$$A_{\xi,\,s}\, f\,(z) = \int_E Df\,(z,\,\Delta)\,\mathcal{L}\,(z,\,s,\,\xi)\,(d\Delta)$$

while $A^*_{\xi,\,s}$ denotes its formal adjoint.

We are now ready to introduce the functional

$$S:\ D\,([0,\,T],\, M_1\,(\mathcal{Z} \times \mathcal{S})) \to [0,\, +\infty]$$

by setting

$$S\,(\mu) = \int_{[0,\,T]\times\mathcal{S}} |||\,\dot{\mu}^s_t - A^*_{\mu_t,\,s}\,|||_{\mu,\,t,\,s}\, m\,(ds)\,dt \tag{3.3}$$

if $\mu_{\mathcal{S}} = m$, $\mu\,(0) = \delta_{z_o} \otimes m$ and $\mu^s\,(\,\cdot\,)$ is absolutely continuous in the sense of definition (3.1), for $m$-almost every $s$ in $\mathcal{S}$, and $S\,(\mu) = +\infty$ otherwise (*see* [DaG], (4.9)).

For all $\mu$ in $D\,([0,\,T],\, M_1\,(\mathcal{Z} \times \mathcal{S}))$, the Orlicz space

$$L^{\tau^*}_\mu := L^{\tau^*}\,([0,\,T] \times \mathcal{Z} \times \mathcal{S} \times E,\ \mathcal{L}\,(z,\,s,\,\mu_t;\,d\Delta)\,\mu_t\,(dz\,ds)\,dt)$$

is endowed with its natural norm $\|\cdot\|_{\tau^*,\,\mu} := \|\cdot\|_{\tau^*,\,\mathcal{L}\,(z,\,s,\,\mu_t;\,d\Delta)\,\mu_t\,(dz\,ds)\,dt}$. The function $\tau^*$ is defined at (2.3). Let us denote $Df\,(z,\,s,\,t,\,\Delta) = f\,(z+\Delta,\,s,\,t) - f\,(z,\,s,\,t)$ for any $f$ in $B_b\,(\mathcal{Z} \times \mathcal{S} \times [0,\,T])$ and let us define

$\mathcal{H}_\mu$ is the $\sigma\,(L^{\tau^*}_\mu,\, C_c)$-closure of $\{e^{Df} - 1;\ f \in C_c\,(]0,\, T[\times \mathcal{Z} \times \mathcal{S})\}$.

3.4. DEFINITION. – *One says that* $\mu \in D\,([0,\,T],\, M_1\,(\mathcal{Z} \times \mathcal{S}))$ *is a* $S$-path *if one can find a function* $h_\mu$ *in* $\mathcal{H}_\mu$ *such that* $\mu$ *is a solution to the following evolution with initial condition* $\mu^s\,(0) = \delta_{z_o}$

$$\begin{aligned} \langle \mu^s_t,\, f\rangle = {} & \langle \mu^s\,(0),\, f\rangle \\ & + \int_0^t \Big\langle \mu^s_u\,(dz),\ \int_E Df\,(z,\,\Delta)\,[h_\mu\,(z,\,s,\,u,\,\Delta) + 1] \\ & \qquad \times \mathcal{L}\,(z,\,s,\,\mu_u;\,d\Delta) \Big\rangle\, du \end{aligned} \tag{3.5}$$

*for all* $0 \le t \le T$, *all* $f \in C_c\,(\mathbb{R}^d)$ *and* $m$-*almost every* $s \in \mathcal{S}$.

The space $D([0, T], M_1(\mathcal{Z} \times \mathcal{S}))$ is endowed with the following metric $\rho$ [*see* (2.10)]

$$\rho(\lambda, \nu) = \sup_{0 \le t \le T} \delta(\mu_t, \nu_t), \ \mu, \nu \in D([0, T], M_1(\mathcal{Z} \times \mathcal{S})),$$

where $\delta(\alpha, \beta)$, $\alpha, \beta \in M_1(\mathcal{Z} \times \mathcal{S})$ is a metric on $M_1(\mathcal{Z} \times \mathcal{S})$ which generates the weak topology $\sigma(M_1(\mathcal{Z} \times \mathcal{S}), C_b(\mathcal{Z} \times \mathcal{S}))$.

The main result of this section is the following theorem.

THEOREM 3.1. – *Let us suppose that the assumptions* $(A.0)$-$(A.4)$ *are satisfied. Then, the sequence of probability measures* $(P_N)_{N \ge 1}$ *obeys a large deviation principle in* $D([0, T], M_1(\mathcal{Z} \times \mathcal{S}))$ *with speed* $N$ *and the rate function* $S$ *given at* (3.3).

*If* $\mu \in D([0, T], M_1(\mathcal{Z} \times \mathcal{S}))$ *satisfies* $S(\mu) < +\infty$, *then* ($\mu_S \equiv m$ *and)* $\mu$ *is absolutely continuous in the sense of* (3.1) *and there exists a unique* $h_\mu$ *in* $\mathcal{H}_\mu$ *[up to almost everywhere equality with respect to* $\mathcal{L}(z, s, \mu_t; d\Delta)\, \mu_t(dz\, ds)\, dt$*], such that* $\mu$ *is a solution to the weak equation* (3.5).

*For any S-path* $\mu \in D([0, T], M_1(\mathcal{Z} \times \mathcal{S}))$ [*see* (3.4)] *such that* $\mu_S \equiv m$ *and* $\mu(0) = \delta_{z_o}$, *the* $[0, +\infty]$*-valued function* $S$ *is given by*

$$\begin{aligned}(3.6)\ S(\mu) = &\int_{[0, T] \times \mathcal{Z} \times \mathcal{S}} \\ &\times \left( \int_E \tau^*(h_\mu(z, s, t, \Delta))\, \mathcal{L}(z, s, \mu_t; d\Delta) \right) \mu_t(dz\, ds)\, dt \\ = &\int_0^T \int_{\mathcal{Z} \times \mathcal{S}} I((h_\mu(z, s, t, \cdot) + 1)\, \mathcal{L}(z, s, \mu_t; \cdot) \\ &\times \mid \mathcal{L}(z, s, \mu_t; \cdot))\, \mu_t^s(dz)\, m(ds)\, dt\end{aligned}$$

*where* $h_\mu$ *is associated with* $\mu$ *by* (3.5).

**c) The proof of Theorem 3.1.** As in section 2, we are going to prove our theorem in the $s$-independent case. The proof in the $s$-dependent case follows easily from the $s$-independent one; it is left to the reader.

We first derive a representation of $S$ which is a result similar to ([DaG], Lemma 4.8).

LEMMA 3.2. – *Let* $\mu \in D([0, T], M_1(\mathcal{Z}))$ *be such that* $\mu(0) = \delta_{z_o}$. *Then,*

$$S(\mu) = \sup_{f \in C} H(\mu, f)$$

*where*

$$H(\mu, f) = -\int_0^T \left\langle \mu_t, \left(\frac{\partial}{\partial t} + A_{\mu_t}\right) f(t, \cdot)\right\rangle dt$$
$$-\int_0^T \left\langle \mu_t, \int_E \tau(Df(t, \cdot, \Delta))\, \mathcal{L}(\cdot, \mu_t)(d\Delta)\right\rangle dt.$$

*Moreover, if* $\mu$ *satisfies* $S(\mu) < +\infty$*, then one can find a unique function* $h_\mu$ *in* $\mathcal{H}_\mu$ *such that* $\mu$ *is a weak solution of* (3.5) *and* $S(\mu)$ *satisfies* (3.6).

*Conversely, if* $\mu$ *is a S-path,* $S(\mu)$ *is given by* (3.6) *where* $h_\mu$ *is associated with* $\mu$ *by the equation* (3.5).

*Proof of Lemma* 3.2. – We fix $\mu$ and we write $A_t$ for $A(\mu_t)$ [*see* (1.4)], $|||\cdot|||_t$ for $|||\cdot|||_{\mu, t}$ and $\mathcal{L}_t(\cdot)$ for $\mathcal{L}(\cdot, \mu_t)$. Let us set for all $0 \leq r < t \leq T$ and $f \in \mathcal{C}$

$$l_{r,t}(f) = \langle \mu_t, f(t, \cdot)\rangle - \langle \mu_r, f(r, \cdot)\rangle - \int_r^t \left\langle \mu_u, \left(\frac{\partial}{\partial u} + A_u\right) f(u, \cdot)\right\rangle du$$

and

$$H_{r,t}(f) = l_{r,t}(f) - \int_r^t \left\langle \mu_u, \int_E \tau(Df(u, \cdot, \Delta))\, \mathcal{L}_u(\cdot; d\Delta)\right\rangle du.$$

($l_{r,t}$ and $H_{r,t}$ depend on $\mu$ which is fixed.)

Computing $H_{r,t}(f/c)$ and $H_{r,t}(-f/c)$ with $c > 0$, we obtain

$$|l_{r,t}(f)| \leq c\left(\sup_{g \in \mathcal{C}} H_{r,t}(g) + \int_{[r,t]\times\mathcal{Z}\times E} \tau\left(\frac{|Df(u, z, \Delta)|}{c}\right) \lambda_\mu(du\, dz\, d\Delta)\right)$$

with

$$\lambda_\mu(du\, dz\, d\Delta) = \mathcal{L}_u(z; d\Delta)\, \mu_u(dz)\, du.$$

Choosing $c$ to be equal to the Orlicz norm

$$\|Df\|_{\tau, r, t} = \inf\left\{a > 0; \int_{[r,t]\times\mathcal{Z}\times E} \tau\left(\frac{|Df(u, z, \Delta)|}{a}\right) \lambda_\mu(du\, dz\, d\Delta) \leq 1\right\}$$

we get for any $0 \leq r \leq T$ and $f \in \mathcal{C}$,

$$|l_{r,t}(f)| \leq 1\left(1 + \sup_{g \in \mathcal{C}} H_{r,t}(g)\right) \|Df\|_{\tau, r, t}.$$

An approximation procedure (*see* [DaG], p. 277) leads us to $\sup_{g \in \mathcal{C}} H_{r,t}(g) \le \sup_{g \in \mathcal{C}} H(\mu, g)$ where $H(\mu, g) = H_{0,T}(g)$. Hence, for any $f \in \mathcal{C}$ and any $0 \le r < t \le T$

$$|l_{r,t}(f)| \le (1 + \sup_{g \in \mathcal{C}} H(\mu, g)) \, \| Df \|_{\tau, r, t}. \tag{3.7}$$

It is easy to prove (*see* [DaG], (4.27)) by means of the integration by parts formula (3.2) that

$$S(\mu) \ge \sup_{f \in \mathcal{C}} H(\mu, f).$$

Now, let us show that

$$S(\mu) \le \sup_{f \in \mathcal{C}} H(\mu, f).$$

This is obvious if $\sup_{f \in \mathcal{C}} H(\mu, f) = +\infty$. Therefore, let us suppose that

$$\sup_{f \in \mathcal{C}} H(\mu, f) < +\infty. \tag{3.8}$$

By (3.7) and (3.8), $l_{r,t}(f)$ only depends on $Df$ and $Df \mapsto l_{r,t}(f)$ is a $\| \cdot \|_{\tau, r, t}$-continuous linear form on the closure $D^{\tau}_{r,t}$ of $\{Df;\ f \in B_b([r, t] \times \mathcal{Z} \times E)\}$ in the Orlicz space $(L^{\tau}([r, t] \times \mathcal{Z} \times E, \lambda_\mu), \|\cdot\|_{\tau, r, t})$. By Hahn-Banach theorem and the Riesz representation theorem for Orlicz spaces, there exists at least one $k_{r,t}$ in $L^{\tau^*}([0, T] \times \mathcal{Z} \times E, \lambda_\mu)$ such that: $l_{r,t}(f) = \int_{[r,t] \times \mathcal{Z} \times E} k_{r,t} \, Df \, d\lambda_\mu$, $\forall 0 \le r < t \le T$. Since $l_{r,t}(f)$ does not depend on the time values $s \notin [r, t]$, it is easy to show that one can choose the $k_{r,t}$'s such that $\mathbf{1}_{[r,t]} k_{r,t} = \mathbf{1}_{[r,t]} k_\mu$, $\forall 0 \le r < t \le T$ where $k_\mu = k_{0,T}$ (*see* [DaG], p. 279). This gives us, for any $f \in \mathcal{C}$ and $0 \le r < t \le T$,

$$l_{r,t}(f) = \int_{[r,t] \times \mathcal{Z} \times E} k_\mu \, Df \, d\lambda_\mu \tag{3.9}$$

and

$$H(\mu, f) = \int_{[0,T] \times \mathcal{Z} \times E} (k_\mu \, Df - \tau(Df)) \, d\lambda_\mu.$$

Therefore,

$$\sup_{f \in \mathcal{C}} H(\mu, f) = \sup_{f \in \mathcal{C}} \left\{ \int_{[0,T] \times \mathcal{Z} \times E} k_\mu \, Df \, d\lambda_\mu - G(Df) \right\} = G^*(k_\mu)$$

is the Legendre transform $G^*$ at point $k_\mu$ of the nonnegative convex function

$$G(Df) = \int_{[0,T]\times \mathcal{Z} \times E} \tau(Df)\, d\lambda_\mu$$

on

$$D\mathcal{C} = \{Df;\ f \in \mathcal{C}\}.$$

It is proved in ([Lé3], Theorem 6.1) that $G^*(k_\mu)$ is finite if and only if there exists $h_\mu \in \mathcal{H}_\mu$ such that

$$\int k_\mu\, Df\, d\lambda_\mu = \int h_\mu\, Df\, d\lambda_\mu, \quad \forall\, Df \in D\mathcal{C}; \tag{3.10}$$

such a $h_\mu$ is unique in $\mathcal{H}_\mu$ (with respect to $\lambda_\mu$-almost everywhere equality) and

$$G^*(k_\mu) = G^*(h_\mu) = \int_{[0,T]\times \mathcal{Z} \times E} \tau^*(h_\mu)\, d\lambda_\mu \tag{3.11}$$

$\left(a\ priori, \int_{[0,T]\times \mathcal{Z} \times E} \tau^*\, d\lambda_\mu \geq \int_{[0,T]\times \mathcal{Z} \times E} \tau^*(h_\mu)\, d\lambda_\mu\right.$ if $k_\mu$ does not belong to $\left.\mathcal{H}_\mu.\right)$

To complete the proof of lemma 3.2, it remains to notice that (3.9) and (3.10) give (3.5) and that (3.5) together with the integration by parts formula (3.2) easily lead us to $G^*(h_\mu) = S(\mu)$. ■

Let us define for all $\mu \in D([0, T], M_1(\mathcal{Z}))$

$$V(\mu) = \inf \{I(Q \mid R(Q));\ Q \in M_1(D([0, T], \mathcal{Z})),\ \Pi(Q) = \mu\}$$

where $I(Q \mid R(Q))$ is the Kullback information and $\Pi$ is defined at (2.11).

Lemma 3.3. – *We have:* $V = S$.

*Proof.* (From an idea communicated by the referee). – Let $\mu \in D([0, T], M_1(\mathcal{Z}))$ be fixed. Since $R(Q) = R(\Pi(Q))$, we have

$$V(\mu) = \inf \{I(Q \mid R(\mu));\ Q \in M_1(D([0, T], \mathcal{Z})),\ \Pi(Q) = \mu\}.$$

Let $(X^{(i)})_{i\geq 1}$ be a sequence of i.i.d. processes with common law $R(\mu)$. By Sanov's theorem, the empirical measures $\frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{(i)}}$ obey a large deviation principle as $N$ tends to infinity with speed $N$ and rate function

$Q \in M_1(D([0, T], \mathcal{Z})) \mapsto I(Q \mid R(\mu))$. As in Lemma 2.8, one shows that the projection $\Pi$ is continuous at any $Q$ such that $I(Q \mid R(\mu)) < +\infty$. Therefore, one is allowed to apply the contraction principle to the large deviation principle of Theorem 2.1 (*see* [DZe], Theorem 4.2.1, Remark (*c*)) to obtain that the sequence $\left(t \in [0, T] \mapsto \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{(i)}}\right)_{N \geq 1}$ obeys a large deviation principle as $N$ tends to infinity with speed $N$ and rate function

$$\nu \mapsto S_\mu(\nu) = \inf\{I(Q \mid R(\mu));\ Q \in M_1(D([0, T], \mathcal{Z})),\ \Pi(Q) = \nu\}$$

from which it follows that

$$V(\mu) = S_\mu(\mu). \tag{3.12}$$

In order to evaluate the function $S_\mu$, let us consider the time-homogeneous Feller process $(\tau(s), \hat{X}_s)_{s \geq 0}$ where $\tau(s)$ belongs to the circle $\mathcal{T} = \mathbb{R}/(T\mathbb{Z}) = [0, T[$ is such that $s = kT + \tau(s)$ for some $k \in \mathbb{Z}$ and $\hat{X}_s = X_s^{(i)}$ if $s \in [(i-1)T, iT[$.

Thanks to a well-known Donsker and Varadhan's theorem (*see* [DeS], theorem 4.2.43), the sequence $\frac{1}{NT} \int_0^{NT} \delta_{(\tau(s), \hat{X}_s)} ds$ obeys a large deviation principle as $N$ tends to infinity with speed $N$ and rate function $U_\mu$ defined, for all $\tilde{\nu} \in M_1(\mathcal{T} \times \mathcal{Z})$, by

$$\begin{aligned}
U_\mu(\tilde{\nu}) &= T \sup\left\{-\int \frac{L(u)}{u}\, d\tilde{\nu};\ u \in D_L \cap C_b,\ L(u) \in C_b,\ u \geq 1\right\} \\
&= T \sup\left\{-\int e^{-f} L(e^f)\, d\tilde{\nu};\ f \in C_b^1,\ f \geq 0\right\} && (a) \\
&= T \sup\left\{\left\langle -L(f) - \int \tau(Df)\, d\mathcal{L}_\mu,\ \tilde{\nu}\right\rangle;\ f \in C_b,\ f \geq 0\right\} \\
&= T \sup\left\{\left\langle \mathbf{1}_{t=0=T}(f(T, \cdot) - f(0, z_o)) - \left(\frac{\partial}{\partial t} + A(\mu.)\right)(f)\right.\right. \\
&\qquad \left.\left. - \int \tau(Df)\, d\mathcal{L}_{\mu.},\ \tilde{\nu}\right\rangle;\ f \in C_b^1\right\} && (b) \\
&= T \sup\left\{\left\langle \mathbf{1}_{t=0=T}(f(T, \cdot) - f(0, z_o)) - \left(\frac{\partial}{\partial t} + A(\mu.)\right)(f)\right.\right. \\
a &\qquad \left.\left. - \int \tau(Df)\, d\mathcal{L}_{\mu.},\ \tilde{\nu}\right\rangle;\ f \in C_c^1\right\} && (c)
\end{aligned}$$

where $L$ is the generator of $(\tau(s), \hat{X}(s))$, $D_L$ is its domain, $C_b^1$ stands for the set of all continuous bounded functions on $\mathcal{T} \times \mathcal{Z}$ which are $t$-differentiable, $C_c^1$ stands for the set of all functions in $C_b^1$ with a compact support and $A(\mu_t)$ is the generator acting on $z \in \mathcal{Z}$ associated with the Lévy kernel $\mathcal{L}_{\mu_t}(d\Delta) = (\mathcal{L}(z, \mu_t; d\Delta))_{0 \leq t \leq T}$ (*see* (1.4)]. Notice that $f(0, \cdot) = f(T, \cdot)$, $\forall f \in C_b$.

One obtains equality $(a)$ taking $u = e^f$, equality $(b)$ comes from

$$L f(t, z) = \mathbf{1}_{t=0=T}(f(0, z_o) - f(T, z)) + \left(A(\mu_t) + \frac{\partial}{\partial t}\right) f(t, z)$$

and $L(f) = L(f - \inf_{\mathcal{T} \times \mathcal{Z}} f)$, equality $(c)$ is a consequence of the uniform boundedness of $(\mathcal{L}_{\mu_t})_{0 \leq t \leq T}$ (in the area of large jumps) thanks to assumptions $(A.1)$ and $(A.2)$.

If $U_\mu(\tilde{\nu}) < +\infty$, considering functions of the form $f(t, z) = \gamma g(t)$ with $g \in C^1(\mathcal{T})$ and $\gamma \in \mathbb{R}$, we get

$$\begin{aligned} &-\gamma \int_{\mathcal{T}} g'(t)\, \tilde{\nu}_{\mathcal{T}}(dt) \\ &\quad = -\gamma \int_{\mathcal{T} \times \mathcal{Z}} g'(t)\, \tilde{\nu}(dt\, dz) \leq T U_\mu(\tilde{\nu}) < +\infty, \quad \forall \gamma \in \mathbb{R}. \end{aligned}$$

Letting $|\gamma|$ tend to infinity, this implies that the $t$-marginal $\tilde{\nu}_{\mathcal{T}}$ of $\tilde{\nu}$ is

$$\tilde{\nu}_{\mathcal{T}} = \frac{1}{T} \mathbf{1}_{\mathcal{T}}(t)\, dt. \tag{3.13}$$

Hence, denoting $\nu_t$ for the $z$-marginal of $\tilde{\nu}$ conditionnally on $t$,

$$\begin{aligned} U_\mu(\tilde{\nu}) = \sup_{f \in C_c^1} \Bigg\{ \int_0^T \Bigg\langle &-\left(\frac{\partial}{\partial t} + A(\mu_t)\right)(f)(t, z) \\ &- \int_E \tau(Df(t, z, \Delta))\, \mathcal{L}_{\mu_t}(d\Delta), \nu_t(dz) \Bigg\rangle dt \Bigg\} \end{aligned} \tag{3.14}$$

if (3.13) holds, and $U_\mu(\tilde{\nu}) = +\infty$ otherwise.

Thanks to a convolution argument, one shows that the knowledge of $\frac{1}{NT} \sum_{i=1}^N \int_0^{NT} f(t, X_t^{(i)})\, dt$ for all $f \in C_c(\mathcal{T} \times \mathcal{Z})$ is equivalent to the knowledge of $\frac{1}{N} \sum_{i=1}^N g(t, X_t^{(i)})$, for all $t \in [0, T[$ and all

$g \in C_c(\mathcal{T} \times \mathcal{Z})$. As a consequence, there exists a one-one transport from $\frac{1}{NT} \int_0^{NT} \delta_{(\tau(s), \hat{X}_s)}\, ds \subset M_1(\mathcal{T} \times \mathcal{Z})$ to

$$\left( t \in [0, T[ \mapsto \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{(i)}} \right)_{N \geq 1} \subset D([0, T], M_1(\mathcal{Z}))$$

(paths are left continuous at $T$) which is continuous once $D([0, T], M_1(\mathcal{Z}))$ is endowed with the image topology. Therefore, the contraction principle and the uniqueness of a lower semicontinuous rate function of a large deviation principle (*see* [DeS], Lemma 2.1.1) provide us with

$$S_\mu(\nu) = U_\mu(\nu_t(dz)\, dt), \qquad \nu \in D([0, T], M_1(\mathcal{Z})),$$

[the lower semicontinuity of $S_\mu$ follows easily from the compactness of the Kullback information and Lemma 2.8, while the lower semicontinuity of $\nu \mapsto U_\mu(\nu_t(dz)\, dt)$ follows from the Legendre transform shape of $U_\mu$ and the continuity of the map $\nu \mapsto \nu_t(dz)\, dt$].

Together with (3.12), (3.14) and Lemma 3.2, this leads us to the desired result. ■

*The proof of Theorem* 3.1. – We have proved at Lemma 2.8 that the application

$$\Pi: \quad Q \in M_1(D([0, T], \mathcal{Z}) \times \mathcal{S}) \mapsto (Q_t)_{0 \leq t \leq T} \in D([0, T], M_1(\mathcal{Z} \times \mathcal{S}))$$

is continuous at any $Q$ such that $J(Q) < +\infty$ and *a fortiori* at any $Q$ such that $I(Q) < +\infty$. Therefore, one is allowed to apply the contraction principle to the large deviation principle of Theorem 2.1 (*see* [DZe], Theorem 4.2.1, remark (*c*)) to obtain that $(P_N)_{N \geq 1}$ obeys a large deviation principle in $D([0, T], M_1(\mathcal{Z}))$ with speed $N$ and rate function $V$. One concludes with Lemmas 3.2 and 3.3. The statement about absolute continuity is a simple consequence of the Orlicz space version of Hölder's inequality (with $L^\tau$ and $L^{\tau^*}$). This completes the proof of Theorem 3.1. ■

## APPENDIX

### Some definitions and results about Orlicz spaces

For basic definitions and results about Orlicz spaces one can read [KRu] and the appendix of [Nev].

**Some definitions** ([KRu], § 8, 9 and 10). The function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is called an *Orlicz function* if it is continuous, non-decreasing, convex and if it also satisfies:

$$\theta(0) = 0 \qquad \text{and} \qquad \lim_{t \to \infty} \frac{\theta(t)}{t} = +\infty.$$

In the following, $S$ is a Polish space, $\mathcal{F}$ is its Borel $\sigma$-field and $\nu$ is a nonnegative bounded measure on $(S, \mathcal{F})$.

The *Orlicz space* associated with $\theta$ is defined by:

$$\tilde{L}^\theta(S, \nu) = \left\{ f : S \to \mathbb{R};\ f \text{ is } \mathcal{F}\text{-measurable} \right.$$
$$\left. \text{and} \quad \int_S \theta(|f(x)|)\, d\nu(x) < +\infty \right\}.$$

In general, $\tilde{L}^\theta$ is not a vector space.

The *Orlicz space* associated with $\theta$ is defined by:

$$L^\theta(S, \nu) = \{ f : S \to \mathbb{R};\ f \text{ is } \mathcal{F}\text{-measurable and } \| f \|_\theta < +\infty \}$$

with

$$\| f \|_\theta = \inf \left\{ a > 0;\ \int_S \theta\left( \frac{|f(x)|}{a} \right) d\nu(x) \le 1 \right\}.$$

$\| \cdot \|_\theta$ is a norm, $L^\theta$ is a vector space and $\tilde{L}^\theta \subset L^\theta$. Let us also define: $E^\theta(S, \nu)$ is the closure of the set of $\mathcal{F}$-measurable bounded functions in $(L^\theta(S, \nu), \| \cdot \|_\theta)$.

The Orlicz spaces have been introduced to extend the notion of $L^p$-spaces. Indeed, $\theta_p(t) = \dfrac{t^p}{p}$ with $1 < p < +\infty$, is an Orlicz function, $L^{\theta_p} = L^p$ and $\| \cdot \|_{\theta_p}$ is equivalent to $\| \cdot \|_p$.

**Some properties.** $(L^\theta, \| \cdot \|_\theta)$ is a Banach space, the imbeddings: $L^\infty \hookrightarrow L^\theta \hookrightarrow L^1$ are continuous, $E^\theta \subset \tilde{L}^\theta \subset L^\theta$ and $L^\theta$ is the vector space spanned by $\tilde{L}^\theta$.

(Ap.1) $C_b$ is a dense subset of $(E^\theta, \| \cdot \|_\theta)$

(Ap.2) $f \in E^\theta \Leftrightarrow \forall \lambda \in \mathbb{R},\ \lambda f \in \tilde{L}^\theta$.

(Ap.3) Two Orlicz functions $\theta_1$ and $\theta_2$ determine the same Orlicz spaces with equivalent norms, if and only if:

$$\exists\, 0 < k_1 \le k_2, t_o \ge 0 \quad \text{such that} : \theta_1(k_1 t) \le \theta_2(t) \le \theta_1(k_2 t), \quad \forall t \ge t_o.$$

One says that the Orlicz function $\theta$ satisfies the $\Delta^2$-*condition* if

($\Delta_2$) There exist $k > 0$ and $t_o \ge 0$ such that: $\theta(2t) \le k\, \theta(t)$, $\forall t \ge t_o$.

We have: $\theta$ satisfies $(\Delta_2) \Leftrightarrow E^\theta = \tilde{L}^\theta = L^\theta$.

Notice that $\theta_p(t) = \dfrac{t^p}{p}$ $(1 < p < +\infty)$ satisfies $(\Delta_2)$ but that $\tau(t) = e^t - t - 1$ does not satisfy $(\Delta_2)$, although its Legendre conjugate $\tau^*$ satisfies $(\Delta_2)$.

**Some duality results.** Dualify for Orlicz spaces is closely related to Legendre conjugation. Let $\theta^*$ be the Legendre conjugate of $\theta$ on $\mathbb{R}_+$, that is:

$$\theta^*(u) = \sup_{t \geq 0} \{ut - \theta(t)\}, \qquad u \geq 0.$$

$\theta^*$ is also an Orlicz function so that one can consider $L^{\theta^*}(S, \nu)$.

(Ap.4) *Hölder's inequality:* For any $f \in L^\theta$, $g \in L^{\theta^*}$, $fg$ belongs to $L^1$ and

$$\| fg \|_1 \leq 2 \| f \|_\theta \| g \|_{\theta^*}.$$

Therefore, any $g$ in $L^{\theta^*}$ defines a continuous linear form on $L^\theta$, by the relation:

$$\langle f, g \rangle = \int_S f(x)\, g(x)\, d\nu(x), \quad \forall f \in L^\theta.$$

(Ap.5) *Representation of the dual space of $E^\theta$:* The strong dual space of $(E^\theta, \| \cdot \|_\theta)$ can be identified with $(L^{\theta^*}, \| \cdot \|_\theta^*)$, where for any $g \in L^{\theta^*}$

$$\| g \|_\theta^* = \sup \left\{ \int_S f(x) g(x) d\nu(x);\, f \text{such that } \int_S \theta(|f(x)|) d\nu(x) \leq 1 \right\},$$

and $\| g \|_{\theta^*} \leq \| g \|_\theta^* \leq 2 \| g \|_{\theta^*}$. Hence the strong dual space of $(E^\theta, \| \cdot \|_\theta)$ can be identified with $(L^{\theta^*}, \| \cdot \|_{\theta^*})$.

**References for the appendix.** Some of the above results are not exactly stated in [KRu], but they can be easily derived from the results of [KRu] which we refer to:

(Ap.1): § 10.2 + Lusin theorem, (Ap.2): *see* p. 84, (Ap.3): Theorems 13.2 and 13.3, (Ap.4): Theorem 9.3, (Ap.5): Theorem 14.2.

For (Ap.4) and (Ap.5) *see* also [Nev], Appendix.

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